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VISCOSITY METHOD FOR HIERARCHICAL FIXED POINT APPROACH TO VARIATIONAL INEQUALITIES

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Abstract. A viscosity method for a hierarchical fixed point approach to variational inequality problems is presented. This method is used to solve variational inequalities where the involving operators are complements of nonexpansive mappigs and the solutions are sought in the set of the fixed points of another nonexpansive mapping. Such variational inequalities include monotone inclusions and convex optimization problems to be solved over the fixed point sets of nonexpansive mappings.

1. Introduction

A fairly common method in solving some nonlinear problems is to replace the original problem by a family of regularized (perturbed) problems and each of these regularized problems has a unique solution. A particular (viscosity) solution of the original problem will be obtained as a limit of these unique solutions of the regularized problems. We will use this idea to provide a viscosity method for the hierarchical fixed point approach to solving variational inequality problems.

Let C be a closed convex subset of a real Hilbert space H and $F : C \to H$ be a nonlinear mapping. Consider the variational inequality problem (VIP) of finding a point x^* with the property

$$x^* \in C \quad \text{such that} \quad \langle Fx^*, x - x^* \rangle \geq 0, \quad x \in C. \tag{1.1}$$

The VIP (1.1) is equivalent to the fixed point equation

$$x^* = P_C(I - \gamma F)x^* \tag{1.2}$$

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where $\gamma > 0$ and P_C is the metric projection of H onto C which assigns to each $x \in H$ the only point in C, denoted $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is well-known that if F is Lipschitzian and strongly monotone (i.e., $\langle Fx - Fy, x - y\rangle \geq \alpha\|x - y\|^2$ for $x, y \in C$ and for some $\alpha > 0$), then for small enough $\gamma > 0$, the mapping $P_C(I - \gamma F)$ is a contraction on C and so the sequence $\{x_n\}$ of Picard iterates, given by $x_n = P_C(I - \gamma F)x_{n-1}$ ($n \geq 1$), converges strongly to the unique solution of the VIP (1.1).

It is also known that if F is inversely strongly monotone (i.e., there is a constant $\mu > 0$ such that $\langle Fx - Fy, x - y\rangle \geq \mu\|Fx - Fy\|^2$ for $x, y \in C$), then the mapping $P_C(I - \gamma F)$ is an averaged mapping (namely, there are $\beta \in (0, 1)$ and a nonexpansive mapping T such that $P_C(I - \gamma F) = (1 - \beta)I + \beta T$), then the sequence of Picard iterates, $\{(P_C(I - \gamma F))^n x_0\}$, converges weakly to a solution of the VIP (1.1) (if such solutions exist). (Recall that a mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for $x, y \in C$.)

In this paper we are concerned with the VIP (1.1) in the case where C is the set $Fix(T)$ of the fixed points of a nonexpansive self-mapping T of C and F is of the form $F = I - V$, with V another nonexpansive self-mapping of C. In other words, our VIP is of the form

(1.3) Find $x^* \in Fix(T)$ such that $\langle x^* - Vx^*, x - x^*\rangle \geq 0, \quad x \in Fix(T).$

Equivalently, x^* is a fixed point of $P_{Fix(T)}V$, $x^* = P_{Fix(T)}Vx^*$. Let S denote the solution set of the VIP (1.3) (i.e., the set of fixed points of $P_{Fix(T)}V$) and assume that $S \neq \emptyset$. (More general variational inequalities and other methods (e.g. hybrid extragradient and steepest-descent) can be found in [3, 13, 22, 23].)

The following two special choices of the mapping V in the VIP (1.3) have been studied in the literature:

(a) V is a constant mapping on C: $Vx \equiv u$ for some $u \in C$ and all $x \in C$.

(b) V is a contraction with coefficient $\rho \in [0, 1)$; that is,

$$\|Vx - Vy\| \leq \rho\|x - y\|, \quad x, y \in C.$$

The first case is equivalent to the VIP:

(1.4) Find $x^* \in Fix(T)$ such that $\langle x^* - u, x - x^*\rangle \geq 0, \quad x \in Fix(T)$

or equivalent to finding the fixed point of T closest to u; that is,

$$x^* = P_{Fix(T)}u = \operatorname{argmin}_{x \in Fix(T)} \frac{1}{2}\|u - x\|^2.$$

This problem has widely been investigated; see [2, 5, 6, 14, 16, 18, 19]. There are two ways to solve it: one implicit and one explicit.

The implicit method, initiated in [2], defines, for each fixed element $u \in C$, a curve: $z : (0,1) \to C$ satisfying the fixed point equation

$$z_t = tu + (1-t)Tz_t. \tag{1.5}$$

(This equation has a unique solution because the mapping $z \mapsto tu + (1-t)Tz$ is a contraction on C.)

The explicit method, initiated in [5], generates a sequence $\{x_n\}$ by the recursive formula:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.6}$$

where $u, x_0 \in C$ and $\{\alpha_n\} \subset [0,1]$.

The following results are commonly known.

Theorem 1.1. [2]. *Assume $Fix(T)$ is nonempty. Then $z_t \to P_{Fix(T)}u$ in norm.*

Theorem 1.2. [5, 6, 15, 11, 12, 17, 16]. *Assume $Fix(T)$ is nonempty. Suppose that the sequence $\{\alpha_n\}$ satisfies the conditions $(C1)$ and $(C2)$:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$.

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Suppose, in addition, that $\{\alpha_n\}$ satisfies one of the following conditions:

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

(C4) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n|/\alpha_{n+1} = 0$.

Then the sequence $\{x_n\}$ generated by the algorithm (1.6) converges in norm to $P_{Fix(T)}u$.

The VIP (1.3) covers several topics recently investigated in literature. Some of them are listed below.

(i) (Monotone inclusions.) Yamada [21] studied the VIP (1.1) by assuming that $C = Fix(T)$ for some nonexpansive mapping T in H, where the operator F is Lipschitzian and strongly monotone. This corresponds to the VIP (1.3) where $V = I - \gamma F$, with $\gamma > 0$ sufficiently small.

(ii) (Convex optimization.) Let φ be a proper lower semicontinuous convex function on H and let ψ be a convex function on H so that $\nabla\psi$ is strongly monotone. Take

$$V = \mathrm{prox}_{\lambda\varphi} := \mathrm{argmin}\left\{\varphi(z) + \frac{1}{2\lambda}\|\cdot - z\|^2\right\}.$$

Then the VIP (1.3) is reduced to the hierarchical minimization problem

$$\min_{x \in \operatorname{argmin}\varphi} \psi(x).$$

(iii) (Quadratic minimization over fixed point sets [8].) Consider the minimization problem

$$\min_{x \in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.7}$$

where A is a linear bounded strongly positive operator on H, h is a potential for γf (i.e., $h'(x) = \gamma f(x)$), where $\gamma > 0$ is a constant and f is a contraction on H. The optimality condition for the minimization (1.7) is the VIP of finding a fixed point of T so that

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in Fix(T). \tag{1.8}$$

Taking $V = I - \tilde{\gamma}(A - \gamma f)$, where $\gamma > 0$ is appropriately chosen so that V is nonexpansive, we find that the VIP (1.8) is reduced to the VIP (1.3).

The viscosity extension of the above results was first studied by Moudafi [9] and further developed by the author [20]. In the viscosity approximation method, the implicit and explicit schemes (1.5) and (1.6) are replaced respectively by the following schemes:

$$x_t = tf(x_t) + (1-t)Tx_t \tag{1.9}$$

and

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)Tx_n \tag{1.10}$$

where f is a contraction on C, $t \in (0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 1]$.

Theorem 1.3. [9, 20]. *Assume $Fix(T)$ is nonempty and let x_t be given by (1.9). Then $s - \lim_{t\to 0} x_t =: x^*$ exists and x^* solves the variational inequality*

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \quad x \in Fix(T). \tag{1.11}$$

Theorem 1.4. [9, 20]. *Assume $Fix(T)$ is nonempty and let $\{x_n\}$ be the sequence generated by the algorithm (1.10). Assume conditions (C1) and (C2), and in addition, either (C3) or (C4), hold. Then $x_n \to x^*$ in norm, where x^* is the unique solution of the variational inequality (1.11).*

Recently, Moudafi and Mainge [10] considered the viscosity method for hierarchical fixed point problems of nonexpansive mappings as follows.

Given on C a contraction f and two nonexpansive mappings V and T. Then for $s, t \in (0,1)$, the mapping

$$x \mapsto sf(x) + (1-s)[tVx + (1-t)Tx]$$

is a contraction on C. So it has a unique fixed point, denoted $x_{s,t} \in C$; thus,

$$x_{s,t} = sf(x_{s,t}) + (1-s)[tVx_{s,t} + (1-t)Tx_{s,t}]. \tag{1.12}$$

In [10], Moudafi and Mainge studied the convergence of the hierarchical scheme (1.12) under certain assumptions.

It is the purpose of the present article to further study the convergence of the implicit hierarchical scheme (1.12). We will show in section 3 that $\{x_{s,t}\}$, defined by the implicit scheme (1.12), converges, for each fixed $t \in (0,1)$, strongly as $s \to 0$ to a point x_t which in turns converges in norm as $t \to 0$ to a solution of the variational inequality (1.11).

2. Preliminaries

In this section we assume that C is a closed convex subset of a real Hilbert space H. Recall that f is a contraction on C with coefficient $\rho \in [0,1)$ if f is a self-mapping of C and satisfy the property:

$$\|f(x) - f(x')\| \le \rho\|x - x'\|, \quad x, x' \in C.$$

Recall also that a mapping $T : C \to C$ is nonexpansive provided

$$\|Tx - Tx'\| \le \|x - x'\|, \quad x, x' \in C.$$

Lemma 2.1. [16]. *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n + \beta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ and $\{\beta_n\}$ are sequences in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} \gamma_n|\delta_n| < \infty$;

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. (cf. [20]). *Let $f : C \to C$ be a contraction with coefficient $\rho \in [0, 1)$ and $T : C \to C$ be a nonexpansive mapping.*

(i) *$I - f$ is strongly monotone with coefficient $1 - \rho$; that is,*

$$\langle x - y, (I - f)x - (I - f)y\rangle \geq (1 - \rho)\|x - y\|^2, \quad x, y \in C.$$

(ii) *$I - T$ is monotone; that is,*

$$\langle x - y, (I - T)x - (I - T)y\rangle \geq 0, \quad x, y \in C.$$

Lemma 2.3. (Demiclosedness Principle) (cf. [4]). *Let $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Recall that if K is a closed convex subset of real Hilbert space H, then we can define the (nearest point) projection from H onto K by assigning to each $x \in H$ the unique point in K, denoted $P_K x$, in such a way that

$$\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}.$$

Lemma 2.4. *Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0 \quad \text{for all } y \in K.$$

The following straightforward inequality will be used.

Lemma 2.5. *There holds the following inequality in an inner product space X:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad x, y \in X.$$

Notation. Let $\{x_n\}$ be a sequence and x be a point in a normed space X. Then we use $x_n \to x$ and $x_n \rightharpoonup x$ to denote strong and weak convergence to x of the sequence $\{x_n\}$, respectively.

3. Hierarchical Fixed Point Approach

In solving nonlinear problems which are not well-posed, a quite common method is to consider a family of perturbed (regularized) problems such that each of these regularized problems is well-posed. Then viscosity approximation method is applied to seek a particular solution of the original problem as a limit of the solutions of the regularized problems. In this section we will use this idea to provide a viscosity method for the hierarchical fixed point approach to solving variational inequality problems.

Now let C be a closed convex subset of a real Hilbert space H. Consider two nonexpansive mappings $T, V : C \to C$ with $Fix(T) \neq \emptyset$ and the associated the variational inequality problem (VIP)

(3.1) $$\text{find } \tilde{x} \in Fix(T) \text{ such that } \langle (I-V)\tilde{x}, x-\tilde{x}\rangle \geq 0, \quad x \in Fix(T).$$

We use S to denote the solution set of (3.1). It is known that S is also the fixed point set of the nonexpansive mapping $P_{Fix(T)}V$, $S = Fix(P_{Fix(T)}V)$, where $P_{Fix(T)}$ is the metric projection onto $Fix(T)$. [We will always assume that S is nonempty.]

Next let $f : C \to C$ be a contraction with contraction coefficient $\rho \in [0,1)$. For each $s,t \in (0,1)$, define two mappings W_t and $f_{s,t}$ by

$$W_t = tV + (1-t)T, \quad f_{s,t} = sf + (1-s)W_t.$$

It is easily seen that W_t is nonexpansive and $f_{s,t}$ is a contraction with coefficient $1-(1-\rho)s$; that is,

$$\|f_{s,t}(x) - f_{s,t}(y)\| \leq [1-(1-\rho)s]\|x-y\|, \quad x,y \in C.$$

Let $x_{s,t}$ be the unique fixed point of $f_{s,t}$ in C. Namely, $x_{s,t}$ is the unique solution in C to the equation

(3.2) $$x_{s,t} = sf(x_{s,t}) + (1-s)W_t x_{s,t} = sf(x_{s,t}) + (1-s)[tVx_{s,t} + (1-t)Tx_{s,t}].$$

It is interesting to know the behavior of $\{x_{s,t}\}$ when s,t tend to 0 separately or jointly. In [10], Moudafi and Mainge initiated the investigation of the behavior of $\{x_{s,t}\}$ as $s \to 0$ first and then as $t \to 0$. They employed the following assumptions:

(A1) For each $t \in (0,1)$, the fixed point set of W_t, $Fix(W_t)$, is nonempty and the set $\{Fix(W_t) : 0 < t < 1\}$ is bounded;

(A2) The solution set S of the variational inequality (3.1) is nonempty; and

(A3) $\emptyset \neq S \subset \|\cdot\| - \liminf_{t\to 0} Fix(W_t) := \{z : \exists z_t \in Fix(W_t) \text{ such that } z_t \to z$ in norm as $t \to 0\}$.

Under these assumptions, Moudafi and Mainge [10] proved that, as s goes to 0, $\{x_{s,t}\}$ converges strongly to a point x_t, and as t goes to 0, $\{x_t\}$ converges *weakly* to a point x_∞ which solve the variational inequality

(3.3) $$\langle x_\infty - f(x_\infty), x - x_\infty\rangle \geq 0, \quad x \in S.$$

(Note that the strong convergence as $s \to 0$ to x_t of $\{x_{s,t}\}$ has actually also been proved in [20].)

Before stating the main theorem, we discuss the behavior as $t \to 0$ of z_t of fixed points of W_t.

Proposition 3.1. *Let $t \in (0,1)$ and let z_t be a fixed point of the mapping $W_t = tV + (1-t)T$; namely, $z_t = tVz_t + (1-t)Tz_t$. Assume $\{z_t\}$ remains bounded as $t \to 0$.*

- (i) *The solution set S of the variational inequality (3.1) is nonempty and each weak limit point (as $t \to 0$) of $\{z_t\}$ solves the VI (3.1).*
- (ii) *If $I - V$ is strictly monotone, then the net $\{z_t\}$ converges weakly to the solution of the VI (3.1).*
- (iii) *If $I - V$ is strongly monotone (e.g., V is a contraction), then the net $\{z_t\}$ converges strongly to the solution of the VI (3.1).*

Proof. (Part (i) was proved in [7] under the additional assumption that $S \neq \emptyset$ and by the graph convergence theory [1]. Here we give an alternative (elementary) proof.)

Let W be the set of all weak accumulation points of $\{z_t\}$ as $t \to 0$; that is, $W = \{z : z_{t_n} \rightharpoonup z$ for some sequence $\{t_n\}$ in $(0,1)$ such that $t_n \to 0\}$.

To prove (i), we notice that the boundedness of $\{z_t\}$ implies that $W \neq \emptyset$ and

$$\|z_t - Tz_t\| = t\|Vz_t - Tz_t\| \to 0 \quad \text{as } t \to 0.$$

It thus follows from Lemma 2.3 that $W \subset Fix(T)$.

Using Lemma 2.5, we derive that, for any $\hat{x} \in Fix(T)$,

$$\begin{aligned}
\|z_t - \hat{x}\|^2 &= \|(1-t)(Tz_t - \hat{x}) + t(Vz_t - \hat{x})\|^2 \\
&\leq (1-t)^2\|Tz_t - \hat{x}\|^2 + 2t\langle Vz_t - \hat{x}, z_t - \hat{x}\rangle \\
&\leq (1-t)^2\|z_t - \hat{x}\|^2 + 2t(\langle Vz_t - z_t, z_t - \hat{x}\rangle + \|z_t - \hat{x}\|^2) \\
&= (1+t^2)\|z_t - \hat{x}\|^2 + 2t\langle Vz_t - z_t, z_t - \hat{x}\rangle.
\end{aligned}$$

It follows that

$$\langle z_t - Vz_t, z_t - \hat{x}\rangle \leq \frac{t}{2}\|z_t - \hat{x}\|^2. \tag{3.4}$$

Since $I - V$ is monotone, we have

$$\langle (I-V)z_t, z_t - \hat{x}\rangle \geq \langle (I-V)\hat{x}, z_t - \hat{x}\rangle.$$

This together with (3.4) implies that

$$\langle (I-V)\hat{x}, z_t - \hat{x}\rangle \le \frac{t}{2}\|z_t - \hat{x}\|^2. \tag{3.5}$$

Now if $\tilde{x} \in W \subset Fix(T)$ and if $t_n \to 0$ is such that $x_{t_n} \rightharpoonup \tilde{x}$, then we obtain from (3.5) that

$$\langle (I-V)\hat{x}, \tilde{x} - \hat{x}\rangle \le 0, \quad \hat{x} \in Fix(T). \tag{3.6}$$

Replacing the $\hat{x}$ in (3.6) with $\tilde{x} + \lambda(x - \tilde{x}) \in Fix(T)$, where $\lambda \in (0,1)$ and $x \in Fix(T)$, we get

$$\langle (I-V)(\tilde{x} + \lambda(x-\tilde{x})), \tilde{x} - x\rangle \le 0.$$

Letting $\lambda \to 0$ yields

$$\langle (I-V)\tilde{x}, \tilde{x} - x\rangle \le 0 \tag{3.7}$$

for all $x \in Fix(T)$. Hence, $\hat{x} \in S$.

To see (ii), we assume that $\{t'_n\}$ is another null sequence in (0,1) such that $x_{t'_n} \rightharpoonup \hat{x}$. Then $\hat{x} \in Fix(T)$ and by replacing the x in (3.7) with $\hat{x}$, we get

$$\langle (I-V)\tilde{x}, \tilde{x} - \hat{x}\rangle \le 0. \tag{3.8}$$

We can interchange $\tilde{x}$ and $\hat{x}$ to get

$$\langle (I-V)\hat{x}, \hat{x} - \tilde{x}\rangle \le 0. \tag{3.9}$$

Adding up (3.8) and (3.9) yields

$$\langle (I-V)\tilde{x} - (I-V)\hat{x}, \tilde{x} - \hat{x}\rangle \le 0.$$

So the strict monotonicity of $I - V$ implies that $\tilde{x} = \hat{x}$ and $\{z_t\}$ converges weakly.

Finally to prove (iii), we observe that the strong monotonicity of $I-V$ and (3.4) imply that

$$\alpha\|z_t - \hat{x}\|^2 + \langle (I-V)\hat{x}, z_t - \hat{x}\rangle \le \frac{t}{2}\|z_t - \hat{x}\|^2 \tag{3.10}$$

where $\alpha > 0$ is the strong monotonicity coefficient of $I - V$; that is,

$$\langle (I-V)x - (I-V)y, x - y\rangle \ge \alpha\|x-y\|^2, \quad x, y \in C.$$

A straightforward consequence of (3.10) is that if $\hat{x} \in W$ and if $z_{t_n} \rightharpoonup \hat{x}$ for some null sequence $\{t_n\}$ in $(0,1)$, then we must have $z_{t_n} \to \hat{x}$. This shows that $\{z_t\}$ is relatively compact in the norm topology, and each of its limit points solves the VI (3.1). Finally repeating the argument in the weak convergence case of (ii), we see that $\{z_t\}$ can have exactly one limit point; hence $\{z_t\}$ converges in norm. ∎

The following is the main result of the present paper in which we improve the result of Moudafi and Mainge [10] by proving that $\{x_t\}$ actually converges *strongly* and also by removing the boundedness of the set $\{Fix(W_t) : 0 < t < 1\}$ in assumption (A1). Our proof presented below is different from that of [10].

Theorem 3.2. *Let the above assumptions (A2) and (A3) hold. Assume also that, for each* $t \in (0,1)$, $Fix(W_t)$ *is nonempty (but not necessarily bounded). Then the strong* $\lim_{s\to 0} x_{s,t} =: x_t$ *exists for each* $t \in (0,1)$. *Moreover the strong* $\lim_{t\to 0} x_t =: x_\infty$ *exists and solves the variational inequality (3.3). Hence, for any null sequence* $\{s_n\}$ *in (0,1), there is another null sequence* (t_n) *in* $(0,1)$ *such that* $x_{s_n,t_n} \to x_\infty$ *in norm, as* $n \to \infty$.

Proof. Since, for each fixed $t \in (0,1)$, the fixed point set $Fix(W_t)$ of W_t is nonempty, we can apply Theorem 1.3 to get that

$$x_t := \|\cdot\| - \lim_{s\to 0} x_{s,t}$$

exists in $Fix(W_t)$ and solves the following variational inequality

$$\langle (I-f)x_t, x - x_t\rangle \geq 0, \quad x \in Fix(W_t). \tag{3.11}$$

Equivalently, $x_t = (P_{Fix(W_t)} f)x_t$, where $P_{Fix(W_t)}$ is the metric projection from H onto $Fix(W_t)$.

It follows from (3.11) that, for $z \in Fix(W_t)$,

$$\begin{aligned}\|x_t - z\|^2 &\leq \langle f(x_t) - z, x_t - z\rangle \\ &= \langle f(x_t) - f(z), x_t - z\rangle + \langle f(z) - z, x_t - z\rangle \\ &\leq \rho\|x_t - z\|^2 + \langle f(z) - z, x_t - z\rangle.\end{aligned}$$

Hence,

$$\|x_t - z\|^2 \leq \frac{1}{1-\rho}\langle f(z) - z, x_t - z\rangle, \quad z \in Fix(W_t). \tag{3.12}$$

This particularly implies that

$$\|x_t - z\| \leq \frac{1}{1-\rho}\|f(z) - z\|, \quad z \in Fix(W_t). \tag{3.13}$$

(3.13) is yet to imply the boundedness of $\{x_t\}$ since z may depend on t. However, since the solution set S of the VIP (3.1) is nonempty, we can take (an arbitrary) $v \in S$, and use assumption (A3) to find $z_t \in Fix(W_t)$ such that $z_t \to v$ in norm as $t \to 0$. Hence (z_t) must be bounded (as $t \to 0$). Now (3.13) implies

$$\begin{aligned}\|x_t\| &\le \|x_t - z_t\| + \|z_t - v\| + \|v\| \\ &\le \frac{1}{1-\rho}\|f(z_t) - z_t\| + \|z_t - v\| + \|v\|\end{aligned}$$

and this is sufficient to ensure that $\{x_t\}$ is bounded (as t close 0).

Now the boundedness of $\{x_t\}$ allows us to apply Proposition 3.1(i) to conclude that every weak limit point $\tilde{x}$ of $\{x_t\}$ belongs to the solution set S of the VIP (3.1). Then (3.12) guarantees that every such weak limit point $\tilde{x}$ of $\{x_t\}$ is also a strong limit point of $\{x_t\}$. Indeed, if $\{t_n\}$ is a null sequence in $(0, 1)$ and if $x_{t_n} \to \tilde{x}$ weakly, then $\tilde{x} \in S$. Use assumption (A3) to get a sequence $\{z_n\}$ such that $z_n \in Fix(W_{t_n})$ for all n and $z_n \to \tilde{x}$ in norm. From (3.12) we get

$$\begin{aligned}\|x_{t_n} - \tilde{x}\|^2 &= \|(x_{t_n} - z_n) + (z_n - \tilde{x})\|^2 \\ &\le 2(\|x_{t_n} - z_n\|^2 + \|z_n - \tilde{x}\|^2) \\ &\le \frac{2}{1-\rho}\langle f(z_n) - z_n, x_{t_n} - z_n\rangle + 2\|z_n - \tilde{x}\|^2.\end{aligned} \tag{3.14}$$

However, $\langle f(z_n) - z_n, x_{t_n} - z_n\rangle \to 0$ since $f(z_n) - z_n \to f(\tilde{x}) - \tilde{x}$ in norm and $x_{t_n} - z_n \to 0$ weakly, and we find that the right-hand side of (3.14) tends to zero. Hence, $x_{t_n} \to \tilde{x}$ in norm.

So to prove the strong convergence of the entire net $\{x_t\}$, it remains to prove that $\{x_t\}$ can have only one strong limit point. Let $\tilde{x}$ and $\tilde{x}'$ be two strong limit points of $\{x_t\}$ and assume that $x_{t_n} \to \tilde{x}$ and $x_{t'_n} \to \tilde{x}'$ both in norm, where $\{t_n\}$ and $\{t'_n\}$ are null sequences in $(0, 1)$. It remains to verify that $\tilde{x} = \tilde{x}'$.

Since $\tilde{x}' \in S$, by assumption (A3), we can find $z_t \in Fix(W_t)$ such that $z_t \to \tilde{x}'$ in norm as $t \to 0$. The variational inequality (3.11) implies

$$\langle (I - f)x_{t_n}, z_{t_n} - x_{t_n}\rangle \ge 0.$$

Taking the limit as $n \to \infty$ yields

$$\langle (I - f)\tilde{x}, \tilde{x}' - \tilde{x}\rangle \ge 0. \tag{3.15}$$

Similarly we have

$$\langle (I - f)\tilde{x}', \tilde{x} - \tilde{x}'\rangle \ge 0. \tag{3.16}$$

Adding up (3.15) and (3.17) gives

$$(3.17)\qquad \langle (I-f)\tilde{x}-(I-f)\tilde{x}', \tilde{x}-\tilde{x}'\rangle \le 0.$$

By Lemma 2.2(i), we obtain $\tilde{x}=\tilde{x}'$ and so $\{x_t\}$ converges in norm to (say) x_∞.

Now for any $v\in S$, since by assumption (A3), we can find $z_t\in Fix(W_t)$ such that $z_t\to v$ in norm, (3.11) then implies

$$\langle (I-f)x_t, v-x_t\rangle \ge \langle (I-f)x_t, v-z_t\rangle \to 0$$

which in turns implies

$$(3.18)\qquad \langle (I-f)x_\infty, v-x_\infty\rangle \ge 0,\quad v\in S.$$

That is, $x_\infty=(P_S f)x_\infty$, the unique fixed point of the contraction $P_S f$. Finally, for any null sequence $\{s_n\}$ in (0,1), using a diagonalization argument (cf. [1]), we can find another null sequence (t_n) in $(0,1)$ such that $x_{s_n,t_n}\to x_\infty$ in norm, as $n\to\infty$. ■

Remark 3.3. Theorem 3.2 shows that for any null sequence $\{s_n\}$ in (0,1), there is another null sequence (t_n) in $(0,1)$ such that $x_{s_n,t_n}\to x_\infty$ in norm, as $n\to\infty$, and x_∞ is a solution to the VIP (3.18). Below we present a general result. We can show that as long as t_s is taken so that $t_s=o(s)$ (i.e., $\lim_{s\to 0} t_s/s=0$), then $x_{s,t_s}\to z_\infty$ in norm, and moreover, z_∞ solves the variational inequality (3.18) on the larger set $Fix(T)$ (i.e., z_∞ is the unique fixed point in $Fix(T)$ of the contraction $P_{Fix(T)}f$), without the assumptions $(A2)$ and $(A3)$. However, for such a general choice of $\{t_s\}$, this solution z_∞ may differ from the solution x_∞ of the VIP (3.18) on the smaller set S (i.e., x_∞ is the unique fixed point in S of the contraction $P_S f$). We will verify this by taking $t_s=s^2$ for simplicity (the argument however works for any net (t_s) in (0,1) such that $\lim_{s\to 0} t_s/s=0$).

Theorem 3.4. *Let, for each $s\in(0,1)$, x_s be the unique solution in C to the equation*

$$(3.19)\qquad x_s=sf(x_s)+(1-s)(s^2Vx_s+(1-s^2)Tx_s).$$

Then, as $s\to 0$, x_s converges in norm to the solution of the VIP

$$z_\infty\in Fix(T),\quad \langle (I-f)z_\infty, z-z_\infty\rangle \ge 0,\quad z\in Fix(T);$$

equivalently, $z_\infty=(P_{Fix(T)}f)z_\infty$.

Proof. Write W_s (instead of W_{s^2}) for $s^2V+(1-s^2)T$; then

$$x_s=sf(x_s)+(1-s)W_sx_s.$$

Take a fixed point z of T to derive that

$$\begin{aligned}\|x_s - z\|^2 &= s\langle f(x_s) - z, x_s - z\rangle + (1-s)\langle W_s x_s - z, x_s - z\rangle \\ &= s\langle f(x_s) - f(z), x_s - z\rangle + s\langle f(z) - z, x_s - z\rangle \\ &\quad +(1-s)\langle W_s x_s - W_s z, x_s - z\rangle + (1-s)\langle W_s z - z, x_s - z\rangle \\ &\le s\rho\|x_s - z\|^2 + s\langle f(z) - z, x_s - z\rangle \\ &\quad +(1-s)\|x_s - z\|^2 + (1-s)s^2\langle Vz - z, x_s - z\rangle.\end{aligned}$$

It follows that, for $z \in Fix(T)$,

$$\|x_s - z\|^2 \le \frac{1}{1-\rho}(\langle f(z) - z, x_s - z\rangle + s\langle Vz - z, x_s - z\rangle). \tag{3.20}$$

This implies that

$$\|x_s - z\| \le \frac{1}{1-\rho}(\|f(z) - z\| + \|Vz - z\|).$$

In particular, $\{x_s\}$ is bounded, and from (3.19), we further get

$$\|x_s - Tx_s\| = s\|f(x_s) + s(1-s)Vx_s - (1+s-s^2)Tx_s\| \to 0 \quad \text{as } s \to 0. \tag{3.21}$$

The demiclosedness principle for nonexpansive mappings (Lemma 2.3) then ensures that every weak limit point, as $s \to 0$, of $\{x_s\}$ is a fixed point of T. Going back to (3.20) we find that each weak limit point of $\{x_s\}$ is actually a strong limit point of $\{x_s\}$. So to prove the strong convergence of $\{x_s\}$, we need only to show the uniqueness of strong limit points of $\{x_s\}$. Assuming $\{s_n\}$ and $\{s'_n\}$ are null sequences in (0,1) such that $x_{s_n} \to v$ and $x_{s'_n} \to v'$, both in norm. Observing that (3.19) implies

$$(I - f)x_s = -\frac{1-s}{s}(I - W_s)x_s,$$

we obtain by virtue of the monotonicity of $I - W_s$, for $z \in Fix(T)$,

$$\begin{aligned}\langle (I-f)x_s, x_s - z\rangle &= -\frac{1-s}{s}\langle (I - W_s)x_s, x_s - z\rangle \\ &= -\frac{1-s}{s}[\langle (I-W_s)x_s - (I-W_s)z, x_s - z\rangle \\ &\quad +\langle (I-W_s)z, x_s - z\rangle] \\ &\le -s(1-s)\langle (I-V)z, x_s - z\rangle.\end{aligned}$$

In particular, we have

$$\langle (I-f)x_{s_n}, x_{s_n} - v'\rangle \le -s_n(1-s_n)\langle (I-V)v', x_{s_n} - v'\rangle.$$

So letting $n \to \infty$ yields

$$\langle (I-f)v, v-v' \rangle \le 0. \tag{3.22}$$

Repeating the above argument obtains

$$\langle (I-f)v', v'-v \rangle \le 0. \tag{3.23}$$

Adding up (3.22) and (3.23) gives us that

$$\langle (I-f)v-(I-f)v', v-v' \rangle \le 0. \tag{3.24}$$

The strong monotonicity of $I-f$ (Lemma 2.2) then implies $v=v'$. Finally taking the limit as $s \to 0$ in (3.22) and letting $z_\infty = \|\cdot\| - \lim_{s\to 0} x_s$, we find that z_∞ solve the variational inequality

$$z_\infty \in Fix(T), \quad \langle (I-f)z_\infty, z_\infty - z \rangle \le 0, \quad z \in Fix(T).$$

Equivalently, $z_\infty = (P_{Fix(T)}f)z_\infty$. The proof is therefore complete. ■

Remark 3.5. If T and V have a common fixed point, then it is not hard to see that $Fix(W_t) = Fix(T) \cap Fix(V)$ for all $t \in (0,1)$. Indeed, it suffices to show the inclusion $Fix(W_t) \subset Fix(T) \cap Fix(V)$.To see this, take $p \in Fix(T) \cap Fix(V)$ and let $z \in Fix(W_t)$. It follows that

$$\begin{aligned} \|z-p\|^2 &= \|W_t z - p\|^2 \\ &= \|t(Vz-p) + (1-t)(Tz-p)\|^2 \\ &= t\|Vz-p\|^2 + (1-t)\|Tz-p\|^2 - t(1-t)\|Vz-Tz\|^2 \\ &\le \|z-p\|^2 - t(1-t)\|Vz-Tz\|^2. \end{aligned}$$

This implies $Vz = Tz = z$; that is $z \in Fix(T) \cap Fix(V)$.

Then assumption (A2) is satisfied for any common fixed point of T and V; hence $Fix(T) \cap Fix(V) \subset S$. While assumption (A3) is reduced to the assumption $S \subset Fix(T) \cap Fix(V)$. Therefore, (A2) and (A3) are equivalent to the assumption $S = Fix(T) \cap Fix(V)$, and they both are superfluous, as shown in the following result.

Corollary 3.6. *Assume that T and V have a common fixed point. Then the conclusion of Theorem 3.2 holds. Namely, the strong* $\lim_{s\to 0} x_{s,t} =: x_t$ *exists for each fixed* $t \in (0,1)$*, and moreover the strong* $\lim_{t\to 0} x_t =: x_\infty$ *exists and solves the variational inequality* (3.3).

Proof. Since $Fix(W_t) = Fix(T) \cap Fix(V)$ is independent of t, the z in both relations (3.12) and (3.13) does not depend on t. Hence it is immediately clear that $\{x_t\}$ is bounded, which then implies via (3.12) that every weak accumulation point of $\{x_t\}$ is also a strong accumulation point of $\{x_t\}$. Eventually, $\{x_t\}$ converges in norm as shown in the final part of the proof of Theorem 3.2. ■

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