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ON COUPLED NONLINEAR WAVE EQUATIONS OF KIRCHHOFF TYPE WITH DAMPING AND SOURCE TERMS

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Abstract. The initial boundary value problem for a system of nonlinear wave equations of Kirchhoff type with strong damping in a bounded domain is considered. The existence, asymptotic behavior and blow-up of solutions are discussed under some conditions. The decay estimates of the energy function and the estimates for the lifespan of solutions are given.

1. Introduction

We consider the initial boundary value problem for the following nonlinear coupled wave equations of Kirchhoff type :

$$u_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta u + h_1(u_t) = f_1(u) \quad \text{in } \Omega\times[0,\infty), \tag{1.1}$$

$$v_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta v + h_2(v_t) = f_2(v) \quad \text{in } \Omega\times[0,\infty), \tag{1.2}$$

with initial conditions,

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x\in\Omega, \tag{1.3}$$

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \quad x\in\Omega, \tag{1.4}$$

and boundary conditions,

$$u(x,t) = v(x,t) = 0, x\in\partial\Omega, t>0, \tag{1.5}$$

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where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial\Omega$ so that Divergence theorem can be applied. Let $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ be the Laplace operator, $h_1(u_t) = -\Delta u_t, h_2(v_t) = -\Delta v_t$ and $M(r)$ be a nonnegative locally Lipschitz function for $r \geq 0$ like $M(r) = m_0 + br^\gamma$, with $m_0 \geq 0$, $b \geq 0$, $m_0 + b > 0$, $\gamma \geq 1$, and $f_i(s)$, $i = 1, 2$, $s \in \mathbb{R}$, be a nonlinear function. We denote $\|\cdot\|_p$ to be L^p-norm.

The existence and nonexistence of solutions for a single wave equation of Kirchhoff type:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + h(u_t) = f(u) \quad \text{in } \Omega \times [0, \infty), \tag{1.6}$$

have been discussed by many authors and the references cited therein. The function h in (1.6) is considered in three different cases. For $h(u_t) = \delta u_t$, $\delta > 0$, the global existence and blow-up results can be found in $[3, 5, 12, 17]$; for $h(u_t) = -\Delta u_t$, some global existence and blow-up results are given in $[4, 5, 10, 13, 14, 17]$; for $h(u_t) = |u_t|^m u_t$, $m > 0$, the main results of existence and blow-up are in $[1, 2, 8, 11, 17]$. As a model it describes the nonlinear vibrations of an elastic string. When $h \equiv f \equiv 0$, Kirchhoff [6] was the first one to study the equation, so that (1.6) is named the wave equation of Kirchhoff type. For the system of wave equations related to $(1.1)-(1.5)$, Park and Bae $[15, 16]$ considered the system of $(1.1)-(1.5)$ with $h_i(s) = |s|^\alpha s, f_i(s) = |s|^\beta s$, $i = 1, 2$, $\alpha, \beta \geq 0$, $s \in \mathbb{R}$ and showed the global existence and asymptotic behavior of solutions under some restrictions on initial energy. Recently, Liu and Wang [7] considered the system $(1.1)-(1.5)$ with $M(r) = m_0 + br$, $h_i(r) = |r|^{\lambda_i} r$, $m_0 \geq 0$, $b \geq 0$, $m_0 + b > 0$, $\lambda_i \geq 0$, $i = 1, 2$ and obtain the global existence for the nonlinear damping with $\lambda_1 \geq \lambda_2$. Concerning blowing up property, Benaissa and Messaoudi [2] studied blowing up properties for the system $(1.1)-(1.5)$ with negative initial energy. Later, Wu and Tsai [18] studied the system $(1.1)-(1.5)$ with $M = M\left(\|\nabla u\|_2^2\right)$ and $M = M\left(\|\nabla v\|_2^2\right)$ in (1.1), (1.2), respectively. In that paper, we consider more general function f and obtain the blow-up result for small positive initial energy. Liu and Wang [7] considered blow-up properties of solutions for $(1.1)-(1.5)$ with linear damping.

The first purpose of this paper is to study the global existence and to derive decay properties of solutions to problem $(1.1)-(1.5)$. We obtain the solution decay at an exponential rate as $t \to \infty$ in the non-degenerate case ($m_0 > 0$) and a certain algebraic rate in the degenerate case ($m_0 = 0$) by using Nako's inequality [9]. The second purpose is to show blowing up of a local solution to problem $(1.1)-(1.5)$. We shall prove that the local solution blows up in finite time by applying the concave method, that is, we show that there exists a finite time $T^* > 0$ such that $\lim_{t \to T^{*-}} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx = \infty$. Estimates for the blow-up time T^* are also given. In this way, we extend the nonexistence result in [18] for more general M. This work also improves early one [13] in which the global existence and non-existence results have been established only for a single equation. The paper is organized as follows. In section 2, we present the preliminaries and some lemmas. In section 3, we will show the existence of a unique local solution (u, v) of our problem $(1.1) - (1.5)$ with $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ by applying the Banach fixed point theorem. In section 4, we first define an energy function $E(t)$ and show that it is a nonincreasing function. Then the global existence and decay property are derived in Theorem 4.5. Finally, the blow-up properties of $(1.1) - (1.5)$ are obtained in the case of the initial energy being non-positive.

2. Preliminaries

Let us begin by stating the following lemmas, which will be used later.

Lemma 2.1. (Sobolev-Poincaré inequality [13]). *If* $1 \leq p \leq \frac{2N}{[N-2m]^+}$ $(1 \leq p < \infty$ *if* $N \leq 2m)$, *then*

$$\|u\|_p \leq c_* \left\| (-\Delta)^{\frac{m}{2}} u \right\|_2, \quad \text{for } u \in D((-\Delta)^{\frac{m}{2}})$$

holds with some positive constant c_*, *where* $[a]^+ = \max\{a, 0\}$, $a \in \mathbb{R}$.

Lemma 2.2. [9]. *Let* $\phi(t)$ *be a non-increasing and nonnegative function on* $[0, T]$, $T > 1$, *such that*

$$\phi(t)^{1+r} \leq \omega_0 \left(\phi(t) - \phi(t+1)\right) \text{ on } [0, T],$$

where ω_0 *is a positive constant and* r *is a nonnegative constant. Then we have*

(i) *if* $r > 0$, *then*

$$\phi(t) \leq \left(\phi(0)^{-r} + \omega_0^{-1} r [t-1]^+\right)^{-\frac{1}{r}}.$$

(ii) *If* $r = 0$, *then*

$$\phi(t) \leq \phi(0) e^{-\omega_1 [t-1]^+} \text{ on } [0, T],$$

where $\omega_1 = \ln(\frac{\omega_0}{\omega_0 - 1})$, *here* $\omega_0 > 1$.

3. Local Existence

In this section we shall discuss the local existence of solutions to problem $(1.1) - (1.5)$ by method of Banach fixed point theorem. In the sequal, for the sake of simplicity we will omit the dependence on t, when the meaning is clear.

Assume that

(A1) $f_i(0) = 0,\ i = 1, 2$ and for any $\rho > 0$ there exists a constant $k(\rho) > 0$ such that

$$|f_1(s) - f_1(t)| \le k(\rho)\left(|s|^p + |t|^p\right)|s - t|,$$

and

$$|f_2(s) - f_2(t)| \le k(\rho)\left(|s|^q + |t|^q\right)|s - t|,$$

where $|s|,\ |t| \le \rho$, for $s, t \in \mathbb{R}$, and $0 \le p, q \le \frac{4}{N-2}$, $(0 \le p, q < \infty$, if $N \le 2)$.

An important step in the proof of local existence Theorem 3.2 below is the study of the following simpler problem :

$$\begin{aligned} &u'' - m(t)\Delta u - \Delta u' = f(t) \text{ in } \Omega \times [0, T], \\ &u(0) = u_0,\ u'(0) = u_1,\ x \in \Omega, \\ &u(x, t) = 0,\ x \in \partial\Omega,\ t > 0, \end{aligned} \tag{3.1}$$

here $u' = \frac{\partial u}{\partial t}$ and $T > 0$..

Theorem 3.1. ([13]). *Let $m(t)$ be a nonnegative Lipschitz function and $f(t)$ be a Lipschitz function on $[0, T], T > 0$. If $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$, then there exists a unique solution u of (3.1) satisfying*

$$u(t) \in C\left([0, T]; H^2(\Omega) \cap H_0^1(\Omega)\right),$$

and

$$u'(t) \in C\left([0, T]; L^2(\Omega)\right) \cap L^2\left((0, T); H_0^1(\Omega)\right).$$

Theorem 3.2. *Assume (A1) holds and $M(r)$ is a nonnegative locally Lipschitz function for $r \ge 0$ with the Lipschitz constant L. If $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$, then there exist a unique local solution (u, v) of (1.1) – (1.5) satisfying*

$$u(t), v(t) \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)),$$

and

$$u'(t), v'(t) \in C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega)), \text{ for } T > 0.$$

Moreover, at least one of the following statements hold :

(i) $T = \infty$.

(ii) $e(u(t), v(t)) \equiv \|u_t\|_2^2 + \|\Delta u\|_2^2 + \|v_t\|_2^2 + \|\Delta v\|_2^2 \to \infty$ *as* $t \to T^-$.

Proof. We set $w(t) = (u(t), v(t))$, and define the following two-parameter space :

$$X_{T,R_0} = \left\{ \begin{array}{c} u(t), v(t) \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)), \\ u_t(t), v_t(t) \in C([0, T]; L^2(\Omega)) \cap L^2((0, T); H_0^1(\Omega)) : \\ e(u(t), v(t)) \le R_0^2, \text{ with } w(0) = (u_0, v_0),\ w_t(0) = (u_1, v_1). \end{array} \right\},$$

for $T > 0,\ R_0 > 0$. Then X_{T,R_0} is a complete metric space with the distance

$$d(y,z) = \sup_{0 \le t \le T} \left\{ \|(\mu-\varphi)_t\|_2^2 + \|\Delta(\mu-\varphi)\|_2^2 + \|(\xi-\psi)_t\|_2^2 + \|\Delta(\xi-\psi)\|_2^2 \right\}^{\frac{1}{2}}, \tag{3.2}$$

where $y(t) = (\mu(t), \xi(t))$, $z(t) = (\varphi(t), \psi(t)) \in X_{T,R_0}$.
Given $\widehat{w}(t) = (\widehat{u}(t), \widehat{v}(t)) \in X_{T,R_0}$, we consider the linear system

$$u_{tt} - M\left(\|\nabla \widehat{u}\|_2^2 + \|\nabla \widehat{v}\|_2^2\right)\Delta u - \Delta u_t = f_1(\widehat{u}) \ \text{ in } \Omega \times [0,T), \tag{3.3}$$

$$v_{tt} - M\left(\|\nabla \widehat{u}\|_2^2 + \|\nabla \widehat{v}\|_2^2\right)\Delta v - \Delta v_t = f_2(\widehat{v}) \ \text{ in } \Omega \times [0,T), \tag{3.4}$$

with initial conditions,

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3.5}$$

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \quad x \in \Omega, \tag{3.6}$$

and boundary conditions,

$$u(x,t) = v(x,t) = 0,\ x \in \partial\Omega,\ t > 0. \tag{3.7}$$

By Theorem 3.1, there exists a unique solution $w(t) = (u(t), v(t))$ of $(3.3)-(3.7)$. We define the nonlinear mapping $S\widehat{w} = w$, and then, we will show that there exist $T > 0$ and $R_0 > 0$ such that

(i) $S : X_{T,R_0} \to X_{T,R_0}$,
(ii) S is a contraction mapping in X_{T,R_0} with respect to the metric $d(\cdot,\cdot)$ defined in (3.2).

Indeed, multiplying (3.3) by $2u_t$ and integrating it over Ω, and then by Divergence theorem, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|u_t\|_2^2 + M\left(\|\nabla \widehat{u}\|_2^2 + \|\nabla \widehat{v}\|_2^2\right)\|\nabla u\|_2^2\right\} + 2\|\nabla u_t\|_2^2 = I_{u1} + I_{u2}, \tag{3.8}$$

where

$$I_{u1} = \left(\frac{\mathrm{d}}{\mathrm{d}t} M\left(\|\nabla \widehat{u}\|_2^2 + \|\nabla \widehat{v}\|_2^2\right)\right)\|\nabla u\|_2^2, \tag{3.9}$$

$$I_{u2} = \int_\Omega 2 f_1(\widehat{u}) u_t \mathrm{d}x. \tag{3.10}$$

Similarly, we also have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|v_t\|_2^2 + M\left(\|\nabla\widehat{u}\|_2^2 + \|\nabla\widehat{v}\|_2^2\right)\|\nabla v\|_2^2\right\} + 2\|\nabla v_t\|_2^2 = I_{v1} + I_{v2}, \tag{3.11}$$

where

$$I_{v1} = \left(\frac{\mathrm{d}}{\mathrm{d}t}M\left(\|\nabla\widehat{u}\|_2^2 + \|\nabla\widehat{v}\|_2^2\right)\right)\|\nabla v\|_2^2,$$
$$I_{v2} = \int_\Omega 2f_2(\widehat{v})v_t dx.$$

From Divergence theorem, $\widehat{w} \in X_{T,R_0}$ and Lemma2.1, we have

$$\begin{aligned} |I_{u1}| &\le 2L\left(\|\Delta\widehat{u}\|_2\|\widehat{u}_t\|_2 + \|\Delta\widehat{v}\|_2\|\widehat{v}_t\|_2\right)\|\nabla u\|_2^2 \\ &\le c_0LR_0^2e(u,v), \end{aligned} \tag{3.12}$$

and

$$|I_{v1}| \le c_0LR_0^2e(u,v). \tag{3.13}$$

where $c_0 = 4c_*^2$.
By (A1), Lemma 2.1 and Hölder inequality, we have from (3.10)

$$\begin{aligned} |I_{u2}| &\le 2k\left(c_*\|\Delta\widehat{u}\|_2\right)^{p+1}\|u_t\|_2 \\ &\le 2kc_*^{p+1}R_0^{p+1}e(u,v)^{\frac{1}{2}}, \end{aligned} \tag{3.14}$$

and

$$|I_{v2}| \le 2kc_*^{q+1}R_0^{q+1}e(u,v)^{\frac{1}{2}}. \tag{3.15}$$

Combining (3.8) and (3.11) together, and using (3.12) – (3.15), we arrive at

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|u_t\|_2^2 + \|v_t\|_2^2 + M\left(\|\nabla\widehat{u}\|_2^2 + \|\nabla\widehat{v}\|_2^2\right)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\right\} \\ &+2\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) \\ \le\ & 2c_0LR_0^2e(u,v) + c_1\left(R_0^{p+1} + R_0^{q+1}\right)e(u,v)^{\frac{1}{2}}, \end{aligned} \tag{3.16}$$

where $c_1 = 2k\max\left(c_*^{p+1}, c_*^{q+1}\right)$. On the other hand, multiplying (3.3) by $-2\Delta u$ and (3.4) by $-2\Delta v$ and integrating them over Ω and adding them together, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\Delta u\|_2^2 + \|\Delta v\|_2^2 - 2\left(\int_\Omega u_t\Delta u\mathrm{d}x + \int_\Omega v_t\Delta v\mathrm{d}x\right)\right\} \\ &+2M\left(\|\nabla\widehat{u}\|_2^2 + \|\nabla\widehat{v}\|_2^2\right)\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) \\ &\le 2\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) + c_1\left(R_0^{p+1} + R_0^{q+1}\right)e(u,v)^{\frac{1}{2}}, \end{aligned} \tag{3.17}$$

the last inequality in (3.17) is obtained by following the argument as in (3.14) and (3.15). Multiplying (3.17) by ε, $0<\varepsilon\leq 1$, and adding (3.16) together, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}e^*_{\widehat{u},\widehat{v}}(u,v)+2(1-\varepsilon)\left[\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right]\\&\leq 2c_0LR_0^2e(u,v)+2(1+\varepsilon)c_1\left(R_0^{p+1}+R_0^{q+1}\right)e(u,v)^{\frac{1}{2}},\end{aligned}\tag{3.18}$$

where

$$\begin{aligned}&e^*_{\widehat{u},\widehat{v}}(u,v)\\&=\|u_t\|_2^2+\|v_t\|_2^2+M\left(\|\nabla\widehat{u}\|_2^2+\|\nabla\widehat{v}\|_2^2\right)\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\\&-2\varepsilon\left(\int_\Omega u_t\Delta u\mathrm{d}x+\int_\Omega v_t\Delta v\mathrm{d}x\right)+\varepsilon\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2\right).\end{aligned}\tag{3.19}$$

By Young's inequality, we get $\left|2\varepsilon\int_\Omega u_t\Delta u dx\right|\leq 2\varepsilon\|u_t\|_2^2+\frac{\varepsilon}{2}\|\Delta u\|_2^2$. Hence

$$\begin{aligned}e^*_{\widehat{u},\widehat{v}}(u,v)\geq&(1-2\varepsilon)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\frac{\varepsilon}{2}\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2\right)\\&+M\left(\|\nabla\widehat{u}\|_2^2+\|\nabla\widehat{v}\|_2^2\right)\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right).\end{aligned}$$

Choosing $\varepsilon=\frac{2}{5}$, we have

$$e^*_{\widehat{u},\widehat{v}}(u,v)\geq\frac{1}{5}e(u,v).\tag{3.20}$$

Then, from (3.18), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}e^*_{\widehat{u},\widehat{v}}(u(t),v(t))\leq&10c_0LR_0^2e^*_{\widehat{u},\widehat{v}}(u(t),v(t))\\&+\frac{14\sqrt{5}}{5}c_1\left(R_0^{p+1}+R_0^{q+1}\right)e^*_{\widehat{u},\widehat{v}}(u(t),v(t))^{\frac{1}{2}}.\end{aligned}$$

By Gronwall Lemma, we deduce

$$e^*_{\widehat{u},\widehat{v}}(u(t),v(t))\leq\left(e^*_{\widehat{u}(0),\widehat{v}(0)}(u_0,v_0)^{\frac{1}{2}}+\frac{7\sqrt{5}}{5}c_1\left(R_0^{p+1}+R_0^{q+1}\right)T\right)^2\mathrm{e}^{10c_0LR_0^2T}.\tag{3.21}$$

Thanks to Young's inequality, we observe that

$$e^*_{\widehat{u}(0),\widehat{v}(0)}(u_0,v_0)\leq c_2,\tag{3.22}$$

where

$$c_2 = 2\left(\|u_1\|_2^2 + \|v_1\|_2^2\right) + \|\Delta u_0\|_2^2 + \|\Delta v_0\|_2^2 \\ +M\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right)\left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2\right).$$

Thus, from (3.21) and using (3.20) and (3.22), we obtain for any $t \in [0,T]$,

$$\begin{aligned} e(u(t),v(t)) &\leq 5e^*_{\widehat{u},\widehat{v}}(u(t),v(t)) \\ &\leq \chi(u_0,u_{1,}v_0,v_1,R_0,T)^2 \mathrm{e}^{10c_0LR_0^2T}, \end{aligned} \tag{3.23}$$

where

$$\chi(u_0,u_{1,}v_0,v_1,R,T) = c_2^{\frac{1}{2}} + \frac{7\sqrt{5}}{5}c_1\left(R_0^{p+1} + R_0^{q+1}\right)T.$$

In order that S maps X_{T,R_0} into itself, it will be enough that the parameters T and R_0 satisfy

$$\chi(u_0,u_{1,}v_0,v_1,R_0,T)^2 \mathrm{e}^{10c_0LR_0^2T} \leq R_0^2. \tag{3.24}$$

Moreover, by Theorem 3.1, $w \in C^0([0,T];H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$ and it follows from (3.24) that $u', v' \in L^2((0,T);H_0^1(\Omega))$.

Next, we will show that S is a contraction mapping with respect to the metric $d(\cdot,\cdot)$. Let $(\widehat{u}_i,\widehat{v}_i) \in X_{T,R_0}$ and $\left(u^{(i)},v^{(i)}\right) \in X_{T,R_0}$, $i = 1,2$, be the corresponding solution to (3.3) – (3.7). Setting $w_1(t) = (u^{(1)} - u^{(2)})(t)$, $w_2(t) = (v^{(1)} - v^{(2)})(t)$, then w_1 and w_2 satisfy the following system:

$$\begin{aligned} &(w_1)_{tt} - M\left(\|\nabla\widehat{u}_1\|_2^2 + \|\nabla\widehat{v}_1\|_2^2\right)\Delta w_1 - \Delta(w_1)_t \\ &= f_1(\widehat{u}_1) - f_1(\widehat{u}_2) + \left[M\left(\|\nabla\widehat{u}_1\|_2^2 + \|\nabla\widehat{v}_1\|_2^2\right)\right. \\ &\quad \left. -M\left(\|\nabla\widehat{u}_2\|_2^2 + \|\nabla\widehat{v}_2\|_2^2\right)\right]\Delta u^{(2)} \end{aligned} \tag{3.25}$$

$$\begin{aligned} &(w_2)_{tt} - M\left(\|\nabla\widehat{u}_1\|_2^2 + \|\nabla\widehat{v}_1\|_2^2\right)\Delta w_2 - \Delta(w_2)_t \\ &= f_2(\widehat{v}_1) - f_2(\widehat{v}_2) + \left[M\left(\|\nabla\widehat{u}_1\|_2^2 + \|\nabla\widehat{v}_1\|_2^2\right)\right. \\ &\quad \left. -M\left(\|\nabla\widehat{u}_2\|_2^2 + \|\nabla\widehat{v}_2\|_2^2\right)\right]\Delta v^{(2)}, \end{aligned} \tag{3.26}$$

$$w_1(0) = (w_1)_t(0) = w_2(0) = (w_2)_t(0) = 0. \tag{3.27}$$

Multiplying (3.25) by $2(w_1)_t$, and integrating it over Ω, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|(w_1)_t\|_2^2 + M\left(\|\nabla\widehat{u}_1\|_2^2 + \|\nabla\widehat{v}_1\|_2^2\right)\|\nabla w_1\|_2^2\right\} + 2\|\nabla(w_1)_t\|_2^2 \\ &= I_{u3} + I_{u4} + I_{u5}, \end{aligned} \tag{3.28}$$

where

$$I_{u3} = \left(\frac{\mathrm{d}}{\mathrm{d}t} M\left(\|\nabla \widehat{u}_1\|_2^2 + \|\nabla \widehat{v}_1\|_2^2\right)\right) \|\nabla w_1\|_2^2, \tag{3.29}$$

$$\begin{aligned} I_{u4} &= 2[M\left(\|\nabla \widehat{u}_1\|_2^2 + \|\nabla \widehat{v}_1\|_2^2\right) \\ &\quad - M\left(\|\nabla \widehat{u}_2\|_2^2 + \|\nabla \widehat{v}_2\|_2^2\right)] \int_\Omega \Delta u^{(2)} (w_1)_t \, \mathrm{d}x, \end{aligned} \tag{3.30}$$

$$I_{u5} = 2\int_\Omega (f_1(\widehat{u}_1) - f_1(\widehat{u}_2)) (w_1)_t \, \mathrm{d}x. \tag{3.31}$$

Similarly, we also have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|(w_2)_t\|_2^2 + M\left(\|\nabla \widehat{u}_1\|_2^2 + \|\nabla \widehat{v}_1\|_2^2\right)\|\nabla w_2\|_2^2\right\} + 2\|\nabla (w_2)_t\|_2^2 \\ &= I_{v3} + I_{v4} + I_{v5}, \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} I_{v3} &= \left(\frac{\mathrm{d}}{\mathrm{d}t} M\left(\|\nabla \widehat{u}_1\|_2^2 + \|\nabla \widehat{v}_1\|_2^2\right)\right) \|\nabla w_2\|_2^2, \\ I_{v4} &= 2\left[M\left(\|\nabla \widehat{u}_1\|_2^2 + \|\nabla \widehat{v}_1\|_2^2\right) - M\left(\|\nabla \widehat{u}_2\|_2^2 + \|\nabla \widehat{v}_2\|_2^2\right)\right] \int_\Omega \Delta v^{(2)} (w_2)_t \, \mathrm{d}x, \\ I_{v5} &= 2\int_\Omega (f_2(\widehat{v}_1) - f_2(\widehat{v}_2)) (w_2)_t \, \mathrm{d}x. \end{aligned}$$

To proceed the estimation, it follows from (3.29) that

$$\begin{aligned} |I_{u3}| &\le 2L\left(\|\Delta \widehat{u}_1\|_2 \|(\widehat{u}_1)_t\|_2 + \|\Delta \widehat{v}_1\|_2 \|(\widehat{v}_1)_t\|_2\right) \|\nabla w_1\|_2^2 \\ &\le c_0 L R_0^2 e(w_1, w_2). \end{aligned} \tag{3.33}$$

Note that by Lemma 2.1, we have

$$\begin{aligned} &\left|M\left(\|\nabla \widehat{u}_1\|_2^2 + \|\nabla \widehat{v}_1\|_2^2\right) - M\left(\|\nabla \widehat{u}_2\|_2^2 + \|\nabla \widehat{v}_2\|_2^2\right)\right| \\ &\le L\left(\|\nabla \widehat{u}_1\|_2 + \|\nabla \widehat{u}_2\|_2 + \|\nabla \widehat{v}_1\|_2 + \|\nabla \widehat{v}_2\|_2\right)\left(\|\nabla \widehat{u}_1 - \nabla \widehat{u}_2\|_2 + \|\nabla \widehat{v}_1 - \nabla \widehat{v}_2\|_2\right) \\ &\le 4c_*^2 R_0 L e(\widehat{u}_1 - \widehat{u}_2, \widehat{v}_1 - \widehat{v}_2)^{\frac{1}{2}}. \end{aligned}$$

Then, from (3.30), we obtain

$$|I_{u4}| \le 8c_*^2 L R_0^2 e(\widehat{u}_1 - \widehat{u}_2, \widehat{v}_1 - \widehat{v}_2)^{\frac{1}{2}} e(w_1, w_2)^{\frac{1}{2}}. \tag{3.34}$$

And by (A1), we see that

$$|I_{u5}| \leq 4kc_*^{p+2}R_0^p e(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}}, \tag{3.35}$$

By the same procedure, we have the inequality for I_{v3}, I_{v4} and I_{v5}. Hence, combining (3.28) and (3.32) together and using (3.33) – (3.35), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|(w_1)_t\|_2^2+\|(w_2)_t\|_2^2+M\left(\|\nabla\widehat{u}_1\|_2^2+\|\nabla\widehat{v}_1\|_2^2\right)\left(\|\nabla w_1\|_2^2+\|\nabla w_2\|_2^2\right)\right\}\\
&+2\left(\|\nabla(w_1)_t\|_2^2+\|\nabla(w_2)_t\|_2^2\right)\\
\leq\ & 2c_0LR_0^2e(w_1,w_2)+16c_*^2LR_0^2e(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}}\\
&+c_3\left(R_0^p+R_0^q\right)e(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}},
\end{aligned} \tag{3.36}$$

where $c_3=4k\max\left(c_*^{p+2},c_*^{q+2}\right)$. On the other hand, multiplying (3.25) by $-2\Delta w_1$ and (3.26) by $-2\Delta w_2$, and integrating them over Ω and adding them together, we deduce

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\Delta w_1\|_2^2+\|\Delta w_2\|_2^2-2\left(\int_\Omega (w_1)_t\,\Delta w_1\mathrm{d}x+\int_\Omega (w_2)_t\,\Delta w_2\mathrm{d}x\right)\right\}\\
&+2M\left(\|\nabla\widehat{u}_1\|_2^2+\|\nabla\widehat{v}_1\|_2^2\right)\left(\|\Delta w_1\|_2^2+\|\Delta w_2\|_2^2\right)\\
\leq\ & 2\left[\|\nabla(w_1)_t\|_2^2+\|\nabla(w_2)_t\|_2^2\right]+(16c_*^2LR_0^2e(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)^{\frac{1}{2}}\\
&+c_3\left(R_0^p+R_0^q\right)e(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)^{\frac{1}{2}})e(w_1,w_2)^{\frac{1}{2}}.
\end{aligned} \tag{3.37}$$

Multiplying (3.37) by ε, $0<\varepsilon\leq 1$, and adding it to (3.36), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}e^*_{\widehat{u}_1,\widehat{v}_1}(w_1,w_2)+2(1-\varepsilon)\left[\|\nabla(w_1)_t\|_2^2+\|\nabla(w_2)_t\|_2^2\right]\\
\leq\ & 2c_0LR_0^2e(w_1,w_2)+16c_*^2(1+\varepsilon)LR_0^2e(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}}\\
&+(1+\varepsilon)c_3\left(R_0^p+R_0^q\right)e(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)^{\frac{1}{2}}e(w_1,w_2)^{\frac{1}{2}},
\end{aligned} \tag{3.38}$$

where $e^*_{\widehat{u}_1,\widehat{v}_1}(w_1,w_2)$ is given by (3.19) with $u=w_1, v=w_2$, $\widehat{u}=\widehat{u}_1$ and $\widehat{v}=\widehat{v}_1$. Taking $\varepsilon=\frac{2}{5}$ in (3.38), and as in (3.17) – (3.20), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}e^*_{\widehat{u}_1,\widehat{v}_1}(w_1,w_2) \leq\ & LR_0^2 10c_0e^*_{\widehat{u}_1,\widehat{v}_1}(w_1,w_2)\\
&+c_5\left(R_0^p+R_0^q\right)e(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)^{\frac{1}{2}}e^*_{\widehat{u}_1,\widehat{v}_1}(w_1,w_2)^{\frac{1}{2}}\\
&+c_4LR_0^2e(\widehat{u}_1-\widehat{u}_2,\widehat{v}_1-\widehat{v}_2)^{\frac{1}{2}}e^*_{\widehat{u}_1,\widehat{v}_1}(w_1,w_2)^{\frac{1}{2}},
\end{aligned}$$

where $c_4 = \frac{112\sqrt{5}}{5}c_*^2$ and $c_5 = \frac{7\sqrt{5}}{5}c_3$. Noting that $e^*_{\widehat{u}_1(0),\widehat{v}_1(0)}(w_1(0), w_2(0)) = 0$, and by applying Gronwall Lemma, we get

$$e^*_{\widehat{u}_1,\widehat{v}_1}(w_1, w_2) \leq \left[\frac{c_4}{2}LR_0^2 + \frac{c_5}{2}\left(R_0^p + R_0^q\right)\right]^2 T^2 e^{10c_0LR_0^2T} \sup_{0\leq t\leq T} e(\widehat{u}_1 - \widehat{u}_2, \widehat{v}_1 - \widehat{v}_2).$$

Thus, by (3.2), we have

$$d\left(\left(u^{(1)}, v^{(1)}\right), \left(u^{(2)}, v^{(2)}\right)\right) \leq C(T, R_0)^{\frac{1}{2}} d\left(\left(\widehat{u}_1, \widehat{v}_1\right), \left(\widehat{u}_2, \widehat{v}_2\right)\right),$$

where

$$C(T, R_0) = \sqrt{5}\left[\frac{c_4}{2}LR_0^2 + \frac{c_5}{2}\left(R_0^p + R_0^q\right)\right] T e^{5c_0LR_0^2T}. \tag{3.39}$$

Hence, under inequality (3.24), S is a contraction mappingn if $C(T, R_0) < 1$. Indeed, we choose R_0 sufficient large and T sufficient small so that (3.24) and (3.39) are satisfied at the same time. By applying Banach fixed point theorem, we obtain the local existence result.

4. Global Existence

In this section, we shall consider the global existence and the asymptotic behavior of the solution for the following equations :

$$u_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta u - \Delta u_t = |u|^p u, \tag{4.1}$$

$$v_{tt} - M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\Delta v - \Delta v_t = |v|^q v, \tag{4.2}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\quad x \in \Omega, \tag{4.3}$$

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\quad x \in \Omega, \tag{4.4}$$

$$u(x,t) = v(x,t) = 0,\ x \in \partial\Omega,\ t > 0, \tag{4.5}$$

where $M(s) = m_0 + bs^\gamma$, with $m_0 \geq 0$, $b > 0$, $\gamma \geq 1$, $s \geq 0$ and $2\gamma < p, q \leq \frac{4}{N-2}$.

Let

$$\begin{aligned} I(u,v) \equiv I(t) = m_0\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + b\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1} \\ - \|u\|_{p+2}^{p+2} - \|v\|_{q+2}^{q+2}, \end{aligned} \tag{4.6}$$

and

$$J(u,v) \equiv J(t) = \frac{m_0}{2}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{b}{2(\gamma+1)}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1}$$

$$-\frac{1}{p+2}\|u\|_{p+2}^{p+2} - \frac{1}{q+2}\|v\|_{q+2}^{q+2}. \tag{4.7}$$

We define the energy function of the solution $(u(t), v(t))$ of $(4.1) - (4.5)$ by

$$E(u,v) \equiv E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + J(t). \tag{4.8}$$

Lemma 4.1. *$E(t)$ is a nonincreasing function on $[0,T)$ and we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|\nabla u_t\|_2^2 - \|\nabla v_t\|_2^2. \tag{4.9}$$

Proof. By differenting (4.8) and using $(4.1) - (4.5)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\|\nabla u_t\|_2^2 - \|\nabla v_t\|_2^2.$$

Thus, Lemma 4.1 follows at once.

Lemma 4.2. *Let $(u(t), v(t))$ be the solution of $(4.1) - (4.5)$ with $u_0, v_0 \in W \cap H^2(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$*, where

$$W = \left\{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega);\ I(u,v) > 0\right\} \cup \{0\}.$$

Assume that

$$(i) \qquad \alpha_1 < 1, \textit{ for } m_0 > 0, \tag{4.10}$$

$$(ii) \qquad \alpha_2 < 1 \textit{ and } p \geq q, \textit{ for } m_0 = 0, \tag{4.11}$$

here

$$\alpha_1 = \frac{1}{m_0}\max\left\{c_*^{p+2}\left(\frac{2(\gamma+1)}{\gamma}E(0)\right)^{\frac{p}{2}}, c_*^{q+2}\left(\frac{2(\gamma+1)}{\gamma}E(0)\right)^{\frac{q}{2}}\right\}$$

and

$$\alpha_2 = \frac{1}{b}\max\left\{c_*^{p+2}\left(\frac{2(\gamma+1)(q+2)}{b(q-2\gamma)}E(0)\right)^{\frac{p-2\gamma}{2(\gamma+1)}} c_*^{q+2}\left(\frac{2(\gamma+1)(q+2)}{b(q-2\gamma)}E(0)\right)^{\frac{q-2\gamma}{2(\gamma+1)}}\right\}.$$

Then $I(t) > 0$, *for all* $t \geq 0$.

Proof. Since $I(0) > 0$, then it follows from the continuity of $u(t)$ and $v(t)$ that

$$I(t) \geq 0, \tag{4.12}$$

for some interval near $t = 0$. Let $t_{\max} > 0$ be a maximal time (possibly $t_{\max} = T$), when (4.12) holds on $[0, t_{\max})$.
From (4.7) and (4.6), we observe that if $m_0 > 0$, then

$$\begin{aligned} J(t) &= \frac{\gamma}{2(\gamma+1)}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + \frac{p-2\gamma}{2(\gamma+1)(p+2)}\|u\|_{p+2}^{p+2} \\ &\quad + \frac{q-2\gamma}{2(\gamma+1)(q+2)}\|v\|_{q+2}^{q+2} + \frac{1}{2(\gamma+1)}I(t) \\ &\geq \frac{\gamma}{2(\gamma+1)}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right), \end{aligned} \tag{4.13}$$

and if $m_0 = 0$, then

$$\begin{aligned} J(t) &= \frac{1}{q+2}I(t) + \frac{q-2\gamma}{2(\gamma+1)(q+2)}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1} \\ &\quad + \frac{p-q}{(q+2)(p+2)}\|u\|_{p+2}^{p+2} \\ &\geq \frac{q-2\gamma}{2(\gamma+1)(q+2)}\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1}. \end{aligned} \tag{4.14}$$

Thus, by Lemma 4.1, we have that if $m_0 > 0$, then

$$\begin{aligned} &\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \\ &\leq \frac{2(\gamma+1)}{\gamma}J(t) \leq \frac{2(\gamma+1)}{\gamma}E(t) \leq \frac{2(\gamma+1)}{\gamma}E(0), t \in [0, t_{\max}), \end{aligned} \tag{4.15}$$

and if $m_0 = 0$, then

$$\begin{aligned} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1} &\leq \frac{2(\gamma+1)(q+2)}{q-2\gamma}J(t) \leq \frac{2(\gamma+1)(q+2)}{q-2\gamma}E(t) \\ &\leq \frac{2(\gamma+1)(q+2)}{q-2\gamma}E(0), t \in [0, t_{\max}). \end{aligned} \tag{4.16}$$

Note that (4.10), it follows from (4.15) that, when $m_0 > 0$,

$$\begin{aligned}
&\|u\|_{p+2}^{p+2} + \|v\|_{q+2}^{q+2} \\
&\leq c_*^{p+2}\|\nabla u\|_2^{p+2} + c_*^{q+2}\|\nabla v\|_2^{q+2} \\
&\leq c_*^{p+2}\left(\frac{2(\gamma+1)}{\gamma}E(0)\right)^{\frac{p}{2}}\|\nabla u\|_2^2 + c_*^{q+2}\left(\frac{2(\gamma+1)}{\gamma}E(0)\right)^{\frac{q}{2}}\|\nabla v\|_2^2 \\
&\leq \alpha_1 m_0\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\
&< m_0\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \text{ on } [0, t_{\max}).
\end{aligned} \tag{4.17}$$

Similarly, when $m_0 = 0$, by (4.16) and (4.11), we have

$$\begin{aligned}
&\|u\|_{p+2}^{p+2} + \|v\|_{q+2}^{q+2} \\
&\leq c_*^{p+2}\|\nabla u\|_2^{p-2\gamma}\|\nabla u\|_2^{2(\gamma+1)} + c_*^{q+2}\|\nabla v\|_2^{q-2\gamma}\|\nabla v\|_2^{2(\gamma+1)} \\
&\leq \alpha_2 b\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1} \\
&< b\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1} \text{ on } [0, t_{\max}).
\end{aligned} \tag{4.18}$$

Therefore, whether $m_0 > 0$ or $m_0 = 0$, we deduce that $I(t) > 0$ on $[0, t_{\max})$. This implies that we can take $t_{\max} = T$.

Lemma 4.3. *Suppose that the assumptions of Lemma 4.2 are satisfied, then there exists* $0 < \eta_i < 1$ *,* $i = 1, 2$ *such that*

$$\|u\|_{p+2}^{p+2} + \|v\|_{q+2}^{q+2} \leq \begin{cases} (1-\eta_1)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right), \ m_0 > 0 \\ (1-\eta_2)\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1}, m_0 = 0 \end{cases} \text{ on } [0, T],$$

where $\eta_i = 1 - \alpha_i$, $i = 1, 2$.

Proof. From (4.17), we have

$$\|u\|_{p+2}^{p+2} + \|v\|_{q+2}^{q+2} \leq \alpha_1 m_0\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right), \ t \in [0, T].$$

Let $\eta_1 = 1 - \alpha_1$, then we have the result for $m_0 > 0$. Similarly, from (4.18), we get the result for $m_0 = 0$.

Remark. It follows from Lemma 4.3 that if $m_0 > 0$, then

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \le \frac{1}{\eta_1} I(t),\ t \in [0, T]. \tag{4.19}$$

and if $m_0 = 0$, then

$$\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\gamma+1} \le \frac{1}{\eta_2} I(t),\ t \in [0, T]. \tag{4.20}$$

Theorem 4.4. (Energy decay). *Suppose that* $u_0, v_0 \in W \cap H_0^2(\Omega),\ u_1, v_1 \in L^2(\Omega)$ *and the conditions of Lemma 4.2 are satisfied. Let* $(u(t), v(t))$ *be the solution of the problem* $(4.1) - (4.5)$*, then we have the following decay estimates:*

(i) *when* $m_0 > 0$,

$$E(t) \le E(0)\mathrm{e}^{-\tau_1 t},\ \text{on } [0, T).$$

(ii) *When* $m_0 = 0$,

$$E(t) \le \left(E(0)^{-\frac{\gamma}{\gamma+1}} + \frac{\gamma\tau_2}{\gamma+1}[t-1]^+\right)^{-\frac{\gamma+1}{\gamma}} \quad \text{on } [0, T),$$

where τ_i, $i = 1, 2$, *is some positive constant given in the proof.*

Proof. By integrating (4.9) over $[t, t+1]$, $t > 0$, we have

$$E(t) - E(t+1) \equiv D(t)^2, \tag{4.21}$$

where

$$D(t)^2 = \int_t^{t+1} \left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) \mathrm{d}t. \tag{4.22}$$

Then, there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|\nabla u_t(t_i)\|_2^2 + \|\nabla v_t(t_i)\|_2^2 \le 4D(t)^2,\ i = 1, 2. \tag{4.23}$$

Next, multiplying (4.1) by u and (4.2) by v and integrating them over $\Omega \times [t_1, t_2]$ and adding them together, we get

$$\int_{t_1}^{t_2} I(t)\mathrm{d}t = -\int_{t_1}^{t_2}\int_\Omega (u_{tt}u + v_{tt}v)\,\mathrm{d}x\mathrm{d}t + \int_{t_1}^{t_2}\int_\Omega (\Delta u_t u + \Delta v_t v)\,\mathrm{d}x\mathrm{d}t. \tag{4.24}$$

Integrating by parts on the first term of the right hand side of (4.24) and then using Divergence theorem and Lemma 2.1, we obtain

$$
\begin{aligned}
&\int_{t_1}^{t_2} I(t)\mathrm{d}t \\
&\le c_*^2 \sum_{i=1}^{2} \left(\|\nabla u_t(t_i)\|_2 \|\nabla u(t_i)\|_2 + \|\nabla v_t(t_i)\|_2 \|\nabla v(t_i)\|_2\right) \\
&\quad + c_*^2 \int_{t_1}^{t_2} \left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) \mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \left(\|\nabla u_t\|_2 \|\nabla u\|_2 + \|\nabla v_t\|_2 \|\nabla v\|_2\right) \mathrm{d}t.
\end{aligned} \tag{4.25}
$$

To proceed further estimation, we note that from (4.15)-(4.16) and (4.22),

$$
\begin{aligned}
&\int_{t_1}^{t_2} \left(\|\nabla u_t\|_2 \|\nabla u\|_2 + \|\nabla v_t\|_2 \|\nabla v\|_2\right) \mathrm{d}t \\
&\le \begin{cases} c_1 D(t) \sup\limits_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}}, \text{ if } m_0 > 0, \\ c_2 D(t) \sup\limits_{t_1 \le s \le t_2} E(s)^{\frac{1}{2(\gamma+1)}}, \text{ if } m_0 = 0. \end{cases}
\end{aligned} \tag{4.26}
$$

And by (4.23) , we have

$$
\begin{aligned}
&\|\nabla u_t(t_i)\|_2 \|\nabla u(t_i)\|_2 + \|\nabla v_t(t_i)\|_2 \|\nabla v(t_i)\|_2 \\
&\le \begin{cases} 2c_1 D(t) \sup\limits_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}}, \text{ if } m_0 > 0, \\ 2c_2 D(t) \sup\limits_{t_1 \le s \le t_2} E(s)^{\frac{1}{2(\gamma+1)}}, \text{ if } m_0 = 0, \end{cases}
\end{aligned} \tag{4.27}
$$

where $c_1 = 2\left(\frac{2(\gamma+1)}{\gamma}\right)^{\frac{1}{2}}$ and $c_2 = 2\left(\frac{2(\gamma+1)(q+2)}{q-2\gamma}\right)^{\frac{1}{2(\gamma+1)}}$.
Thus, from (4.20) and by $(4.26) - (4.27)$, (4.22) , we deduce that if $m_0 > 0$, then

$$
\int_{t_1}^{t_2} I(t)\mathrm{d}t \le c_3 D(t) \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2}} + c_*^2 D(t)^2, \tag{4.28}
$$

and if $m_0 = 0$, then

$$
\int_{t_1}^{t_2} I(t)\mathrm{d}t \le c_4 D(t) \sup_{t_1 \le s \le t_2} E(s)^{\frac{1}{2(\gamma+1)}} + c_*^2 D(t)^2, \tag{4.29}
$$

where $c_3 = 4c_*^2c_1 + c_1$ and $c_4 = 4c_*^2c_2 + c_2$.
On the other hand, from (4.8) and Poincaré inequality, we note that if $m_0 > 0$, then using (4.19) and (4.17),

$$\begin{aligned}
&E(t)\\
&\le \frac{c_*^2}{2}\left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)+\frac{1}{2(\gamma+1)}I(t)+\frac{\gamma}{2(\gamma+1)}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\\
&\quad+\frac{p-2\gamma}{2(\gamma+1)\,(p+2)}\,\|u\|_{p+2}^{p+2}+\frac{q-2\gamma}{2(\gamma+1)\,(q+2)}\,\|v\|_{q+2}^{q+2}\\
&\le \frac{c_*^2}{2}\left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)+c_5I(t)+c_6\left(\|u\|_{p+2}^{p+2}+\|v\|_{q+2}^{q+2}\right)\\
&\le \frac{c_*^2}{2}\left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)+c_7I(t),
\end{aligned} \tag{4.30}$$

and if $m_0 = 0$, then using (4.20) and (4.18),

$$\begin{aligned}
E(t) &\le \frac{c_*^2}{2}\left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)+c_8I(t)+\frac{p-q}{2(q+1)\,(p+2)}\,\|u\|_{p+2}^{p+2}\\
&\le \frac{c_*^2}{2}\left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)+c_9I(t),
\end{aligned} \tag{4.31}$$

where $c_5 = \left(\frac{\gamma}{2(\gamma+1)\eta_1}+\frac{1}{2\gamma+2}\right)$, $c_6 = \max\left\{\frac{p-2\gamma}{2(\gamma+1)(p+2)}, \frac{q-2\gamma}{2(\gamma+1)(q+2)}\right\}$, $c_7 = c_5 + \frac{c_6\alpha_1 m_0}{\eta_1}$, $c_8 = \frac{q-2\gamma}{2(\gamma+1)(q+2)\eta_2}+\frac{1}{q+2}$ and $c_9 = c_8 + \frac{\alpha_2 b(p-q)}{2(q+1)(p+2)\eta_2}$.

Hence, by integrating (4.30) over (t_1, t_2) and using (4.22) and (4.28), we obtain

$$\int_{t_1}^{t_2} E(t)\mathrm{d}t \le c_{10}D(t)^2 + c_{11}D(t)\sup_{t_1\le s\le t_2} E(s)^{\frac{1}{2}}, \tag{4.32}$$

where $c_{10} = \frac{c_*^2}{2} + c_*^2c_7$ and $c_{11} = c_3c_7$.
Moreover, integrating (4.8) over (t, t_2), we get

$$E(t) = E(t_2) + \int_t^{t_2}\left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)\mathrm{d}s.$$

Since $t_2 - t_1 \ge \frac{1}{2}$, it follows that

$$E(t_2) \le 2\int_{t_1}^{t_2} E(t)\mathrm{d}t.$$

Then, thanks to (4.22), we arrive at

$$\begin{aligned} E(t) &\leq 2\int_{t_1}^{t_2} E(t)\mathrm{d}t + \int_{t}^{t_2}\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right)\mathrm{d}s \\ &= 2\int_{t_1}^{t_2} E(t)\mathrm{d}t + D(t)^2. \end{aligned}$$

Thus, by using (4.32) and Lemma 4.1, we see that

$$E(t) \leq c_{12}D(t)^2 + c_{13}D(t)E(t)^{\frac{1}{2}},\ t \geq 0,$$

where $c_{11} = 2c_9 + 1$ and $c_{12} = 2c_{10}$.
Hence, by Young's inequality, we deduce

$$\begin{aligned} E(t) &\leq c_{14}D(t)^2, \\ &\leq c_{15}\left[E(t) - E(t+1)\right]. \end{aligned} \tag{4.33}$$

where c_{15} is some positive constant greater than $\max(1, c_{14})$. Therefore, by Lemma 2.2, we have the decay estimate for $m_0 > 0$:

$$E(t) \leq E(0)\mathrm{e}^{-\tau_1 t},\ \text{on } [0, T),$$

where $\tau_1 = \ln\frac{c_{15}}{c_{15}-1}$. Similarly, when $m_0 = 0$, following the arguments as in $(4.32) - (4.33)$, we arrive at

$$\begin{aligned} E(t) &\leq c_{16}\left(1 + D(t)^{2-\frac{2(\gamma+1)}{2\gamma+1}}\right)D(t)^{\frac{2(\gamma+1)}{2\gamma+1}} \\ &\leq c_{16}\left(1 + E(0)^{2-\frac{2(\gamma+1)}{2\gamma+1}}\right)D(t)^{\frac{2(\gamma+1)}{2\gamma+1}}. \end{aligned}$$

This implies that

$$E(t)^{1+\frac{\gamma}{\gamma+1}} \leq \left(c_{17}(E(0))\right)^{\frac{2\gamma+1}{\gamma+1}}\left[E(t) - E(t+1)\right],$$

where $c_{17}(E(0)) = c_{16}\left[1 + E(0)^{2-\frac{2(\gamma+1)}{2\gamma+1}}\right]$ with $\lim\limits_{E(0)\to 0} c_{17}(E(0)) = c_{15} > 0$.
Setting $\tau_2 = (c_{17}(E(0)))^{--\frac{2(\gamma+1)}{\gamma+1}}$, then applying Lemma 2.2 yields

$$E(t) \leq \left(E(0)^{-\frac{\gamma}{\gamma+1}} + \frac{\gamma\tau_2}{\gamma+1}[t-1]^+\right)^{-\frac{\gamma+1}{\gamma}} \quad \text{on } [0, T).$$

Theorem 4.5. *(Global existence and Decay property) Suppose that* $u_0, v_0 \in W \cap H_0^2(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$ *with* $\alpha_1 < 1,$ *for* $m_0 > 0$ *or* $\alpha_2 < 1$ *and* $p \geq q,$ *for* $m_0 = 0.$ *Then the problem* $(4.1) - (4.5)$ *admits a global solution*

$$u(t), v(t) \in C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)),$$

and

$$u'(t), v'(t) \in C([0,\infty); L^2(\Omega)) \cap L^2((0,\infty); H_0^1(\Omega)).$$

Furthermore, we have the following decay estimates :

(i) *if* $m_0 > 0,$ *then*

$$E(t) \leq E(0)\mathrm{e}^{-\tau_1 t}, \text{ on } [0,\infty).$$

(ii) *If* $m_0 = 0,$ *then*

$$E(t) \leq \left(E(0)^{-\frac{\gamma}{\gamma+1}} + \frac{\gamma\tau_2}{\gamma+1}[t-1]^+\right)^{-\frac{\gamma+1}{\gamma}} \text{ on } [0,\infty).$$

Proof. Multiplying (4.1) by $-2\Delta u$ and (4.2) by $-2\Delta v$ and integrating them over Ω and combining them together, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\Delta u\|_2^2 + \|\Delta v\|_2^2 - 2\left(\int_\Omega u_t \Delta u dx + \int_\Omega v_t \Delta v \mathrm{d}x\right)\right\} \\ &+2M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) \\ &\leq 2\left(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right) - 2\int_\Omega |u|^p u \Delta u \mathrm{d}x - 2\int_\Omega |v|^q v \Delta v \mathrm{d}x. \end{aligned} \tag{4.34}$$

Multiplying (4.34) by ε, $0 < \varepsilon \leq 1$, and multiplying (4.8) by 2 and adding them together, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}E^*(t) + 2(1-\varepsilon)\left[\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right] \\ &+2\varepsilon M\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right) \\ &\leq -2\varepsilon\int_\Omega |u|^p u \Delta u \mathrm{d}x - 2\varepsilon\int_\Omega |v|^q v \Delta v \mathrm{d}x, \end{aligned} \tag{4.35}$$

where

$$E^*(t) = 2E(t) - 2\varepsilon\left(\int_\Omega u_t \Delta u \mathrm{d}x + \int_\Omega v_t \Delta v \mathrm{d}x\right) + \varepsilon\left(\|\Delta u\|_2^2 + \|\Delta v\|_2^2\right).$$

By Lemma 4.2 and noting that $\left|2\varepsilon\int_\Omega u_t\Delta u dx\right| \le 2\varepsilon\|u_t\|_2^2+\frac{\varepsilon}{2}\|\Delta u\|_2^2$, we see that

$$E^*(t)\ge(1-2\varepsilon)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\varepsilon\left(\|\Delta u\|_2^2+\|\Delta v\|_2^2\right)$$

Choosing $\varepsilon=\frac{2}{5}$, we have

$$E^*(t)\ge\frac{1}{5}e(u,v).$$

Moreover, we note that

$$\begin{aligned}2\left|\int_\Omega |u|^p u\Delta u\mathrm{d}x\right| &= 2p\int_\Omega |u|^p|\nabla u|^2dx\\ &\le 2p\|u\|_{p\theta_1}^p\|\nabla u\|_{2\theta_2}^2,\end{aligned}$$

and

$$2\left|\int_\Omega |v|^q v\Delta v\mathrm{d}x\right|\le 2q\|v\|_{q\theta_1}^q\|\nabla v\|_{2\theta_2}^2,$$

where $\frac{1}{\theta_1}+\frac{1}{\theta_2}=1$, so that, we put $\theta_1=1$ and $\theta_2=\infty$, if $N=1$; $\theta_1=1+\varepsilon_1$ (for arbitrary small $\varepsilon_1>0$), if $N=2$; and $\theta_1=\frac{N}{2}$, $\theta_2=\frac{N}{N-2}$, if $N\ge3$. Thus, if $m_0>0$, using (4.15), we have

$$\begin{aligned}2\left|\int_\Omega\left(|u|^p u\Delta u+|v|^q v\Delta v\right)\mathrm{d}x\right| &\le 2\left(c_*^{p+2}p\|\nabla u\|_2^p\|\Delta u\|_2^2+c_*^{q+2}q\|\nabla v\|_2^q\|\Delta v\|_2^2\right)\\ &\le c_{18}E^*(t),\end{aligned}$$

and if $m_0=0$, by (4.16), we get

$$2\left|\int_\Omega\left(|u|^p u\Delta u+|v|^q v\Delta v\right)\mathrm{d}x\right|\le c_{19}E^*(t),$$

where $c_{18}=10\max\left(pc_*^{p+2}\left(\frac{2(\gamma+1)}{\gamma}E(0)\right)^{\frac{p}{2}},\ qc_*^{q+2}\left(\frac{2(\gamma+1)}{\gamma}E(0)\right)^{\frac{q}{2}}\right)$ and $c_{19}=10\max\left(pc_*^{p+2}\left(\frac{2(\gamma+1)(q+2)}{q-2\gamma}E(0)\right)^{\frac{p}{2(\gamma+1)}},\ qc_*^{q+2}\left(\frac{2(\gamma+1)(q+2)}{q-2\gamma}E(0)\right)^{\frac{q}{2(\gamma+1)}}\right)$.

Hence, by integrating (4.35) over $(0,t)$, we obtain

$$E^*(t)\le\begin{cases}E^*(0)+\int_0^t c_{18}E^*(s)\mathrm{d}s, & \text{if } m_0>0,\\ E^*(0)+\int_0^t c_{19}E^*(s)\mathrm{d}s, & \text{if } m_0=0.\end{cases}$$

Then by Gronwall Lemma, we deduce

$$E^*(t)\le E^*(0)\exp(c_it),$$

$i=18,19$, for any $t\ge0$. Therefore by Theorem 3.2, whether $m_0>0$ or $m_0=0$, we have $T=\infty$.

5. Blow-up Property

In this section, we will study blow-up phenomena of solutions for a kind of system $(1.1)-(1.5)$:

$$
\begin{aligned}
&u_{tt}-M\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\Delta u-\Delta u_t=f_1(u),\\
&v_{tt}-M\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\Delta v-\Delta v_t=f_2(v),\\
&u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right),\ x\in\Omega,\\
&v\left(x,0\right)=v_0\left(x\right),\ v_t\left(x,0\right)=v_1\left(x\right),\ x\in\Omega,\\
&u\left(x,t\right)=v\left(x,t\right)=0,\ x\in\partial\Omega,\ t>0.
\end{aligned}
\tag{5.1}
$$

In order to state our results, we make further assumptions on f_i and M:
(A2) there exists a positive constant δ such that

$$uf_1(u)+vf_2(v)\geq(2+4\delta)\left(F_1(u)+F_2(v)\right),\ \text{for all } u,v\in\mathbb{R},$$

and

$$(2\delta+1)\overline{M}(s)\geq M(s)s,\ \text{for all } s\geq0,$$

where

$$F_1(u)=\int_0^u f_1(r)\mathrm{d}r,\ F_2(v)=\int_0^v f_2(r)\mathrm{d}r \text{ and } \overline{M}(s)=\int_0^s M(r)\mathrm{d}r.$$

Remark. (1) In this case, we define the energy function of the solution (u,v) of (5.1) by

$$E(t)=\frac{1}{2}\left(\|u_t\|_2^2+\|v_t\|_2^2\right)+\frac{1}{2}\overline{M}(\|\nabla u\|_2^2+\|\nabla v\|_2^2)-\int_\Omega\left(F_1(u)+F_2(v)\right)\mathrm{d}x, \tag{5.2}$$

for $t\geq0$. Then we have

$$E(t)=E(0)-\int_0^t\left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right)\mathrm{d}t. \tag{5.3}$$

(2) It is clear that $f_1(u)=|u|^p\,u$, $f_2(v)=|v|^q\,v$, $p,q\geq0$ and $M\left(s\right)=m_0+bs^\gamma$ for $m_0\geq0$, $b\geq0$, $m_0+b>0$, $\gamma>0$, $s\geq0$ satisfies (A2) with $\frac{\gamma}{2}<\delta\leq\min(\frac{p}{4},\frac{q}{4})$.

Definition. A solution $w(t)=(u(t),v(t))$ of (5.1) is called blow-up if there exists a finite time T^* such that

$$\lim_{t\to T^{*-}}\int_\Omega\left(|\nabla u|^2+|\nabla v|^2\right)\mathrm{d}x=\infty.$$

Now, let

$$(5.4)\quad \begin{aligned} a(t) \;=\; & \|u\|_2^2+\|v\|_2^2+\int_0^t \left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right) \mathrm{d}t+l(t+\tau)^2 \\ & +(T_1-t)\left(\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2\right), \end{aligned}$$

for $t \geq 0$, here $l \geq 0$, $\tau > 0$ and $T_1 > 0$ are certain constants to be determined later.

Lemma 5.1. *Suppose that (A1) and (A2) hold, then the function* $a(t)$ *satisfies*

$$(5.5)\quad \begin{aligned} & a''(t)-4(\delta+1)\left[l+\|u_t\|_2^2+\|v_t\|_2^2\right] \\ \geq\; & (-2-4\delta)(2E(0)+l)+(4+8\delta)\int_0^t \left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right) \mathrm{d}t. \end{aligned}$$

Proof. Form (5.4), we have

$$(5.6)\quad \begin{aligned} a'(t) \;=\; & 2\int_\Omega (uu_t+vv_t)\,\mathrm{d}x+\|\nabla u\|_2^2+\|\nabla v\|_2^2+2l(t+\tau) \\ & -\left(\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2\right). \end{aligned}$$

By (5.1) and Divergence theorem, we get

$$(5.7)\quad \begin{aligned} a''(t) \;=\; & 2\int_\Omega \left(u_t^2+v_t^2\right) dx-2M\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right) \\ & +2\int_\Omega (uf_1(u)+vf_2(v))\,\mathrm{d}x+2l. \end{aligned}$$

Then, by $(5.1)-(5.3)$, we have

$$\begin{aligned} & a''(t)-4(\delta+1)\left[l+\|u_t\|_2^2+\|v_t\|_2^2\right] \\ \geq\; & (-2-4\delta)(2E(0)+l)+(4+8\delta)\int_0^t \left(\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2\right) \mathrm{d}s \\ & +(2+4\delta)\,\overline{M}\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)-2M\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right) \\ & +2\int_\Omega \left[uf_1(u)+vf_2(v)-(2+4\delta)(F_1(u)+F_2(v))\right] \mathrm{d}x. \end{aligned}$$

Therefore, from (A2), we obtain (5.5).

Now, we will find the estimate for the life span of $a(t)$. Let

$$(5.8)\quad J(t)=a(t)^{-\delta}, \text{ for } t \in [0,T_1].$$

Then we have

$$J'(t) = -\delta J(t)^{1+\frac{1}{\delta}} a'(t),$$

and

$$J''(t) = -\delta J(t)^{1+\frac{2}{\delta}} V(t), \tag{5.9}$$

where

$$V(t) = a''(t)\, a(t) - (1+\delta)\, a'(t)^2. \tag{5.10}$$

For simplicity of calculation, we denote

$$\begin{aligned}
P_u &= \int_\Omega u^2 \mathrm{d}x,\ P_v = \int_\Omega v^2 \mathrm{d}x,\\
Q_u &= \int_0^t \|\nabla u\|_2^2 \,\mathrm{d}t,\ Q_v = \int_0^t \|\nabla v\|_2^2 \,\mathrm{d}t,\\
R_u &= \int_\Omega u_t^2 \mathrm{d}x,\ R_v = \int_\Omega v_t^2 \mathrm{d}x,\\
S_u &= \int_0^t \|\nabla u_t\|_2^2 \,\mathrm{d}t,\ S_v = \int_0^t \|\nabla v_t\|_2^2 \,\mathrm{d}t.
\end{aligned}$$

From (5.6), and Hölder inequality, we get

$$\begin{aligned}
& a'(t)^2\\
&= 4\left(\int_\Omega (uu_t+vv_t)\,\mathrm{d}x + \int_0^t\int_\Omega (\nabla u\nabla u_t+\nabla v\nabla v_t)\,\mathrm{d}x\mathrm{d}t + l(t+\tau)\right)^2\\
&\le 4\left(\sqrt{R_uP_u}+\sqrt{Q_uS_u}+\sqrt{R_vP_v}+\sqrt{Q_vS_v}+\sqrt{l}\sqrt{l}(t+\tau)\right)^2.
\end{aligned} \tag{5.11}$$

By (5.5), we have

$$a''(t) \ge -(2+4\delta)(2E(0)+l) + 4(1+\delta)(R_u+S_u+R_v+S_v+l). \tag{5.12}$$

Thus, by (5.11) and (5.12), we obtain from (5.10)

$$\begin{aligned}
V(t) \ge\ & a(t)\left[-(2+4\delta)(2E(0)+l) + 4(1+\delta)(R_u+S_u+R_v+S_v+l)\right]\\
& -4(1+\delta)\left(\sqrt{R_uP_u}+\sqrt{Q_uS_u}+\sqrt{R_vP_v}+\sqrt{Q_vS_v}+\sqrt{l}\sqrt{l}(t+\tau)\right)^2.
\end{aligned}$$

And by (5.4), we get

$$\begin{aligned}
V(t) \ge\ & -(2+4\delta)(2E(0)+l)\,a(t) + 4(1+\delta)\,\Theta(t)\\
& +4(1+\delta)(R_u+S_u+R_v+S_v+l)(T_1-t)\left(\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2\right),
\end{aligned}$$

where

$$\Theta(t) = (R_u + S_u + R_v + S_v + l)\left(P_u + Q_u + P_v + Q_v + l(t+\tau)^2\right) - \left(\sqrt{R_uP_u} + \sqrt{Q_uS_u} + \sqrt{R_vP_v} + \sqrt{Q_vS_v} + \sqrt{l}\sqrt{l}(t+\tau)\right)^2.$$

By Schwarz inequality, $\Theta(t)$ is nonnegative. Hence, we have

$$V(t) \geq -(2+4\delta)(2E(0)+l)J(t)^{-\frac{1}{\delta}},\ t \in [0, T_1].$$

Therefore, from (5.9), we get

$$J''(t) \leq \delta(2+4\delta)(2E(0)+l)J(t)^{1+\frac{1}{\delta}}. \tag{5.13}$$

Theorem 5.2. *Suppose that* $(A1)$ *and* $(A2)$ *hold and that either one of the following statements is satisfied:*

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $2\delta\int_\Omega (u_0u_1 + v_0v_1) > \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2$,

then the solution $(u(t), v(t))$ *blows up at finite time* $T^* > 0$.

Moreover, the finite time T^* *can be estimated as follows :*

(i) *if* $E(0) < 0$, *then*

$$T^* \leq \frac{\phi + \sqrt{\phi^2 - 8E(0)\delta^2\left(\|u_0\|_2^2 + \|v_0\|_2^2\right)}}{4(-E(0))\delta^2}. \tag{5.14}$$

(ii) *If* $E(0) = 0$, *then*

$$T^* \leq \frac{\|u_0\|_2^2 + \|v_0\|_2^2}{-\phi}, \tag{5.15}$$

where $\phi = \|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 - 2\delta\int_\Omega (u_0u_1 + v_0v_1)\mathrm{d}x$.

Proof. Taking $l = -2E(0)(\geq 0)$ in (5.13) and from (5.8), we see that

$$\left(a(t)^{-\delta}\right)'' \leq 0,\ t \geq 0.$$

Now, we consider two different cases on the sign of the initial energy $E(0)$.

Case 1. $E(0) < 0$. First, we choose τ so large that

$$a'(0) = 2\int_\Omega (u_0u_1 + v_0v_1) - 4E(0)\tau > 0,$$

and select T_1 such that

(5.16) $$\frac{a(0)}{\delta a'(0)} \le T_1,$$

then, we deduce

$$a(t) \ge \left(\frac{a(0)^{1+\delta}}{a(0) - \delta a'(0)t} \right)^{\frac{1}{\delta}}.$$

Therefore, there exists a finite time $T^* \le T_1$ such that

$$\lim_{t \to T^{*-}} \left\{ \int_\Omega \left(u^2 + v^2\right) dx + \int_0^t \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) dt \right\} = \infty.$$

By Poincaré inequality, it implies that

$$\lim_{t \to T^{*-}} \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx = \infty.$$

Moreover, inequality (5.16) holds if and only if

$$T_1(\tau) \equiv \frac{\|u_0\|_2^2 + \|v_0\|_2^2 - 2E(0)\tau^2}{2\delta \left(\int_\Omega (u_0 u_1 + v_0 v_1)\, \mathrm{d}x - 2E(0)\tau \right) - \|\nabla u_0\|_2^2 - \|\nabla v_0\|_2^2} \le T_1.$$

We observe that $T_1(\tau)$ take a minimum at

$$\tau \equiv \tau_0 = \frac{\phi + \sqrt{\phi^2 - 8E(0)\delta^2 \left(\|u_0\|_2^2 + \|v_0\|_2^2 \right)}}{4\,(-E(0))\,\delta},$$

Thus putting $T_1 = T_1(\tau_0)$, we arrive at (5.14).

Case 2. $E(0) = 0$ and $2 \int_\Omega (u_0 u_1 + v_0 v_1) > 0$. Then we see

$$a'(0) = 2 \int_\Omega (u_0 u_1 + v_0 v_1) > 0$$

and

$$a(0) = \|u_0\|_2^2 + \|v_0\|_2^2 + T_1 \left(\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 \right).$$

Thus, we get (5.15), if we choose $T_1 = \frac{a(0)}{\delta a'(0)}$ in (5.16).

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