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ON A NON-COMPACT GENERALIZATION OF FAN'S MINIMAX THEOREM

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Abstract. In this paper, we first introduce the weak convexlike condition which generalizes the convexlike concept due to Fan. Next, using the separation theorem for convex sets, we will prove a non-compact generalization of Fan's minimax theorem by relaxing the concavelike assumption to the weak concavelike condition. Also we give some examples which show that the convex and concave assumptions on Kneser's minimax theorem can not be relaxed with the quasi-convex and quasi-concave conditions simultaneously, and the previous minimax theorems can not be available.

1. INTRODUCTION

In 1928, von Neumann found his celebrated minimax theorem [15] and, in 1937, intersection lemma [16], which was intended to establish his minimax theorem and theorem on optimal balanced growth paths. Since then, several extensions of von Neumann's minimax theorem were established. Among them, in 1952, Kneser proved the following generalization of von Neumann's minimax theorem by weakening the compactness, linearity and continuity assumptions, and it has been very useful in many applications in convex analysis and the theory of games:

Theorem A. [10]. *Let X be a non-empty compact convex subset of a locally convex topological vector space E and Y be a non-empty convex subset of a a locally convex topological vector space F. Let the function $f : X \times Y \to \mathbb{R}$ with the properties*

(1) *for each $y \in Y$, the function $x \mapsto f(x, y)$ is upper semicontinuous and concave;*

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(2) *for each $x \in X$, the function $y \mapsto f(x,y)$ is convex.*

Then we have

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\max_{x\in X} f(x,y).$$

And it has been an interesting question whether the convex and concave assumptions on f can be further relaxed in Theorem A. On this question, in 1953, Fan proved the abstract minimax theorem using general convexity assumptions on f without assuming the linear structures on X and Y as follows:

Theorem B. [4]. *Let X be a non-empty compact topological space, Y a non-empty (discrete) set, and a function $f : X \times Y \to \mathbb{R}$ with the properties*

(1) *for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$, there exists an $x_0 \in X$ such that*

$$f(x_0,y) \geq \lambda f(x_1,y) + (1-\lambda) f(x_2,y) \quad \textit{for all } y \in Y;$$

(2) *for any $y_1, y_2 \in Y$ and $\lambda \in [0,1]$, there exists an $y_0 \in Y$ such that*

$$f(x,y_0) \leq \lambda f(x,y_1) + (1-\lambda) f(x,y_2) \quad \textit{for all } x \in X;$$

(3) *for each $y \in Y$, the function $x \mapsto f(x,y)$ is upper semicontinuous.*

Then we have

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\max_{x\in X} f(x,y).$$

When X and Y are non-empty convex sets, if $x \mapsto f(x,y)$ is concave for each $y \in Y$, and $y \mapsto f(x,y)$ is convex for each $x \in X$, then the assumptions (1) and (2) of Theorem B are clearly satisfied so that Theorem B is an abstract generalization of Theorem A. As is well-known, there have been numerous minimax theorems in abstract settings which generalize von Neumann's minimax theorem, e.g., see [1-9,11,12,14]. Nevertheless, Theorem B can be considered as the basic minimax theorem among numerous generalizations of Kneser's minimax theorem and Fan's minimax theorem. The proofs of Kneser type minimax theorems and Fan type minimax theorems often require arguments involving various kind of equivalent theorems, e.g., Brouwer's fixed point theorem, von Neumann's minimax theorem, the KKM theorem, a separation theorem, or Helly's theorem as shown in [1-12, 14, 15].

In this paper, we first introduce the weak convexlike condition which generalizes the convexlike concept due to Fan [4], and using this concept, we will give a non-compact generalization of Fan's minimax theorem with a Gordan type alternative theorem by applying the separation theorem for convex sets. Next we give some examples which shows that the convexity and concavity assumptions on Kneser's minimax theorem can not be relaxed with the quasi-convex and quasi-concave conditions, and also is suitable for our theorem.

2. Preliminaries

Now we recall some concepts which generalize the convexity as follows: Let X be a non-empty convex subset of a vector space E and let $f : X \to \mathbb{R}$. We say that f is *quasi-convex* if for each $t \in \mathbb{R}$, $\{x \in X \mid f(x) \leq t\}$ is convex; and that f is *quasi-concave* if $-f$ is quasi-convex. It is easy to see that if f is quasi-concave, then

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \min\{f(x_1), f(x_2)\},$$

holds for every $x_1, x_2 \in X$ and every $\lambda \in [0,1]$. It should be noted that if f and g are quasi-concave, then $f+g$ is not quasi-concave in general.

When X and Y are any non-empty sets without linear structures, recall that $f : X \times Y \to \mathbb{R}$ is *convexlike* [4] *on* X if for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$, there exists an $x_0 \in X$ such that

$$f(x_0, y) \leq \lambda f(x_1, y) + (1-\lambda) f(x_2, y) \quad \text{for all } y \in Y;$$

and that f is *concavelike* if $-f$ is convexlike.

It should be noted here that in the convexlike definition, the inequality

$$f(x_0, y) \leq \lambda f(x_1, y) + (1-\lambda) f(x_2, y)$$

must hold for all $y \in Y$. This means that $f(x_0, y)$ is always less than or equal to the values $\lambda f(x_1, y) + (1-\lambda) f(x_2, y)$ for every $y \in Y$. In some sense, this is a rather strong requirement for x_0 so that we relax this condition into a finite subset of Y as follows:

Definition 1. Let X and Y be any non-empty sets and $f : X \times Y \to \mathbb{R}$ be a real-valued function on $X \times Y$. Then f is called *weak convexlike on* X if for any $x_1, x_2 \in X$, $\lambda \in [0,1]$ and for any finite subset $\{y_1, \cdots, y_m\}$ of Y, there exists an $x_0 \in X$ such that

$$f(x_0, y) \leq \lambda f(x_1, y) + (1-\lambda) f(x_2, y) \quad \text{for all } y \in \{y_1, \cdots, y_m\};$$

and that f is *weak concavelike* on X if $-f$ is weak convexlike on X.

It is clear that the convexlike condition implies the weak convexlike condition but the converse may not be true. In the Definition 1, if Y is a finite set, then the weak convexlike condition is actually the same as the convexlike condition due to Fan.

Note that there is no implication between '$x \mapsto f(x, y)$ being quasi-convex for each $y \in Y$' and '$f(x, y)$ being convexlike on X' as follows:

Example 1. Let $X = Y = [-1, 1]$ be convex sets and the function $f : X \times Y \to \mathbb{R}$ is defined by

$$f(x, y) := x^2 y^3 \qquad \text{for each } (x, y) \in X \times Y.$$

Then, for each $y \in (0, 1]$, the function $x \mapsto f(x, y)$ is clearly convex but for each $y \in [-1, 0)$, the function $x \mapsto f(x, y)$ is not quasi-convex. Next, we shall show that the function $x \mapsto f(x, y)$ is convexlike on X. In fact, let $x_1, x_2 \in X$ and $\lambda \in (0, 1)$ be arbitrarily given. We want to show there exists an $x_0 \in X$ such that $f(x_0, y) \leq \lambda f(x_1, y) + (1 - \lambda) f(x_2, y)$ for all $y \in Y$, that is, $x_0^2 y^3 \leq \lambda\, x_1^2 y^3 + (1 - \lambda)\, x_2^2 y^3$ for all $y \in Y$ so that

$$0 \;\leq\; y^3 \big[\lambda\, x_1^2 + (1 - \lambda)\, x_2^2 - x_0^2\big] \qquad \text{for all } y \in [-1, 1].$$

Whenever $y \in (0, 1]$, we have $0 \leq \lambda\, x_1^2 + (1 - \lambda)\, x_2^2 - x_0^2$; and whenever $y \in [-1, 0)$, we have $0 \geq \lambda\, x_1^2 + (1 - \lambda)\, x_2^2 - x_0^2$, which can hold simultaneously only when $\lambda\, x_1^2 + (1 - \lambda)\, x_2^2 = x_0^2$. Therefore, we can find an $x_0 \in X$ satisfying the convexlike condition. Hence, $x \mapsto f(x, y)$ is convexlike on X.

On the other hand, as shown in [12], if we let

$$g(x, y) := \frac{-x^2}{(x - y)^2 + 1} \qquad \text{for each } (x, y) \in X \times Y,$$

then the function $y \mapsto g(x, y)$ is quasi-convex but not convexlike on Y. Therefore, there is no implication between the quasi-convexity and the (weak) convexlike condition.

Next, we shall need the following equivalent to the weak convexlike definition:

Lemma 1. *Let X and Y be any non-empty sets, and $f : X \times Y \to \mathbb{R}$ be a real-valued function on $X \times Y$. Then $f : X \times Y \to \mathbb{R}$ is weak convexlike on X, if and only if, for every $n \geq 2$, whenever $\{x_1, \cdots, x_n\} \subseteq X$ is given and for any finite subset $\{y_1, \cdots, y_m\}$ of Y and any $\lambda_i \in [0, 1], i = 1, \ldots, n,$ with $\sum_{i=1}^{n} \lambda_i = 1$, there exists a point $x_0 \in X$ such that*

$$(*) \quad f(x_0, y) \leq \lambda_1 f(x_1, y) + \cdots + \lambda_n f(x_n, y) \quad \text{for all } y \in \{y_1, \ldots, y_m\}.$$

Proof. The sufficiency is clear. For the necessity, we shall use the induction argument on n. When $n = 2$, the condition $(*)$ is exactly the same as the definition of convexlike condition. Assume that the condition $(*)$ holds for all $k \leq n-1$ $(n \geq 3)$. Let $\{x_1, \cdots, x_n\} \subseteq X$ and a finite subset $\{y_1, \cdots, y_m\}$ of Y be given, and $\lambda_i \in [0,1], i = 1, \ldots, n,$ with $\sum_{i=1}^{n} \lambda_i = 1$ be arbitrarily given. Without loss of generality, we may assume $\sum_{i=1}^{n-1} \lambda_i > 0$ by reindexing i. Denote

$$\mu_j := \frac{\lambda_j}{\sum_{i=1}^{n-1} \lambda_i} \quad \text{for all} \quad j = 1, \cdots, n-1;$$

then each $\mu_j \geq 0$ and $\sum_{j=1}^{n-1} \mu_j = 1$. Then, for the given sets $\{x_1, \cdots, x_{n-1}\}$ and $\{y_1, \cdots, y_m\}$, the induction assumption assures that there exists a point $\bar{x} \in X$ such that

$$f(\bar{x}, y) \leq \mu_1 f(x_1, y) + \cdots + \mu_{n-1} f(x_{n-1}, y) \quad \text{for all} \quad y \in \{y_1, \ldots, y_m\}.$$

Then, by the induction assumption again on two points $\bar{x}, x_n$ with $(\sum_{i=1}^{n-1} \lambda_i), \lambda_n$, and the given set $\{y_1, \cdots, y_m\}$, there exists a point $x_0 \in X$ such that

$$f(x_0, y) \leq \left(\sum_{i=1}^{n-1} \lambda_i \right) f(\bar{x}, y) + \lambda_n f(x_n, y) \quad \text{for all} \quad y \in \{y_1, \ldots, y_m\}.$$

Therefore, we finally have

$$\begin{aligned} f(x_0, y) &\leq \left(\sum_{i=1}^{n-1} \lambda_i \right) f(\bar{x}, y) + \lambda_n f(x_n, y) \\ &\leq \left(\sum_{i=1}^{n-1} \lambda_i \right) (\mu_1 f(x_1, y) + \cdots + \mu_{n-1} f(x_{n-1}, y)) + \lambda_n f(x_n, y) \\ &= \lambda_1 f(x_1, y) + \ldots + \lambda_n f(x_n, y), \end{aligned}$$

for all $y \in \{y_1, \ldots, y_m\}$. Therefore, by the induction, for every $n \geq 2$, we can obtain the desired conclusion. ■

3. A Non-compact Generalization of Fan's Minimax Theorem

Using the separation theorem for convex sets, we will prove a non-compact generalization of Fan's minimax theorem by relaxing the concavelike condition.

First, we begin with the following which is a Gordan type nonlinear alternative theorem in a non-compact setting:

Theorem 1. *Let X be a topological space, D a non-empty compact subset of X, and Y be a non-empty (discrete) set, and c be a given real number. Let $f : X \times Y \to \mathbb{R}$ be a function satisfying the following:*

(1) *for each* $y \in Y$*, the function* $x \mapsto f(x, y)$ *is upper semicontinuous and weak concavelike on* X*;*
(2) *for each* $x \in X$*, the function* $y \mapsto f(x, y)$ *is convexlike on* Y*;*
(3) *there exists* $y_o \in Y$ *such that* $f(x, y_o) < c$ *for all* $x \in X \setminus D$.

Then, either (A) or (B) holds:

(A) *there exists* $\bar{x} \in D$ *such that*

$$f(\bar{x}, y) \geq c \quad for \quad all \quad y \in Y;$$

(B) *there exists* $\bar{y} \in Y$ *such that*

$$f(x, \bar{y}) \leq c \quad for \quad all \quad x \in X.$$

Proof. Suppose (A) were false. Then for each $x \in D$, there exists $y \in Y$ such that $f(x, y) < c$. Since $x \mapsto f(x, y)$ is upper semicontinuous, for each $y \in Y$, the set

$$O_y := \{x \in X \mid f(x, y) < c\}$$

is open and we see that $D \subseteq \cup_{y \in Y} O_y$. Since D is compact, there exists a finite subset $\{y_1, \cdots, y_n\} \subset Y$ such that $D \subseteq \cup_{i=1}^n O_{y_i}$. By the assumption (3), $X \setminus D \subseteq O_{y_o}$ so that we have $X \subseteq \cup_{i=0}^n O_{y_i}$. Therefore, for each $x \in X$, there exists $j \in \{0, \cdots, n\}$ with $x \in O_{y_j}$. Hence, we have that for each $x \in X$,

$$\min_{0 \leq i \leq n} f(x, y_i) < c.$$

Here we note that y_o may coincide with one of $\{y_1, \cdots, y_n\}$. Now, we let

$$K_1 := co\{\big(f(x, y_0), \cdots, f(x, y_n)\big) \in \mathbb{R}^{n+1} \mid x \in X\};$$

$$K_2 := \{(z_0, \cdots, z_n) \in \mathbb{R}^{n+1} \mid z_i \leq c,\ i = 0, \cdots, n\}.$$

Then, it is clear that K_1 is a non-empty convex subset of $\mathbb{R}^{n+1}$ and K_2 is a non-empty closed convex subset of $\mathbb{R}^{n+1}$ with non-empty interior. Now we claim that $K_1 \cap K_2 = \emptyset$. Indeed, suppose that there exists $(z_0, \cdots, z_n) \in K_1 \cap K_2$. Then, there exist $\{x_1, \cdots, x_k\} \subset X$ and $\lambda_i \in (0, 1),\ i = 1, \ldots, k,$ with $\sum_{i=1}^k \lambda_i = 1$ such that

$$\begin{aligned}(z_0, \cdots, z_n) &= \sum_{j=1}^k \lambda_j \left(f(x_j, y_1), \cdots, f(x_j, y_n)\right) \\ &= \left(\sum_{j=1}^k \lambda_j f(x_j, y_1), \cdots, \sum_{j=1}^k \lambda_j f(x_j, y_n)\right).\end{aligned}$$

Since $x \mapsto f(x,y)$ is weak concavelike, for the given sets $\{x_1, \cdots, x_k\}$ and $\{y_1, \cdots, y_n\}$, and given $\lambda_j \in (0,1)$, $j = 1, \ldots, k$, with $\sum_{j=1}^k \lambda_j = 1$, there exists $x_0 \in X$ such that

$$\sum_{j=1}^{k} \lambda_j f(x_j, y) \leq f(x_0, y) \quad \text{for} \quad \text{all} \quad y \in \{y_1, \cdots, y_n\}.$$

Therefore, for each $i \in \{0, \ldots, n\}$, we have

$$c \ \leq \ z_i = \sum_{j=1}^{k} \lambda_j f(x_j, y_i) \ \leq \ f(x_0, y_i).$$

Since $x_0 \in O_{y_j}$ for some $j \in \{0, \cdots, n\}$, we must have $f(x_0, y_j) < c$ which is a contradiction. Therefore, $K_1 \cap K_2 = \emptyset$. By the separation theorem for convex sets (e.g., Theorem 3.4 in [13]), there exists $(u_0, \cdots, u_n) \in \mathbb{R}^{n+1} \setminus \{\mathbb{O}\}$ such that for all $x \in X$ and for all $(z_0, \cdots, z_n) \in K_2$,

$$\sum_{i=0}^{n} u_i \cdot f(x, y_i) \ \leq \ \sum_{i=0}^{n} u_i \cdot z_i.$$

If we let $z_i \to \infty$, we have $u_i \geq 0$ for each $i \in \{0, \cdots, n\}$. Therefore, by letting $u_i' := \frac{u_i}{\sum_{i=0}^n u_i}$, we may assume that $u_i \in [0,1]$, $i = 0, \ldots, n$, with $\sum_{i=0}^n u_i = 1$. If we choose $(z_0, \cdots, z_n) = (c, \cdots, c) \in K_2$, then we have $\sum_{i=0}^n u_i \cdot f(x, y_i) \leq c$ for all $x \in X$. By the assumption (2) of Theorem 1, the function $y \mapsto f(x,y)$ is convexlike on Y so that there exists a point $y_0 \in Y$ such that

$$f(x, y_0) \leq \sum_{i=0}^{n} u_i \cdot f(x, y_i) \leq c \quad \text{for all} \ \ x \in X,$$

which proves (B). ■

Remark. When X is a compact topological space in Theorem 1, the coercive condition (3) is automatically satisfied by letting $D = X$.

Next, using Theorem 1, we can prove a non-compact generalization of Fan's minimax theorem by relaxing the concavelike condition as follow:

Theorem 2. *Let X be a topological space, D a non-empty compact subset of X, and Y be a non-empty (discrete) set. Let $f : X \times Y \to \mathbb{R}$ be a function satisfying the following*

(1) *for each $y \in Y$, the function $x \mapsto f(x,y)$ is upper semicontinuous and weak concavelike on X;*

(2) *for each* $x \in X$, *the function* $y \mapsto f(x,y)$ *is convexlike on* Y*;*

(3) *the inequality* $\inf_{y\in Y}\sup_{x\in X\setminus D} f(x,y) < \inf_{y\in Y}\sup_{x\in X} f(x,y)$ *holds. Then we have*

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\sup_{x\in X} f(x,y).$$

Proof. If $\inf_{y\in Y}\sup_{x\in X} f(x,y) = -\infty$, there is nothing to prove since

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) \geq \inf_{y\in Y}\sup_{x\in X} f(x,y) = -\infty$$

is always holds.

For any constant c with $\inf_{y\in Y}\sup_{x\in X\setminus D} f(x,y) < c < \inf_{y\in Y}\sup_{x\in X} f(x,y)$, if (A) of Theorem 1 holds, then there exists $\bar{x} \in D$ such that

$$\phi(\bar{x}) := \inf_{y\in Y} f(\bar{x},y) \geq c;$$

hence we have

$$(\dagger) \qquad \sup_{x\in X}\phi(x) = \sup_{x\in X}\inf_{y\in Y} f(x,y) \geq c.$$

On the other hand, if (B) of Theorem 1 holds, then there exists $\bar{y} \in Y$ such that

$$\psi(\bar{y}) := \sup_{x\in X} f(x,\bar{y}) \leq c;$$

so that we have

$$\inf_{y\in Y}\psi(y) = \inf_{y\in Y}\sup_{x\in X} f(x,y) \leq c,$$

which can not be true since $c < \inf_{y\in Y}\sup_{x\in X} f(x,y)$.

If $\inf_{y\in Y}\sup_{x\in X} f(x,y) = \infty$, then for any $c \in \mathbb{R}$ with $\inf_{y\in Y}\sup_{x\in X\setminus D} f(x,y) < c$, from the inequality $(\dagger)$, we have $\sup_{x\in X}\inf_{y\in Y} f(x,y) \geq c$. Therefore, we have $\sup_{x\in X}\inf_{y\in Y} f(x,y) = \infty$, and hence we can obtain the conclusion.

Suppose that $\inf_{y\in Y}\sup_{x\in X} f(x,y) < \infty$. Then, for any $\varepsilon > 0$, if we take $c = \inf_{y\in Y}\sup_{x\in X} f(x,y) - \varepsilon$, from the inequality $(\dagger)$ again, we can obtain that

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) \geq \inf_{y\in Y}\sup_{x\in X} f(x,y) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) \geq \inf_{y\in Y}\sup_{x\in X} f(x,y);$$

and the reverse inequality is clear so that we can obtain the conclusion. ∎

When X is a compact topological space in Theorem 2, the coercive condition (3) is automatically satisfied since $\inf_{y\in Y}\sup_{x\in X\setminus D} f(x,y) = -\infty$ by letting $D = X$ so that we can obtain a generalization of Fan's minimax theorem as follow:

Theorem 3. *Let X be a compact topological space and Y be a non-empty (discrete) set. Let $f : X \times Y \to \mathbb{R}$ be a function satisfying the following:*

(1) *for each $y \in Y$, the function $x \mapsto f(x,y)$ is upper semicontinuous and weak concavelike on X;*

(2) *for each $x \in X$, the function $y \mapsto f(x,y)$ is convexlike on Y.*

Then we have

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\max_{x\in X} f(x,y).$$

We now give an example where Theorem A can not be generalized by relaxing the convex and concave assumptions with the quasi-convex and quasi-concave conditions as follow:

Example 2. Let $X := [0,1]$ and $Y := [0,1]$ be compact convex sets in $\mathbb{R}$, and the function $f : X \times Y \to \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} 1, & \text{if } (x,y) \in \{(0,1)\} \cup \{(1,y) \mid 0 \le y < 1\}; \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $y \in Y$, the function $x \mapsto f(x,y)$ is clearly upper semicontinuous and quasi-concave but not weak concavelike on X. Indeed, let $x_1 = 0,\ x_2 = 1, \lambda \in (0,1)$ be given. For a subset $\{\frac{1}{2}, 1\}$ of Y, x_0 in the weak concavelike definition must satisfy the following

$$\begin{aligned} f\left(x_0, \frac{1}{2}\right) &\geq \lambda f\left(0, \frac{1}{2}\right) + (1-\lambda) f\left(1, \frac{1}{2}\right) = (1-\lambda) > 0 \\ f(x_0, 1) &\geq \lambda f(0,1) + (1-\lambda) f(1,1) = \lambda > 0 \end{aligned}$$

so that we have $f(x_0, \frac{1}{2}) = f(x_0, 1) = 1$ which is a contradiction. Therefore, the function $x \mapsto f(x,y)$ is not concavelike nor weak concavelike. Similarly, we can see that for each $x \in X$, the function $y \mapsto f(x,y)$ is quasi-convex but not convexlike nor weak convexlike on Y. Therefore, Theorems A and B can not be applied for $f(x,y)$. Indeed, it is easy to see that

$$0 = \sup_{x\in X}\inf_{y\in Y} f(x,y) \lneqq \inf_{y\in Y}\max_{x\in X} f(x,y) = 1;$$

thus the convex and concave assumptions on Kneser's minimax theorem (Theorem A) can not be relaxed with the quasi-convex and quasi-concave conditions simultaneously.

Next, we will give an example where Theorem 2 can be applied but the previous minimax theorems due to von Neumann, Kneser are not available.

Example 3. Let $X := [0, 3)$ and $Y := (0, 3]$ be convex sets and the function $f : X \times Y \to \mathbb{R}$ be defined by

$$f(x, y) := \begin{cases} 1, & \text{if } x \leq y \leq 2, \ (x, y) \in [0, 2] \times Y; \\ -1, & \text{if } y = 3, \ (x, y) \in (2, 3) \times Y; \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $y \in Y$, it is easy to see that $x \mapsto f(x, y)$ is upper semicontinuous and quasi-concave and concavelike on X. Indeed, for any $x_1, x_2 \in X$ and each $\lambda \in [0, 1]$, there exists an $x_0 = 0 \in X$ such that

$$f(0, y) \geq \lambda f(x_1, y) + (1 - \lambda) f(x_2, y) \quad \text{for each } y \in Y$$

(here, $f(0, y)$ is 0 or 1). Therefore, $x \mapsto f(x, y)$ is concavelike on X so that weak concavelike on X. And, for each $x \in X$, $y \mapsto f(x, y)$ is convexlike but not quasi-convex on Y. Indeed, we can see that for any $y_1, y_2 \in Y$ and each $\lambda \in [0, 1]$, there exists an $y_0 = 3 \in Y$ such that

$$f(x, 3) \leq \lambda f(x, y_1) + (1 - \lambda) f(x, y_2) \quad \text{for each } x \in X$$

(here, $f(x, 3)$ is -1 or 0). Therefore, $y \mapsto f(x, y)$ is convexlike on X so that weak convexlike on X. And the set $\{y \in Y \mid f(\frac{1}{2}, y) \leq \frac{1}{2}\} = (0, \frac{1}{2}) \cup (2, 3]$ is not convex in Y so that $y \mapsto f(x, y)$ is not quasi-convex on Y.

For the compact set $D = [0, 2]$, since

$$-1 = \inf_{y \in Y} \sup_{x \in X \setminus D} f(x, y) < \inf_{y \in Y} \sup_{x \in X} f(x, y) = 0,$$

we know that the assumption (3) of Theorem 2 is satisfied for the compact set D. Therefore, all the hypotheses of Theorem 2 are satisfied so that we have

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y) = 0.$$

Note that since the domain of f is not compact and the map $y \mapsto f(x, y)$ is not quasi-convex on Y, the previous minimax theorems in [4,6,10,12,14,15] can not be applied for this function f.

From the Examples 2 and 3, we finally propose the following question

Question.

(i) In the assumption (2) of Theorem 2 (and Theorem B), can the convexlike assumption on "$y \mapsto f(x, y)$" be relaxed to the weak convexlike condition on Y?

(ii) In Theorem A, can the concave assumption (1) on "$x \mapsto f(x, y)$" be relaxed to the quasi-concave condition, and the convex assumption (2) on "$y \mapsto f(x, y)$" be relaxed to the (weak) convexlike condition?

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