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VARIATIONAL LYAPUNOV METHOD AND STABILITY ANALYSIS FOR PERTURBED SETVALUED IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. In this paper, we develop stability criteria in terms of two measures for perturbed setvalued delay integro-differential equations with fixed moments of impulsive effects. Variational Lyapunov method is employed to establish the stability properties of the perturbed system via a comparison result which connects the solutions of perturbed system and the unperturbed one through the solutions of a comparison system.

1. INTRODUCTION

The study of setvalued differential equations, initiated as an independent subject, has been addressed by many authors, for instance, see [1-5] and the references therein. The interesting feature of the setvalued differential equations is that the results obtained in this new framework become the corresponding results of ordinary differential equations as the Hukuhara derivative and the integral used in formulating the set differential equations reduce to the ordinary vector derivative and integral when the set under consideration is a single valued mapping. Also, the differential equations with delay provide a better approach for mathematical formulation of a physical phenomenon involving a time lag between the cause and the effect, see [6,7].

Stability is one of the major problems encountered in applications and has attracted the attention of many researchers. In the perturbation theory of nonlinear differential systems, a flexible mechanism known as variation of Lyapunov second

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method (variational Lyapunov method) was introduced in [8]. This technique essentially connects the solutions of perturbed system and the unperturbed one through the solutions of a comparison system using a comparison principle. The concept of stability in terms of two measures [9-12] which unifies a number of stability concepts such as Lyapunov stability, partial stability, conditional stability, etc. has become an important area of investigation in the qualitative analysis.

In this paper, we investigate the the stability criteria in terms of two measures for setvalued perturbed impulsive integro-differential equations with delay by employing the variational Lyapunov method. A comparison result which connects the solutions of perturbed system and the unperturbed one through the solutions of a comparison system is also proved.

The theory and applications of integro-differential equations is an important area of investigation as these equations extensively occur in the mathematical modelling of physical problems, for instance, the governing equations in the problems of biological sciences such as spreading of disease by the dispersal of infectious individuals, the reaction-diffusion models in ecology to estimate the speed of invasion, etc. are integro-differential equations.

2. Preliminaries and Comparison Result

Let $K_c(R^n)$ denote the collection of nonempty, compact and convex subsets of R^n. We define the Hausdorff metric as

$$D[X, Y] = \max[\sup_{y \in Y} d(y, X), \sup_{x \in X} d(x, Y)], \tag{1}$$

where $d(y, X) = \inf[d(y, x) : x \in X]$ and X, Y are bounded subsets of R^n. Notice that $K_c(R^n)$ with the metric defined by (1) is a complete metric space. Moreover, $K_c(R^n)$ equipped with the natural algebraic operations of addition and nonnegative scalar multiplication becomes a semilinear metric space which can be embedded as a complete cone into a corresponding Banach space [13, 14]. The Hausdorff metric (1) satisfies the following properties:

$$D[X + Z, Y + Z] = D[X, Y] \text{ and } D[X, Y] = D[Y, X], \tag{2}$$

$$D[\mu X, \mu Y] = \mu D[X, Y], \tag{3}$$

$$D[X, Y] \leq D[X, Z] + D[Z, Y], \tag{4}$$

$\forall\ X, Y, Z \in K_c(R^n)$ and $\mu \in R_+ = [0, \infty)$.

Definition 2.1. The set $Z \in K_c(R^n)$ satisfying $X = Y + Z$ is known as the Hukuhara difference of the sets X and Y in $K_c(R^n)$ and is denoted as $X - Y$.

Definition 2.2. For any interval $I \in R$, the mapping $F : I \to K_c(R^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$ if there exists an element $D_H F(t_0) \in K_c(R^n)$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \text{ and } \lim_{h \to 0^+} \frac{F(t_0) - F(t_0 - h)}{h},$$

exist in the topology of $K_c(R^n)$ and each one is equal to $D_H F(t_0)$.

By embedding $K_c(R^n)$ as a complete cone in a corresponding Banach space and taking into account the result on differentiation of Bochner integral, it is found that if

$$F(t) = X_0 + \int_0^t \Omega(\eta) d\eta, \ X_0 \in K_c(R^n),$$

where $\Omega : I \to K_c(R^n)$ is integrable in the sense of Bochner, then $D_H F(t)$ exists and

$$D_H F(t) = \Omega(t) \text{ a.e. on } I.$$

Moreover, if $F : [t_0, T] \to K_c(R^n)$ is integrable, then

$$\int_{t_0}^{t_2} F(\sigma) d\sigma = \int_{t_0}^{t_1} F(\sigma) d\sigma + \int_{t_1}^{t_2} F(\sigma) d\sigma, \ t_0 \le t_1 \le t_2 \le T,$$

$$\int_{t_0}^{T} \zeta F(\sigma) d\sigma = \zeta \int_{t_0}^{T} F(\sigma) d\sigma, \ \zeta \in R_+.$$

Also, if $F, G : [t_0, T] \to K_c(R^n)$ are integrable, then $D[F(.), G(.)] : [t_0, T] \to R$ is integrable and

$$D[\int_{t_0}^t F(\sigma) d\sigma, \int_{t_0}^t G(\sigma) d\sigma] \le \int_{t_0}^t D[F(\sigma), G(\sigma)] d\sigma.$$

For convenience, we define the following classes of functions:

$\mathcal{K} = \{\nu : [0, \rho) \to R_+$ is continuous, strictly increasing and $\nu(0) = 0, \rho > 0\}$;

$PC = \{\mu : R_+ \to R_+$ is continuous on $(t_{k-1}, t_k]$ and $\mu \to \mu(t_k^+)$ exists as $t \to t_k^+\}$;

$PC\mathcal{K} = \{\phi : R_+ \times [0, \rho) \to R_+, \ \phi(., m) \in PC$ for each $m \in [0, \rho), \phi(t, .) \in \mathcal{K}$ for each $t \in R_+\}$;

$\Gamma = \{h : R_+ \times K_c(R^n) \to R_+, \ \inf_{U \in K_c(R^n)} h(t, U) = 0, \ h(., U) \in PC$, for each $U \in K_c(R^n)$, and $h(t, .) \in C(K_c(R^n), R_+)$ for each $t \in R_+\}$;

$$S(h,\rho) = \{(t,U) \in R_+ \times K_c(R^n) : h(t,U) < \rho, h \in \Gamma\};$$

$$\mathrm{C} = PC([-\tau,0], K_c(R^n)), \tau > 0;$$
$$S(\rho) = \{U \in K_c(R^n) : (t,U) \in S(h,\rho) \text{ for each } t \in R_+\}.$$

Consider the following perturbed setvalued integro-differential equations with fixed moments of impulse

$$(5) \qquad \begin{cases} D_H U(t) = F(t, U_t, L_1 U_t), & t \neq t_k, \\ U_{t_k^+} = U_{t_k} + I_k(U_{t_k}), & k = 1,2,3,..., \\ U_{t_0} = \Phi_0, \end{cases}$$

together with the unperturbed ones

$$(6) \qquad \begin{cases} D_H V(t) = G(t, V_t, L_2 V_t), & t \neq t_k, \\ V_{t_k^+} = V_{t_k} + I_k(V_{t_k}), & k = 1,2,3,..., \\ V_{t_0} = \Phi_0, \end{cases}$$

where $F, G : R_+ \times \mathrm{C} \times \mathrm{C} \to K_c(R^n)$ are continuous on $(t_{k-1}, t_k] \times \mathrm{C} \times \mathrm{C}$ with G smooth enough or containing the linear terms of system (5), $\Phi_0 \in \mathrm{C}$, L_i denote the integral in sense of Hukuhara [22-23] and is defined by $L_i U_t = \int_{t_0}^{t} K_i(t,\eta,U_\eta)d\eta$, $K_i : R_+ \times R_+ \times \mathrm{C} \to K_c(R^n)$ are continuous on $(t_{k-1}, t_k] \times (t_{k-1}, t_k] \times \mathrm{C}$, $i = 1,2$, $I_k, J_k \in C(K_c(R^n), K_c(R^n))$ and $\{t_k\}$ is a sequence of points such that $0 \leq t_0 < t_1 < ...t_k < ...$ with $\lim_{k\to\infty} t_k = \infty$ and $U_t \in \mathrm{C}$ be defined by $U_t(s) = U(t+s)$, $-\tau \leq s \leq 0$. The linear space $PC([-\tau,0], K_c(R^n))$ is equipped with the norm $\|.\|_\tau$ defined by $\|\psi\|_\tau = \sup_{-\tau\leq s\leq 0} \psi(s)$ and $[-\tau,0] = (-\tau,0]$ when $\tau = \infty$.

We denote the solution of (5) by $U(t) = U(t_0,\Phi_0)(t)$ with $U_{t_0} = \Phi_0$ and that of (6) by $V(t) = V(t_0,\Phi_0)(t)$ with $V_{t_0} = \Phi_0$. By a solution of (5) (and that of (6)), we mean a piecewise continuous function $U(t_0,\Phi_0)(t)$ on $[t_0,\infty)$ which is left continuous in every subinterval $(t_k, t_{k+1}]$, $k = 0,1,2,....$

Definition 2.3. Let $W : R_+ \times K_c(R^n) \to R_+$. Then W is said to belong to a class W_0 if $W(t,X)$ is continuous in each $(t_{k-1}, t_k] \times K_c(R^n)$ and for each $X \in K_c(R^n)$, $\lim_{(t,Y)\to(t_k^+,X)} W(t,Y) = W(t_k^+, X)$ exists for $k = 1,2,....$ and $W(t,X)$ is locally Lipscitzian in X.

Definition 2.4. Let $W \in W_0$ and $V(t,\eta,U)$ be any solution of (6). Then for any fixed $t > t_0$, $(\eta,U) \in (t_{k-1}, t_k) \times S(\rho)$, $t_0 \leq \eta < t$, we define

$$\begin{aligned} & D^+W(\eta, V(t,\eta,U)) \\ = \; & \limsup_{h\to 0^+} \frac{1}{h}[W(\eta+h, V(t,\eta+h, U+hF(\eta,U_\eta,L_1U_\eta))) - W(\eta, V(t,\eta,U))], \end{aligned}$$

where $V(t,\eta,U)$ is any solution of (6) such that $V(\eta,\eta,U)=U$.

Definition 2.5. Let $h,\ h_0 \in \Gamma$. We say that

(i) h_0 is finer than h if there exists a $\overline{\lambda} > 0$ and a function $\phi \in PCK$ such that

$$h_0(t,U) < \overline{\lambda} \text{ implies } h(t,U) \leq \phi(t, h_0(t,U));$$

(ii) h_0 is uniformly finer than h if (i) holds for $\phi \in \mathcal{K}$.

Definition 2.6. Let $h,\ h_0 \in \Gamma$ and $W \in W_0$. Then $W(t,U)$ is said to be

(i) $h-$ positive definite if there exists a $\lambda > 0$ and a function $b \in \mathcal{K}$ such that

$$h(t,U) < \lambda \text{ implies } b(h(t,U)) \leq W(t,U);$$

(ii) weakly h_0-decrescent if there exists a $\lambda_1 > 0$ and a function $a \in PCK$ such that

$$h_0(t,U) < \lambda_1 \text{ implies } W(t,U) \leq a(t, h_0(t,U));$$

(iii) h_0-decrescent if (ii) holds with $a \in \mathcal{K}$.

Definition 2.7. For $h_0 \in \Gamma,\ \Phi_0 \in$ C, we define

$$\tilde{h}_0(t,\Phi_0) = \sup_{-\tau \leq s \leq 0} \{h_0(t+s, \Phi_0(s))\}$$

Definition 2.8. Let $h,\ h_0 \in \Gamma$ and $U(t) = U(t_0,\Phi_0)(t)$ be any solution of (5), then the system (5) is said to be

(I) $(\tilde{h}_0, h)-$stable if for each $\epsilon > 0$, there exists a $\delta = \delta(t_0,\epsilon) > 0$ such that

$$\tilde{h}_0(t_0,\Phi_0) < \delta \text{ implies } h(t,U(t)) < \epsilon,\ t \geq t_0;$$

(II) $(\tilde{h}_0, h)$-uniformly stable if (I) holds with δ independent of t_0;

(III) $(\tilde{h}_0, h)$-attractive if there exists a $\delta = \delta(t_0) > 0$ and for each $\epsilon > 0$, there exists $T = T(t_0,\epsilon) > 0$ such that

$$\tilde{h}_0(t_0,\Phi_0) < \delta_0 \text{ implies } h(t,U(t)) < \epsilon,\ t \geq t_0 + T;$$

(IV) $(\tilde{h}_0, h)$-uniformly attractive if (III) holds with δ and T independent of t_0;

(V) $(\tilde{h}_0, h)$-asymptotically stable if it is $(\tilde{h}_0, h)$-stable and $(\tilde{h}_0, h)$-attractive;

(VI) $(\tilde{h}_0, h)$-uniformly asymptotically stable if it is $(\tilde{h}_0, h)$-uniformly stable and $(\tilde{h}_0, h)$-uniformly attractive.

Now, we prove a comparison result which is needed for the sequel.

Lemma 2.1. *Assume that*

($\mathbf{A_1}$) *The solution* $V(t) = V(t, t_0, \Phi_0)$ *of (6) existing for all* $t \geq t_0$ *is unique, continuous with respect to the initial values, locally Lipschitzian in* Φ_0 *and* $V(t_0) = \Phi_0$;

($\mathbf{A_2}$) $W \in C[R_+ \times K(R^n), R_+]$ *satisfies* $D[W(t, X) - W(t, Y)] \leq ND[X, Y]$, *where* N *is the local Lipschitz constant,* $X, Y \in K(R^n),\ t \in R_+$;

($\mathbf{A_3}$) *For* $(\eta, U) \in S(h, \rho),\ t_0 \leq \eta < t,\ W \in W_0$ *satisfies the inequality*

$$\begin{cases} D^+W(\eta, V(t, \eta, U)) \leq g_1(\eta, W(\eta, V(t, \eta, U))), & t \neq t_k, \\ W(t_k^+, V(t, t_k^+, U(t_k^+))) \leq \psi_k(W(t_k, V(t, t_k, U(t_k))), & k = 1, 2, ..., \\ W(t_0^+, V(t, t_0^+, U_0)) \leq x_0, & \end{cases}$$

where $g_1(t, x) \in PC$ *for each* $x \in R_+$ *and* $\psi_k : R_+ \to R_+$ *are nondecreasing functions for all* $k = 1, 2, ...$;

($\mathbf{A_4}$) *The maximal solution* $r(t) = r(t, t_0, x_0)$ *of the following scalar impulsive differential equation exists on* $[t_0, \infty)$

$$(7) \qquad \begin{cases} x' = g_1(t, x), & t \neq t_k, \\ x(t_k^+) = \psi_k(x(t_k)),\ k = 1, 2, ..., & \\ x(t_0^+) = x_0 \geq 0. & \end{cases}$$

Then $W(t, U(t, t_0, \Phi_0)) \leq r(t, t_0, x_0)$.

Proof. Let $U(t) = U(t, t_0, \Phi_0)$ be any solutions of (5) with $(t_0, \Phi_0) \in S(h, \rho)$. We set $m(\eta) = W(\eta, V(t, \eta, U(\eta))),\ \eta \in [t_0, t]$ and $\lim_{\eta \to t-0} m(\eta) = m(t)$. For small $h > 0$, we consider

$$\begin{aligned} m(\eta + h) - m(\eta) &= W(\eta + h, V(t, \eta + h, U(\eta + h))) - W(\eta, V(t, \eta, U(\eta)) \\ &= W(\eta+h, V(t, \eta + h, U(\eta + h))) - W(\eta+h, V(t, \eta+h, U(\eta) + hF(\eta, U_\eta, L_1U_\eta))) \\ &+ W(\eta + h, V(t, \eta + h, U(\eta) + hF(\eta, U_\eta, L_1U_\eta)))) - W(\eta, V(t, \eta, U(\eta))) \\ &\leq ND[V(t, \eta + h, U(\eta + h)), V(t, \eta + h, U(\eta) + hF(\eta, U_\eta, L_1U_\eta)))] \\ &+ W(\eta + h, V(t, \eta + h, U(\eta) + hF(\eta, U_\eta, L_1U_\eta)))) - W(\eta, V(t, \eta, U(\eta))), \end{aligned}$$

where we have used the assumption (A_2). Thus,

$$\begin{aligned} D^+m(t) &= \limsup_{h\to 0^+} \frac{1}{h}[m(t+h)-m(t)] \\ &\le D^+W(\eta, V(t,\eta,U(\eta))) + N^2 \limsup_{h\to 0^+} \frac{1}{h} D[U(\eta+h), U(\eta) \\ &\quad +hF(\eta, U_\eta, L_1U_\eta))]. \end{aligned}$$

Letting $U(\eta+h) = U(\eta)+Z(\eta)$, where $Z(\eta)$ is the Hukuhara difference of $U(\eta+h)$ and $U(\eta)$ for small $h>0$ and is assumed to exist. Hence, employing the properties of $D[.,.]$, it follows that

$$\begin{aligned} &D[U(\eta+h), U(\eta)+hF(\eta,U_\eta,L_1U_\eta))] \\ &= D[U(\eta)+Z(\eta), U(\eta)+hF(\eta,U_\eta,L_1U_\eta))] = D[Z(\eta), hF(\eta,U_\eta,L_1U_\eta))] \\ &= D[U(\eta+h)-U(\eta), hF(\eta,U_\eta,L_1U_\eta))]. \end{aligned}$$

Consequently, we find that

$$\begin{aligned} &\frac{1}{h} D[U(\eta+h), U(\eta)+hF(\eta,U_\eta,L_1U_\eta))] \\ &= D\left[\frac{U(\eta+h)-U(\eta)}{h}, F(\eta,U_\eta,L_1U_\eta))\right], \end{aligned}$$

which, in view of the fact that $U(t)$ is a solution of (5), yields

$$\begin{aligned} &\limsup_{h\to 0^+} \frac{1}{h} D[U(\eta+h), U(\eta)+hF(\eta,U_\eta,L_1U_\eta))] \\ &= \limsup_{h\to 0^+} D\left[\frac{U(\eta+h)-U(\eta)}{h}, F(\eta,U_\eta,L_1U_\eta))\right] \\ &= D[U'_H(\eta), F(\eta,U_\eta,L_1U_\eta))] = 0. \end{aligned}$$

Hence, we have

$$D^+m(\eta) \le g_1(\eta, m(\eta)),\ t \ne t_k.$$

Also

$$m(t_k^+) \le \psi_k(m(t_k)),\ k = 1,2,...,$$
$$m(t_0) \le x_0.$$

Now, by Theorem 1.4.3 [15], it follows that $m(\eta) \le r(\eta, t_0, x_0),\ \eta \in [t_0, t]$, that is, $W(\eta, V(t,\eta,U(\eta))) \le r(\eta,t_0,x_0),\ \eta\in[t_0,t]$. Since $V(t,t,U(t)) = U(t)$, therefore we have

$$W(t, U(t,t_0,\Phi_0)) = W(t, V(t,t,U(t))) \le r(t,t_0,x_0).$$

This proves the assertion of the lemma.

3. Stability Analysis in Terms of Two Measures

Theorem 3.1. *Assume that*

($\mathbf{B_1}$) *The solution* $V(t) = V(t, t_0, \Phi_0) = V(t_0, \Phi_0)(t)$ *of (6) existing for all* $t \geq t_0$ *is unique, continuous with respect to the initial values, locally Lipschitzian in* Φ_0 *and* $V(t_0) = \Phi_0$.

($\mathbf{B_2}$) $K_i(t, s, 0) = 0$ *so that* $G(t, 0, 0) = G(t, 0) = 0,\ g_1(t, 0) = 0$ *and* $J_k(0) = 0, \psi_k(0) = 0,\ k = 1, 2, ...$;

($\mathbf{B_3}$) $h_0, h^*, h \in \Gamma$ *such that* h^* *is finer than* h *and* $h^*(t, U)$ *is nondecreasing in* t;

($\mathbf{B_4}$) $W \in W_0$ *be such that* $W(t, U)$ *is* $h-$*positive definite and weakly* h^*-*decrescent for* $(t, U) \in S(h, \rho)$, *and satisfies the inequality*

$$\begin{cases} D^+W(\eta, V(t, \eta, U)) \leq g_1(\eta, W(\eta, V(t, \eta, U))),\ \eta \neq t_k, \\ (\eta, U) \in S(h, \rho), \eta \in [t_0, t), \\ W(t_k^+, V(t, t_k^+, U(t_k^+))) \leq \psi_k(W(t_k, V(t, t_k, U(t_k)))),\ k = 1, 2, ...; \end{cases}$$

($\mathbf{B_5}$) *There exists a* $\rho_0 \in (0, \rho]$ *such that*

$$h(t_k, U(t_k)) < \rho_0 \textit{ implies that } h(t_k^+, U(t_k^+)) < \rho,\ k = 1, 2, ...$$

Then $(h_0, h^*)-$*stability of the system* (6) *and the asymptotical stability of the trivial solution of* (7) *imply the* $(\tilde{h}_0, h)-$ *asymptotical stability of* (5).

Proof. Let $U = U(t_0, \Phi_0)(t),\ V = V(t_0, \Phi_0)(t)$ and $x(t) = x(t, t_0, x_0)$ be any solutions of (5), (6) and (7) respectively. Since $W(t, U)$ is $h-$positive definite on $S(h, \rho)$, there exists $b \in \mathcal{K}$ such that

(8) $$h(t, U) < \rho \text{ implies } b(h(t, U)) \leq W(t, U).$$

Also $W(t, U)$ is weakly h^*-decrescent and h^* is finer than h, so there exists a $\lambda_0 > 0$ and $a \in PC\mathcal{K},\ \phi \in PC\mathcal{K}$ such that

(9) $$h(t, U) \leq \phi(t, h^*(t, U)) \text{ and } W(t, U) \leq a(t, h^*(t, U)),$$

when $h^*(t, U) < \lambda_0$ and $\phi(t_0^+, \lambda_0) < \rho$. Since the trivial solution of (7) is stable, therefore, for given $b(\epsilon) > 0$, we can find a $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that

(10) $$0 \leq x_0 < \delta_1 \text{ implies that } x(t, t_0, x_0) < b(\epsilon),\ t \geq t_0,$$

where $0 < \epsilon < \rho_0$ and $t_0 \in R_+$. Since the system (6) is $(h_0, h^*)-$stable, so there exists a $\delta_2 = \delta_2(t_0, \epsilon) > 0$ corresponding to δ_1 such that

(11) $$h_0(t_0^+, \Phi_0) < \delta_2 \text{ implies } h^*(t_0^+, V(t)) < a^{-1}(t_0, \delta_1),\ t \geq t_0.$$

Select $\delta = \delta(t_0, \epsilon) > 0$ satisfying $\delta < \min\{\lambda_0, \delta_2\}$. Now if $\tilde{h}_0(t_0^+, \Phi_0) < \delta$, then it follows from (8)-(11) that

$$b(h(t_0^+, \Phi_0)) \leq W(t_0^+, \Phi_0) \leq a(t_0^+, h^*(t_0^+, \Phi_0)) < a(t_0^+, \delta_2) \leq \delta_1 \leq b(\epsilon),$$

which implies that $h(t_0^+, \Phi_0)) < \epsilon$.

Now we claim that

$$h(t, U(t)) < \epsilon \text{ whenever } \tilde{h}_0(t_0^+, \Phi_0) < \delta. \tag{12}$$

For the sake of contradiction, let us assume that (12) is false and there exists $t^* > t_0$ such that $h(t^*, U(t^*)) \geq \epsilon$. For $h \in \Gamma$, there are two cases: (i) $t_0 < t^* \leq t_1$ (ii) $t_k < t^* \leq t_{k+1}$ for some $k = 1, 2, ...$

(i) Without loss of generality, let $t^* = \inf\{t : h(t, U(t)) \geq \epsilon\}$ and $h(t^*, U(t^*)) = \epsilon$. Using Lemma 2.1 and (8)-(9) together with the fact that $r(t, t_0, x_1) \leq r(t, t_0, x_2)$ for $x_1 \leq x_2$, we obtain

$$\begin{aligned} W(t^*, U(t^*)) &\leq r(t^*, t_0, W(t_0^+, V(t^*, t_0, \Phi_0))) \\ &\leq r(t^*, t_0, a(t_0, h^*(t_0^+, V(t^*, t_0, \Phi_0))) \\ &\leq r(t^*, t_0, \delta_1) < b(\epsilon). \end{aligned} \tag{13}$$

On the other hand, it follows from (8) that

$$W(t^*, U(t^*)) \geq b(h(t^*, U(t^*))) = b(\epsilon),$$

which contradicts (12).

(ii) In view of the impulse effect, we have

$$h(t^*, U(t^*)) \geq \epsilon \text{ and } h(t, U(t)) < \epsilon, \ t \in [t_0, t_k].$$

Since $0 < \epsilon < \rho_0$, it follows from assumption (B_5) that

$$h(t_k^+, U(t_k^+)) = h(t_k^+, U(t_k) + I_k(U(t_k)))) < \rho.$$

Consequently, there exists a $t^{**} \in (t_k, t^*]$ such that

$$\epsilon \leq h^*(t^{**}, U(t^{**})) < \rho \text{ and } h(t, U(t)) < \rho, \ t \in [t_0, t_1). \tag{14}$$

Now, by virtue of Lemma 2.1 and (8)-(9), we obtain

$$\begin{aligned} W(t^{**}, U(t^{**})) &\leq r(t^{**}, t_0, W(t_0^+, V(t^{**}, t_0, U_0))) \\ &\leq r(t^{**}, t_0, a(t_0, h(t_0^+, V(t^{**}, t_0, U_0)))) \\ &\leq r(t^{**}, t_0, \delta_1) < b(\epsilon), \end{aligned}$$

whereas (8) and (14) yields

$$W(t^{**}, U(t^{**})) \geq b(h(t^{**}, U(t^{**}))) \geq b(\epsilon),$$

which is again a contradiction. Thus $h(t, U(t)) < \epsilon$ whenever $\tilde{h}_0(t_0^+, \Phi_0) < \delta,\ t \geq t_0$. Hence the system (5) is $(\tilde{h}_0, h)-$ stable.

Next, it is assumed that the trivial solution of (7) is asymptotically stable. In view of $(\tilde{h}_0, h)-$ stability of the system (5), we set $\epsilon = \rho_0$ and $\delta = \delta_3 = \delta_3(t_0, \rho_0) > 0$ in (12) and obtain

$$h(t, U(t)) < \rho_0 < \rho \text{ whenever } \tilde{h}_0(t_0^+, \Phi_0)) < \delta_3,\ t \geq t_0.$$

In order to prove the $(\tilde{h}_0, h)-$ attractive of system (5), let the trivial solution of (7) be attractive, that is, for $t_0 \in R_+$, there exists a $\delta_0^* = \delta_0^*(t_0) > 0$ such that

$$x_0 < \delta_0^* \text{ implies } \lim_{t \to \infty} x(t, t_0, x_0) = 0.$$

Now, for this δ_0^*, there is a $\delta_1^* = \delta_1^*(t_0, \delta_0^*) > 0$ such that

$$\tilde{h}_0(t_0^+, \Phi_0) < \delta_1^* \text{ implies } h^*(t_0^+, V(t)) < a^{-1}(t_0, \delta_0^*).$$

Taking $\delta_0 = \delta_0(t_0)$ (independent of ϵ) such that $0 < \delta_0 < \min\{\delta^*, \delta_0^*, \delta_1^*\}$ and applying the earlier arguments, we find that

$$b(h(t, U(t))) \leq W(t, U(t)) \leq r(t, t_0, W(t_0^+, V(t, t_0, \Phi_0))) \leq r(t, t_0, \delta_0^*) \to 0,$$

as $t \to \infty$ when $\tilde{h}_0(t_0^+, \Phi_0)) < \delta_0$. This implies that $\lim_{t \to \infty} h(t, U(t)) = 0$ when $\tilde{h}_0(t_0^+, \Phi_0)) < \delta_0$, that is, the system (5) is $(\tilde{h}_0, h)-$ attractive. Hence system (5) is $(\tilde{h}_0, h)-$asymptotically stable.

Theorem 3.2. *Assume that all the assumptions of Theorem 3.1 hold except* (B_3) *and* (B_4) *which are modified as*

$(\mathbf{B_3^*})$ h^* *is uniformly finer than* h *instead of finer in* (B_3)*;*

$(\mathbf{B_4^*})$ W *is* h^*- *decrescent instead of weakly* h^*- *decrescent in* (B_4).

Then the $(h_0, h^*)-$*uniform stability of the trivial solution of (6) and the uniform asymptotical stability of the trivial solution of (7) imply the* $(\tilde{h}_0, h)-$*uniform asymptotical stability of (5).*

Proof. From (B_3^*) and (B_4^*), it follows that there exists a $\lambda_0 > 0$ and $a,\ \phi \in \mathcal{K}$ such that

(15) $$h(t, U) \leq \phi(h^*(t, U)) \text{ and } W(t, U) \leq a(h^*(t, U)),$$

when $h^*(t,U) < \lambda_0$ with $\phi(\lambda_0) < \rho$. The trivial solution of (7) is uniformly stable, therefore, for given $b(\epsilon) > 0$, we can find a $\delta_1 = \delta_1(\epsilon) > 0$ independent of t_0 such that

$$(16) \qquad 0 \leq x_0 < \delta_1 \text{ implies } x(t,t_0,x_0) < b(\epsilon),\ t \geq t_0,$$

where $0 < \epsilon < \rho_0$ and $t_0 \in R_+$. From the hypothesis that the trivial solution of (6) is $(h_0, h^*)-$uniformly stable, for the above δ_1, there exists a $\delta_2 = \delta_2(\epsilon) > 0$ independent of t_0 such that

$$(17) \qquad h_0(t_0^+, \Phi_0) < \delta_2 \text{ implies } h^*(t_0^+, V(t)) < a^{-1}(\delta_1).$$

Now, applying the arguments similar to the ones used in the proof of Theorem 2.1 and recalling that $U(t) = U(t_0, \Phi_0)(t)$ is any solution of (5), we conclude that

$$\tilde{h}_0(t_0^+, \Phi_0) < \delta \text{ implies } h(t, U(t)) < \epsilon,\ t \geq t_0,$$

where δ is independent of t_0 and satisfies $0 < \delta = \delta(\epsilon) < \min\{\lambda_0, \delta_2\}$. Thus, the system (5) is $(h_0, h)-$ uniformly stable.

Next, from the hypothesis that the trivial solution of (7) is uniformly asymptotically stable, we can find a $\delta_0^* > 0$ independent of t_0 and any ϵ satisfying $0 < \epsilon < \rho_0$ such that there exists a $\tau = \tau(\epsilon)$ so that

$$(18) \qquad 0 < x_0 < \delta_0^* \text{ implies } x(t,t_0,x_0) < b(\epsilon),\ t \geq t_0 + \tau(\epsilon),\ t_0 \in R_+.$$

In view of the fact that (6) is uniformly stable, there is a δ_1^* independent of t_0 corresponding to δ_0^* such that

$$h_0(t_0^+, \Phi_0) < \delta_1^* \text{ implies } h^*(t, V(t)) < a^{-1}(\delta_0^*),\ t \geq t_0.$$

Since uniform asymptotical stability of (7) implies its asymptotically stability, so system (5) is $(\tilde{h}_0, h)-$ uniformly stable. For $\epsilon = \rho_0$, there exists a $\delta^* = \delta^*(\rho_0)$ such that

$$\tilde{h}_0(t_0^+, \Phi_0) < \delta^* \text{ implies } h(t, U(t)) < \rho_0 < \rho,\ t \geq t_0.$$

Choosing δ_0 such that $0 < \delta_0 < \min\{\delta^*, \delta_0^*, \delta_1^*\}$ and using the arguments employed in Theorem 2.1, we find that $h(t, U(t))) \leq \epsilon,\ t \geq t_0 + \tau$, when $\tilde{h}_0(t_0^+, \Phi_0)) < \delta_0$, where δ_0 and τ are independent of t_0. This implies that the system (5) is $(\tilde{h}_0, h)-$ uniformly attractive. Hence the system (5) is $(\tilde{h}_0, h)-$uniformly asymptotically stable.

Remark. The $(h_0, h)-$ equatability of (5) can be established on the same pattern if we require $\delta = \delta(t_0, \epsilon)$ in Definition 2.8 to be a continuous function in t_0 for each ϵ.

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