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# A CHARACTERIZATION OF THE MULTI-CHOICE SHAPLEY VALUE WITH PARTIALLY CONSISTENT PROPERTY

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**Abstract.** Reducing both the number of players and the number of choices, we define a new reduced game for a multi-choice cooperative game with respect to a solution of the game and an action vector. Then, we characterize the multi-choice Shapley value by applying a partially consistent property of the multi-choice Shapley value.

## 1. Introduction

In real-life, a player might work diligently or work lazily in a coalition. But, a traditional cooperative game can not reflect the above truth. In order to remedy that weak point of the traditional cooperative games, in [3](1992) and [4](1993), Hsiao and Raghavan extended the traditional cooperative game to a multi-choice cooperative game and extended the traditional Shapley value to a multi-choice Shapley value. The multi-choice Shapley value is an extension of both the symmetric and the asymmetric Shapley values. It is symmetric among players and asymmetric among actions, or say choices. The multi-choice Shapley value is monotonic, dummy free of dummy player, dummy free of dummy action [4], transferable utility invariant and independent of non-essential player [6], redundant free [7]. Moreover, in this article, we will prove that it has consistent property that allows a player to reduce some part of his choices instead of all of his choices. Some authors call the multi-choice Shapley value define by Hsiao and Raghavan, the *H&R Shapley value*.

In 1989, Hart and Mas-Colell [2] were the first to introduce the potential approach to traditional TU games. In consequence, they proved that the traditional

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Shapley value [12] can result as the vector of marginal contributions of a potential. The potential approach is also shown to yield an elegant characterization for the Shapley value, particularly in terms of an internal consistency property.

Following Shapley's advice, Hsiao, Yeh and Mo[5](1994) got an explicit formula for the $w$-potential function of multi-choice games. They also found the relationship between the H&R Shapley value and the $w$-potential function. At the end of [5], with respect to a solution and an action vector, Hsiao defined a reduced game which may reduce both the number of players and the number of choices. However, with that reduced game, Hsiao was not able to extend Hart and Mas-Colell's [2] axiomatization of the traditional Shapley value to the multi-choice Shapley value.[2]

Reducing a multi-choice game with respect to its solution and a subset of players rather than an action, in [8] (2008 )[3] Hwang and Liao extended Hart and Mas-Colell's axiomatization to another multi-choice value named the **D&P Shapley value**. However, reducing a multi-choice game with respect to a subset of players is apparently a special case of reducing a multi-choice game with respect to an action vector in [5].

Fortunately, in this article, we finish more than what Hsiao tried to do at the end of [5], we will define a new reduced game which may really reduce both the number of players and the number of choices of a multi-choice game, with respect to its solution and an action vector. Then, we will define the partially consistent property to characterize the *H&R Shapley value*. In Appendix, we give the reasons why we study the multi-choices games.

## 2. Definitions and Notations

We believe that all the readers are familiar with the traditional mathematical symbols. Therefore, from cognitive viewpoint, in this article, we will use the traditional mathematical symbols and notations to modify the multi-choice game.

Let $U$ be the universal set of players. Without loss of generality, given a finite set of $n$ players $N \subset U$ where $N = \{1, 2, ..., n\}$, we have the following definitions and notations. Any subset $S \subset N$ is called a coalition. Other than what we did in [3, 4, 6], we now allow players to have different numbers of choices. We allow player $j$ to have $(m_j + 1)$ actions, say $\sigma_0$, $\sigma_1$, $\sigma_2$, ..., $\sigma_{m_j}$, where $\sigma_0$ is the action to do nothing, while $\sigma_k$ is the option to work at level $k$, which has higher level than $\sigma_{k-1}$. In this article, we assume that there are finitely many players with finitely many choices.

[2]Therefore, Hsiao leave the axiomatization as a conjecture and refused to send [5] for publication. Also, in 1996, Hsiao gave the problem in [5] to his student Liao [8], [9].

[3]As a matter of fact, the formula (3) in Theorem 1 in [8] (2008) is essentially the explicit formula of the potential in [5] (1994).

For convenience, we will use non-negative integers to denote the players' actions. Let $I_+$ denote the set of all finite non-negative integers. Let $\boldsymbol{\beta}_j = \{0, 1, \ldots, m_j\}$, with $m_j > 0$, be the action space of player $j$. Given $\mathbf{m} = (m_1,\ m_2,\ ...,m_n\ ) \in I_+^n$, with $m_j > 0$ for all $j$, the action space of $N$ is defined by $\Gamma(\mathbf{m}) = \prod_{j \in N} \boldsymbol{\beta}_j = \{(x_1, ..., x_n) \mid x_j \leq m_j$ and $x_j \in I_+$, for all $j \in N\}$. Thus $\mathbf{x} = (x_1, ..., x_n)$ is called an action vector of $N$, and $x_j = k$ if and only if player $j$ takes action $\sigma_k$.

**Definition 2.1.** A multi-choice cooperative game in characteristic function form is the pair $(\Gamma(\mathbf{m}), v)$ defined by $v : \Gamma(\mathbf{m}) \to R$, such that $v(\mathbf{0}) = 0$, where $\mathbf{0} = (0, 0, ..., 0)$.

**Definition 2.2.** Given $\mathbf{x} \in I_+^n$, we define $S(\mathbf{x})$=$\{j | x_j \neq 0\}$. Moreover, when $S(\mathbf{x}) = T$, we use the notation $\mathbf{x}^T$ to denote that $S(\mathbf{x}) = T$.

**Note 2.1.** (Identify the players). Even in a traditional cooperative game, a player has at least two choice say, to participate in a coalition or not to participate in a coalition. Therefore, with no risk of confusion, when $m_j = 0$, we will not regard $j$ as a "player" in the multi-choice cooperative game $(\Gamma(\mathbf{m}), v)$. In other words, we do not regard a player who has only one choice $\sigma_0$ - "doing nothing" in a game as a player. Therefore the number of players in $(\Gamma(\mathbf{m}), v)$ is the number of players whose highest action levels are greater than $\sigma_0$ - "doing nothing".

Sometimes, for short, we denote $(\Gamma(\mathbf{m}), v)$ by $(\mathbf{m}, v)$. Whenever we need to emphasis who are the players in $(\mathbf{m}, v)$, we will denote $(\mathbf{m}, v)$ by $(N, \mathbf{m}, v)$ if and only if $S(\mathbf{m}) = N$ ( This notation will be used in section 5). Let $S(\mathbf{m}) = \{k_1, k_2, ..., k_\ell\} = T$, since we do not regard $j$ as a player when $m_j = 0$, with no risk of confusion, when $\mathbf{m}^{\{k_1,k_2,...,k_\ell\}} = (0, \ldots, 0, m_{k_1}, 0, \ldots, 0, m_{k_2}, 0, \ldots, 0, \ldots, 0, m_{k_\ell}, 0, \ldots, 0) = \mathbf{m}^T$, then we define $(\mathbf{m}^{\{k_1,k_2,...,k_\ell\}}, v) \equiv ((0, \ldots, 0, m_{k_1}, 0, \ldots, 0, m_{k_2}, 0, \ldots, 0, \ldots, 0, m_{k_\ell}, 0, \ldots 0), v) \equiv (\mathbf{m}^T, v)$ which means that there are $\ell$ players, player $k_1$, player $k_2$,..., and player $k_\ell$ in the game.

For $(x_1, \ldots, x_n) \in \Gamma((0, \ldots, 0, m_{k_1}, 0, \ldots, 0, m_{k_2}, 0, \ldots, 0, \ldots, 0, m_{k_\ell}, 0, \ldots, 0) = \Gamma(\mathbf{m}^{\{k_1,k_2,...,k_\ell\}})$, we identify

$$(x_{k_1}, x_{k_2}, \ldots, x_{k_\ell}) \equiv (0, \ldots, 0, x_{k_1}, 0, \ldots, 0, x_{k_2}, 0, \ldots, 0, \ldots, 0, x_{k_\ell}, 0, \ldots, 0),$$

and if no confusion may arise we write

$$v((x_{k_1}, x_{k_2}, \ldots, x_{k_\ell})) = v((0, \ldots, 0, x_{k_1}, 0, \ldots, 0, x_{k_2}, 0, \ldots, 0, \ldots, 0, x_{k_\ell}, 0, \ldots, 0)).$$

When $m_j > 0$, for all $j \in N$, we can identify the set of all multi-choice cooperative games defined on $\Gamma(\mathbf{m})$ by $G(\mathbf{m}) \simeq R^{\Pi_{j=1}^n (m_j+1)-1}$.

We may consider $v(\mathbf{x})$ as the payoff or the cost for the players whenever the players take action vector $\mathbf{x}$. Since we allow players to have different numbers of actions, from time to time, we will denote $v(\mathbf{x})$ by $(\Gamma(\mathbf{m}), v)(\mathbf{x})$ or $(\mathbf{m}, v)(\mathbf{x})$ in order to identify the domain. To be consistent with traditional mathematical setup, we allow the existence of the game $(\Gamma(0), v)$ and call it the **null game**. Also, we call the game $(\Gamma(\mathbf{m}), v)$ with $v \equiv 0$ a **zero game**.

Given $\mathbf{z} = (z_1, z_2, ..., z_n)$, $\mathbf{m} = (m_1, m_2, ..., m_n) \in I_+^n$, we define $\mathbf{z} \leq \mathbf{m}$ if and only if $z_j \leq m_j$ for all $j \in N$. It is clear that $\Gamma(\mathbf{z}) \subseteq \Gamma(\mathbf{m})$ whenever $\mathbf{z} \leq \mathbf{m}$.

Given a $\mathbf{z} \in I_+^n$ such that $\mathbf{z} \leq \mathbf{m}$, we may obtain a sub-game of $(\Gamma(\mathbf{m}), v)$ by restricting the domain of $v$ to $\Gamma(\mathbf{z})$. We denote the sub-game by $(\Gamma(\mathbf{z}), v)$. In other words, let $\mathbf{z} \in I_+^n$ with $\mathbf{z} \leq \mathbf{m}$, we call $(\Gamma(\mathbf{z}), v)$ a sub-game of $(\Gamma(\mathbf{m}), v)$ if and only if $(\Gamma(\mathbf{z}), v)(\mathbf{x}) = (\Gamma(\mathbf{m}), v)(\mathbf{x})$ for all $\mathbf{x} \in \Gamma(\mathbf{z})$.

Let $G$ be the set of all multi-choice cooperative games with finitely many players and finitely many actions. A value, or say a solution, is a function $\psi$ defined on $G$ that assigns to each game $(\Gamma(\mathbf{m}), v)$ a $\sum_{j=1}^n m_j$ dimensional vector such that

$$\psi(v) = (\psi_{11}(v), ...\psi_{m_1 1}(v), \psi_{12}(v), ...\psi_{m_2 2}(v), ...\psi_{1n}(v), ...\psi_{m_n n}(v))$$

Sometimes, for a better understanding of the multi-choice value, we would write the solution in a matrix-type table as follows.

$$\begin{pmatrix} \psi_{11}(v) & \psi_{12}(v) & \cdots & \psi_{1n}(v) \\ \psi_{21}(v) & \psi_{22}(v) & \cdots & \psi_{2n}(v) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \psi_{m_n n}(v) \\ \psi_{m_1 1}(v) & \vdots & & \\ & \psi_{m_2 2}(v) & & \end{pmatrix}$$

Please note that, to be consistent with the traditional notation of a matrix, we use the row index $i$ to denote the action $\sigma_i$ and the column index $j$ to denote player $j$. Here $\psi_{ij}(v)$ is the power index or the value of player $j$ when he takes action $\sigma_i$ in game $v$. Also, please note that we always assign $\psi_{0,j}(v) = 0$ for the action $\sigma_0$-"doing nothing". When we need to emphasis the action space of $v$, we denote the value by $\psi_{ij}(\mathbf{m}, v)$.

From the point of view of a benefit-sharing problem or a cost-allocation problem, we regard solution $\psi$ as a **value**, and from the point of view of a multi-choice voting game where players have more than two choices, we regard solution $\psi$ as a **power index**. It is interesting to search for "good" solutions for the multi-choice games, the idea of the following Remark came from Hart and Mas-Colell [2].

**Remark 2.1.** (The motivation of this article). Let $N = \{1, 2, 3\}$, $\mathbf{m} = (11, 7, 9)$, suppose after a solution $\psi$ of the game $(\mathbf{m}, v)$ is given, the following situation happens.

(i) Player 1 is satisfied with all his value $\psi_{i,1}(v)$ for $i = 1, \ldots, 11$. Let $z_1 = 0$.

(ii) Player 2 doubts that his value $\psi_{i,2}(v)$ is not fairly calculated for $i = 1, \ldots 5$, but he is satisfied with $\psi_{6,2}(v)$ and $\psi_{7,2}(v)$. Let $z_2 = 5$.

(iii) Player 3 is satisfied with none of his values, he doubts that none of $\psi_{i,3}(v)$ is fairly calculated for $i = 1, \ldots 9$. Let $z_3 = 9 = m_3$.

Let $\mathbf{z} = (0, 5, 9)$, allow players to reconsider, or say revise their values by a reduced game $(\mathbf{z}, v^{\psi}_{\mathbf{z}})$. It is crucial for a "good" solution to sweep out the players' dissatisfaction by satisfying (a) and (b) as follow.

$(a)$ For player 2, $\psi_{i,2}(\mathbf{m}, v) = \psi_{i,2}(\mathbf{z}, v^{\psi}_{\mathbf{z}})$, for $i = 1, \ldots, 5 = z_2$

$(b)$ For player 3 , $\psi_{i,3}(\mathbf{m}, v) = \psi_{i,3}(\mathbf{z}, v^{\psi}_{\mathbf{z}})$, for $i = 1, \ldots, 9 = z_3$.

Since $z_1 = 0$, player 1 is not a player in the reduced game $(\mathbf{z}, v^{\psi}_{\mathbf{z}})$. Next, since $z_2 = 5$, actions $\sigma_6$ and $\sigma_7$ are not choices for player 2 in the reduced game. Finally, since $z_3 = 9 = m_3$, player 3 has the same number of choices in the reduced $(\mathbf{z}, v^{\psi}_{\mathbf{z}})$ as he does in $(\mathbf{m}, v)$. Therefore, the reduced game $(\mathbf{z}, v^{\psi}_{\mathbf{z}})$ reduces both the players and the choices.

In this article, we will give $(\mathbf{z}, v^{\psi}_{\mathbf{z}})$ a suitable new definition in Section 4, and show that the *H&R Shapley value* is consistent.

We now consider the *H&R Shapley value* for a game where different players may have different number of choices in the game. The H&R Shapley value is the unique solution for multi-choice games which satisfies four axioms that are analogous to the axioms of the traditional Shapley value. Please see [4] for details.

Since we do not assume that the difference between $\sigma_{i-1}$ and $\sigma_i$ is the same as the difference between $\sigma_i$ and $\sigma_{i+1}$, etc., giving weights (discriminations) to actions is necessary. Furthermore, the fact that different players with the same action might have different contribution to a coalition should be reflected in the characteristic function rather than the weights. Therefore, we give weights to the actions instead of the players.

Let $w : I_+ \rightarrow R_+$ be a non-negative function such that $w(0) = 0$, $w(0) < w(1) \leq w(2) \leq \ldots$, then $w$ is called a **weight function** and $w(i)$ is said to be the **weight** of $\sigma_i$.

We need the following definitions and notations from [4]. Given $\mathbf{x}, \mathbf{y} \in \Gamma(\mathbf{m})$, we define $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, ...., x_n \vee y_n)$ where $x_j \vee y_j = \max\{x_j, y_j\}$ for each $j$. Similarly, we define $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, ...., x_n \wedge y_n)$ where $x_j \wedge y_j = \min\{x_j, y_j\}$ for each $j$. Also, we define $\mathbf{x} \leq \mathbf{y}$ if and only if $x_j \leq y_j$ for each $j$.

**Definition 2.3.** A vector $\tilde{\mathbf{x}} \in \Gamma(\mathbf{m})$ is called a **carrier** of $v$, if $v(\tilde{\mathbf{x}} \wedge \mathbf{x}) = v(\mathbf{x})$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$. We call $\mathbf{x}^0$ the *minimal carrier* of $v$ if $\sum x_j^0 = \min\{\sum x_j \mid \mathbf{x}$ is a carrier of $v\}$.
Please note that the minimal carrier is unique.
We denote $(\mathbf{x} \mid x_j = k)$ as an action vector with $x_j = k$.

**Definition 2.4.** Player $j$ is said to be a **dummy player** if $v((\mathbf{x} \mid x_j = k)) = v((\mathbf{x} \mid x_j = 0))$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$ and for all $k = 0, 1, 2, ..., m_j$.

Many solutions of the traditional cooperative games, e.g. cores and the Shapley value, require efficiency. Here is a similar definition.

**Definition 2.5.** A solution $\psi$ is said to be efficient for $(\mathbf{m}, v)$ if

$$\sum_{j \in N} \psi_{m_j, j}(v) = v(\mathbf{m}).$$

**Definition 2.6.** Given a set of players $N$ and a coalition $S \subseteq N$, let $\mathbf{e}^S$ be the binary vector with components $e_j^S$ satisfying

$$e_j^S = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise.} \end{cases}$$

For brevity, we denote the standard unit vectors $\mathbf{e}^{\{j\}} = \mathbf{e}_j$, for all $j \in N$, and let $|S|$ be the number of elements of $S$.

**Definition 2.7.** Given $\Gamma(\mathbf{m})$ and $w(0) = 0$, $w(1)$,..., for any $\mathbf{x} \in \Gamma(\mathbf{m})$, we define

$$\|\mathbf{x}\|_w = \sum_{r=1}^{n} w(x_r)$$

.

**Definition 2.8** Given $\mathbf{x} \in \Gamma(\mathbf{m})$ and $j \in N = \{1, 2, \ldots, n\}$, we define

$$M_j(\mathbf{x}; \mathbf{m}) = \{\ell \mid x_\ell \neq m_\ell, \ell \neq j\}.$$

Given a weight function, rewrite Theorem 2 in [4], we can slightly extend the H&R Shapley value as follows.

$$(\star) \qquad \phi_{ij}^w(v) = \sum_{k=1}^{i} \sum_{\substack{x_j = k \\ \mathbf{x} \neq \mathbf{0} \\ \mathbf{x} \in \Gamma(\mathbf{m})}} \left[ \sum_{T \subseteq M_j(\mathbf{x};\mathbf{m})} (-1)^{|T|} \frac{w(x_j)}{\|\mathbf{x}\|_w + \sum_{r \in T} [w(x_r + 1) - w(x_r)]} \right] \times [v(\mathbf{x}) - v(\mathbf{x} - \mathbf{e}_j)].$$

**Remark 2.2.** For the traditional asymmetric Shapley value, Shapley gives weights (discriminations) to the players. For the H&R Shapley value, we do not give weights( discriminations) to the players. However, as we allow players to have more than two choices, we should expect some differences due to actions. We use a weight function $w$ to modify the differences due to actions.

It is well-known that the traditional Shapley value has applications in many fields such as economics, political sciences, accounting, and even military sciences. Of course, our extended Shapley value also has the same applications as the traditional Shapley value does.

However, the weight function $w$ has different meanings in different fields. In military sciences, we may treat $w(i)$s' as parameters to modify the differences due to different levels of military actions. As a matter of fact, we may treat the parameters $w(i)$s' as prior power indices of the choices while simulating (pre-playing) the multi-choice game. It is an essential assumption that players are allowed to pre-play the game. Nowadays, computer simulation makes the assumption possible and the weight $w(j)$s' become acceptable in studying a multi-choice game. As a matter of fact, in the real world, we use the *H&R Shapley value* to evaluate the power indices of actions taken by a *disease control agent* for each pandemic alert level.

To make this article self-contained, we copy some definitions from [7] as follows.

**Definition 2.9.** Given a game $(\mathbf{m}, v)$, the action $\sigma_{m_j}$ is said to be a **redundant action** for player $j$ if $v((\mathbf{x} \mid x_j = m_j)) = v((\mathbf{x} \mid x_j = m_j - 1))$ for all $\mathbf{x} \in \Gamma(\mathbf{m})$.

Given a solution $\psi$ for $(\Gamma(\mathbf{m}), v)$, suppose we allow player $j$ to have one more action which is redundant for player $j$, say $\sigma_{m_j+1}$,

Let $\hat{\mathbf{m}} = (m_1, m_2, \cdots, m_{j-1}, (m_j+1), m_{j+1}, \cdots, m_n)$, then we have a new action vector space $\Gamma(\hat{\mathbf{m}}) = \{(x_1, \cdots, x_\ell, \cdots, x_n) \mid x_\ell \leq m_\ell, x_\ell \in I_+ \text{ for all } \ell \neq j, \text{ and } x_j = 0, 1, 2, \cdots, m_j + 1$. We may extend $(\Gamma(\mathbf{m}), v)$ to $(\Gamma(\hat{\mathbf{m}}), v^R)$ such that $v^R(\mathbf{x}) = v(\mathbf{x})$, for all $\mathbf{x} \in \Gamma(\mathbf{m})$ and $v^R((\mathbf{x} \mid x_j = m_j + 1)) = v((\mathbf{x} \mid x_j = m_j))$, for all $\mathbf{x} \in \Gamma(\hat{\mathbf{m}})$. The solution $\psi$ is said to be **redundant free** if and only if $\psi_{i,\ell}(v^R) = \psi_{i,\ell}(v)$ for all $\ell \in N$, and $i = 1, 2, ..., m_\ell$, and $\psi_{(m_j+1),j}(v^R) = \psi_{m_j,j}(v)$.

Please note that the definition of *redundant free* in this article is different from the definition of *dummy free of action* and the definition of *dummy free of player* in [3].

In [7], we proved

**Theorem 2.0.** *The H&R Shapley value ($\star$) is redundant free.*

Therefore, from the *H&R Shapley value's* viewpoint, it makes no difference whether the players have the same number of actions or not.

## 3. The Potential for Multi-choice Games

Following [5], we have definitions and notations as follow. Let $G$ be the set of all multi-choice cooperative games where there are finitely many players having finitely many choices. Given a weight function $w$, we define a function $P_w : G \to R$ which associates each game $(\mathbf{x}, v) \in G$ a real number $P_w(\mathbf{x}, v)$.

Given $P_w(\mathbf{x}, v)$, we define the following operators.

$$D_{i,j}P_w(\mathbf{x}, v) = w(i) \cdot \Big[P_w(\ (\mathbf{x}|x_j = i), v) - P_w(\ (\mathbf{x}|x_j = i-1), v)\Big],$$

and

$$H_{x_j,j} = \sum_{r=1}^{r=x_j} D_{r,j}.$$

**Definition 3.1.** ([5]). Given a weight function $w$, a function $P_w : G \to R$ with $P_w(\mathbf{0}, v) = 0$ is called a $w$-potential function if it satisfies the following condition: Given$(\mathbf{m}, v) \in G$

$$(\star\star) \qquad \sum_{j\in S(\mathbf{m})} H_{m_j,j}P_w(\mathbf{m}, v) = (\mathbf{m}, v)(\mathbf{m})$$

We now prove the following lemma which will be used in Theorems 4.1.

**Lemma 3.1.** *Given a weight function* $w$. *Suppose there are two real-valued functions* $\bar{P}_w, P_w : G \to R$ *with* $\bar{P}_w(\ \mathbf{0}, v) = c$ *and* $P_w(\ \mathbf{0}, v) = 0$ *satisfying the following: Given* $(\mathbf{m}, v) \in G$

$$(3.1) \qquad \sum_{j\in S(\mathbf{m})} H_{m_j,j}\bar{P}_w(\mathbf{m}, v) = \sum_{j\in S(\mathbf{m})} H_{m_j,j}P_w(\mathbf{m}, v)$$

Then

$$(3.2) \qquad \bar{P}_w(\mathbf{m}, v) = P_w(\mathbf{m}, v) + c$$

*Proof.* Let $|\mathbf{m}| = \sum_{j\in S(\mathbf{m})} m_j$. We shall claim that if (3.1) holds for $(\mathbf{m}, v)$ then $\bar{P}_w(\mathbf{m}, v)$= $P_w(\mathbf{m}, v) + c$ by mathematical induction on $|\mathbf{m}|$.

For $|\mathbf{m}| = 1$, we have $S(\mathbf{m}) = \{j_0\}$ for some $j_0$ and $m_{j_0} = 1$. Then by equation (3.1)

$$w(m_{j_0})[\bar{P}_w(\mathbf{m}, v) - \bar{P}_w(\mathbf{0}, v)] = w(m_{j_0})[P_w(\mathbf{m}, v) - P_w(\mathbf{0}, v)].$$

By the assumptions $\bar{P}_w(\mathbf{0}, v) = c$ and $P_w(\mathbf{0}, v) = 0$, (3.2) holds for $|\mathbf{m}| = 1$.

Suppose our claim holds for all $\mathbf{m}$ with $|\mathbf{m}| \le k-1$. If $\mathbf{m}$ has $|\mathbf{m}| = k$, then $S(\mathbf{m}) \neq \emptyset$. Now by equation (3.1) we have

$$\begin{aligned}\sum_{j\in S(\mathbf{m})}\sum_{r=1}^{r=m_j} w(r)\cdot\Big[\bar{P}_w(\,(\mathbf{m}|m_j{=}r),v)-\bar{P}_w(\,(\mathbf{m}|m_j{=}r-1),v)\Big]\\ =\sum_{j\in S(\mathbf{m})}\sum_{r=1}^{r=m_j} w(r)\cdot\Big[P_w(\,(\mathbf{m}|m_j{=}r),v)-P_w(\,(\mathbf{m}|m_j{=}r-1),v)\Big]\end{aligned} \tag{3.3}$$

Next for each $j \in S(\mathbf{m})$ and for all $s$ with $0 \le s \le m_j - 1$, since $|(\mathbf{m}|m_j = s)| \le k-1$, by the hypothesis of mathematical induction, we have
$\bar{P}_w(\,\mathbf{m}|m_j = s),v) = P_w(\,(\mathbf{m}|m_j = s),v) + c$.
In equation (3.3), we replace $\bar{P}_w(\,(\mathbf{m}|m_j = s),v)$ by $P_w(\,(\mathbf{m}|m_j = s),v)+c$, for each $j \in S(\mathbf{m})$ and for all $s$ with $0 \le s \le m_j - 1$, then equation (3.3) is reduced to be

$$\begin{aligned}&\sum_{j\in S(\mathbf{m})} w(m_j)\cdot\Big[\bar{P}_w(\mathbf{m},v)-\Big(P_w(\,(\mathbf{m}-\mathbf{e}_j),v)+c\Big)\Big]\\ &=\sum_{j\in S(\mathbf{m})} w(m_j)\cdot\Big[P_w(\mathbf{m},v)-P_w(\,(\mathbf{m}-\mathbf{e}_j),v)\Big]\\ &\Rightarrow\Big[\sum_{j\in S(\mathbf{m})} w(m_j)\Big]\cdot\Big(\bar{P}_w(\mathbf{m},v)-c\Big)=\Big[\sum_{j\in S(\mathbf{m})} w(m_j)\Big]\cdot P_w(\mathbf{m},v)\\ &\Rightarrow\bar{P}_w(\mathbf{m},v)=P_w(\mathbf{m},v)+c.\end{aligned}$$

Hence, our claim holds for $|\mathbf{m}| = k$. By mathematical induction, the proof is completed. ■

Here, we copy Theorem 3.1 and Theorem 3.2 from [5]. Given $j \in N$ and $v(\mathbf{x})$, we define

$$d_j v(\mathbf{x}) = v(\mathbf{x}) - v(\mathbf{x}-\mathbf{e}_j),$$

then $d_j$ is associative, i.e., $d_j(d_\ell v(\mathbf{x})) = d_\ell(d_j v(\mathbf{x}))$. For convenience, we denote $d_j d_\ell = d_{j\ell}$, $d_{j_1,j_2,j_3} = d_{j_1}d_{j_2}d_{j_3}$, ..., etc. We also denote $d_{j_1,j_2,\ldots,j_\ell} = d_T$ whenever $\{j_1, j_2, \ldots, j_\ell\} = T$. Furthermore, for brevity, we denote $d_{S(\mathbf{x})}$by $d_{\mathbf{x}}$.

**Theorem 3.1.** ([5]) *The $w$-potential of multi-choice cooperative games is unique, and*

$$P_w(\mathbf{x},v) = \sum_{\substack{\mathbf{y}\le\mathbf{x}\\ \mathbf{y}\neq\mathbf{0}}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}}(\mathbf{x},v)(\mathbf{y}) \tag{3.4}$$

**Theorem 3.2.** ([5]) *Given a multi-choice cooperative game* $(\mathbf{m}, v)$ *then the H&R Shapley value and the* $w$*-potential of* $(\mathbf{m}, v)$ *have the following relationship.*

$$\phi_{ij}^{w}(\mathbf{m}, v) = H_{ij}P_w(\mathbf{m}, v). \tag{3.5}$$

In the proof of Theorem 1 in [6], the first formula in page 428 in [6], we see a reformulation of the H&R Shapley value as follows. To make this article-self contained, we give a simple proof as follows.

**Proposition 3.3.** *The H&R Shapley value (⋆) can be reformulated as follows.*

$$H_{ij}P_w(\mathbf{m}, v) = \sum_{k=1}^{i} w(k) \cdot \left[ \sum_{\substack{y_j=k \\ \mathbf{y}\in\Gamma(\mathbf{x})}} \frac{1}{||\mathbf{y}||_w} \sum_{T\subseteq S(\mathbf{y})} (-1)^{|T|} v(\mathbf{y} - \sum_{r\in T} \mathbf{e}_r) \right] \tag{3.6}$$

*Proof.* By Theorems 3.1 and 3.2,

$$\begin{aligned}
&H_{i,j}\Bigg( \sum_{\substack{\mathbf{y}\leq\mathbf{x} \\ \mathbf{y}\neq\mathbf{0}}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}}(\mathbf{x}, v)(\mathbf{y}) \Bigg) \\
&= \sum_{k=1}^{i} w(k) \cdot \Bigg[ \sum_{\substack{\mathbf{y}\leq(\mathbf{x}|x_j=k) \\ \mathbf{y}\neq\mathbf{0}}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}} v(\mathbf{y}) - \sum_{\substack{\mathbf{y}\leq(\mathbf{x}|x_j=k-1) \\ \mathbf{y}\neq\mathbf{0}}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}} v(\mathbf{y}) \Bigg] \\
&= \sum_{k=1}^{i} w(k) \cdot \Bigg[ \sum_{\substack{y_j=k \\ \mathbf{y}\in\Gamma(\mathbf{x})}} \frac{1}{||\mathbf{y}||_w} d_{\mathbf{y}} v(\mathbf{y}) \Bigg] \\
&= \sum_{k=1}^{i} w(k) \cdot \Bigg[ \sum_{\substack{y_j=k \\ \mathbf{y}\in\Gamma(\mathbf{x})}} \frac{1}{||\mathbf{y}||_w} \sum_{T\subseteq S(\mathbf{y})} (-1)^{|T|} v(\mathbf{y} - \sum_{r\in T} \mathbf{e}_r) \Bigg]
\end{aligned}$$

■

**Remark 3.1.** Taking the following part of (3.6) as a value for player $j$ with action $\sigma_k$, we get the so called weighted consistent value in [9].

$$w(k) \cdot \Bigg[ \sum_{\substack{y_j=k \\ \mathbf{y}\in\Gamma(\mathbf{x})}} \frac{1}{||\mathbf{y}||_w} \sum_{T\subseteq S(\mathbf{y})} (-1)^{|T|} v(\mathbf{y} - \sum_{r\in T} \mathbf{e}_r) \Bigg].$$

## 4. Consistency Property of the Multi-choice Shapley Value

We now start to prove the main results of this article. Given a multi-choice cooperative game $(\mathbf{m}, v)$ and its solution,

$$(\psi^w_{11}(v), .., \psi^w_{m_1 1}(v), \psi^w_{12}(v), .., \psi^w_{m_2 2}(v), .., \psi^w_{1n}(v), .., \psi^w_{m_n n}(v))$$

for each $\mathbf{z} \in \Gamma(\mathbf{m})$, we define an action vector $\mathbf{z}^* = \mathbf{z}^*(\mathbf{m}) = (z_1^*, z_2^*, ..., z_n^*)$ where

$$\begin{cases} z_j^* = m_j \text{ if } z_j < m_j \\ z_j^* = 0 \ \text{ if } z_j = m_j. \end{cases}$$

In [5], Hsiao suggested the following reduced game which reduces both the number of players and the number of choices.

**Definition 4.0.** Let $\psi$ be a solution defined on G. Given $(\mathbf{m}, v) \in G$ and $\mathbf{z} \leq \mathbf{m}$, the reduced game $v^\psi_{\mathbf{z}}$ of $(\mathbf{m}, v)$ with respect to $\mathbf{z}$ and the solution $\psi$ is defined by

$$v^\psi_{\mathbf{z}}(\mathbf{y}) = v(\mathbf{y} \vee \mathbf{z}^*) - \sum_{z_j^* \neq 0} \Big[\psi_{m_j,j}((\mathbf{y} \vee \mathbf{z}^*), v)\Big],$$

where $\mathbf{y} \leq \mathbf{z}$.

However, in the above Definition, if $0 < z_j < m_j$, then player $j$ is a **dummy player** in $(\mathbf{z}, v^\psi_{\mathbf{z}})$. This oversight kept Hsiao from completing what he tried to do at the end of [5]. We suggest authors who cited the above Definition in their paper should revise their results carefully.

Observing the *H&R Shapley value* in the matrix-type form, we get the following new definition of a reduced game.

**Definition 4.1.** Let $\psi$ be a solution defined on G. Given $(\mathbf{m}, v) \in G$ and $\mathbf{z} \leq \mathbf{m}$, the reduced function $v^\psi_{\mathbf{z}}$ of $(\mathbf{m}, v)$ with respect to $\mathbf{z}$ and the solution $\psi$ is defined by

$$(4.1.1) \qquad v^\psi_{\mathbf{z}}(\mathbf{y}) = v(\mathbf{y} \vee \mathbf{z}^*) - \sum_{z_j^* \neq 0} \Big[\psi_{m_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) - \psi_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*), v)\Big],$$

where $\mathbf{y} \leq \mathbf{z}$. Furthermore, if $v^\psi_{\mathbf{z}}$ satisfies

$$v^\psi_{\mathbf{z}}(\mathbf{0}) = v(\mathbf{z}^*) - \sum_{z_j^* \neq 0} \psi_{m_j,j}(\mathbf{z}^*, v) = 0$$

then we call $v^\psi_{\mathbf{z}}$ a reduced game.

Please note that if $z_j = 0$ then $z_j^* = m_j$ and $y_j \leq z_j = 0$, hence $[\psi_{m_j,j}((\mathbf{y} \vee \mathbf{z}^*, v) - \psi_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*, v)]$ =$[\psi_{m_j,j}((\mathbf{y} \vee \mathbf{z}^*, v) - 0]$ Furthermore, we allow $\mathbf{z} = \mathbf{m}$, i.e., we allow

$$(\mathbf{m}, v^{\psi}_{\mathbf{m}})(\mathbf{y}) = v(\mathbf{y} \vee \mathbf{0}) - 0 = v(\mathbf{y}),$$

for all $\mathbf{y} \in \Gamma(\mathbf{m})$. Therefore, we identify $(\mathbf{m}, v^{\psi}_{\mathbf{m}}) \equiv (\mathbf{m}, v)$.

Now, we extend the consistency defined in [2](1989) as follows.

**Definition 4.2.** $\psi$ is said to be *partially consistent* if the following holds: Let $(\mathbf{m}, v)$ be a game. Whenever a reduced function $v^{\psi}_{\mathbf{z}}$ is a reduced game, we have that for $j \in S(\mathbf{z})$

$$\psi_{i,j}(\mathbf{z}, v^{\psi}_{\mathbf{z}}) = \psi_{i,j}(\mathbf{m}, v) \text{ for each } i \leq z_j \tag{4.2}$$

If every reduced function $v^{\psi}_{\mathbf{z}}$ is a reduced game for every game $(\mathbf{m}, v)$ and every action vector $\mathbf{z}$, and (4.2) holds for all $i \leq z_j$ and $j \in N$, then $\psi$ is said to be *consistent*.

**Note 4.1.** There are three ways to define a reduced game in order to characterize the *H&R Shapley value*.

(i) First, we may assume that $\psi$ is efficient, this will make the characterization much less desirable.

(ii) Secondly, we may define the reduced game as follows.

$$v^{\psi}_{\mathbf{z}}(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{y}=\mathbf{0} \\ v(\mathbf{y}\vee\mathbf{z}^*) - \sum_{z^*_j \neq 0} \left[\psi_{m_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) - \psi_{y_j,j}((\mathbf{y}\vee\mathbf{z}^*), v)\right] & \text{if } \mathbf{0}\neq\mathbf{y}\leq\mathbf{z}. \end{cases} \tag{4.1.2}$$

But, then we are actually adding at least $[\Pi^{n}_{j=1} m_j] - 1$ additional assumptions (equations) indexed by $\mathbf{z}$, say $v^{\psi}_{\mathbf{z}}(\mathbf{0}) = 0$ for each $\mathbf{z} \leq \mathbf{m}$ with $\mathbf{z} \neq \mathbf{0}$, to define the "consistent property" (4.2). This way is not much less than assuming the efficiency of $\psi$.

(iii) We choose the third way to characterize the *H&R Shapley value*, we will use partial consistency and *simple Pareto optimal* property which needs only one natural simple equation to characterize that value.

**Remark 4.1.** Without loss of generality, let $m_1 \geq 2$, $\mathbf{m} = (m_1, m_2, m_3)$, $\tilde{\mathbf{m}} = (m_1 - 1, m_2, m_3)$ and let $(\tilde{\mathbf{m}}, v)$ be a sub-game of $(\mathbf{m}, v)$ such that $\sigma_{m_1}$ is a redundant action, i.e., $v((m_1, m_2, m_3))$=$v((m_1 - 1, m_2, m_3))$. Suppose $\psi$ is efficient then it is easy to see that any reduced function with respect to $\psi$ is a reduced game. Take $\mathbf{z} = (m_1 - 1, 0, 0)$, we have:

**Case 1.** For the reduced game $v_{\mathbf{z}}^{\psi}$ of $(\mathbf{m}, v)$ with respect to $\mathbf{z}$ and $\psi$, we have $\mathbf{z}^* = \mathbf{m}$. Since $\psi$ is efficient, then

$$(1) \qquad (\mathbf{z}, v_{\mathbf{z}}^{\psi})((y_1, 0, 0)) = \psi_{y_1,1}((m_1, m_2, m_3), v) \ \text{ for all } y_1 \le m_1 - 1.$$

**Case 2.** For the reduced game $v_{\mathbf{z}}^{\psi}$ of $(\tilde{\mathbf{m}}, v)$ with respect to $\mathbf{z}$ and $\psi$, we have $\mathbf{z}^* = (0, m_2, m_3)$. Since $\psi$ is efficient, then

$$(2) \qquad (\mathbf{z}, v_{\mathbf{z}}^{\psi})((y_1, 0, 0)) = \psi_{y_1,1}((y_1, m_2, m_3), v) \ \text{ for all } y_1 \le m_1 - 1.$$

Please note that even when $\psi$ is efficient, the reduced games (1) and (2) are not necessarily the same. In particular $\psi_{m_1-1,1}(\mathbf{m}, v)$ not necessarily equals $\psi_{m_1-1,1}(\tilde{\mathbf{m}}, v)$.

As a matter of fact, to be more precise, we should denote the reduced game (1) by$(\mathbf{z}, v_{\mathbf{z}|\mathbf{m}}^{\psi})$ and denote the reduced game (2) by $(\mathbf{z}, v_{\mathbf{z}|\tilde{\mathbf{m}}}^{\psi})$. However, since in Definition 4.1 $(\mathbf{m}, v)$ is given at the beginning, with no risk of confusion, we leave the notation as it is in Definition 4.1.

**Remark 4.2. More general analogy to Hart and Mas-Colell's consistency**. Here, we have an interpretation which is analogous to Hart and Mas-Colell's interpretation for consistency. Given a multi-choice game $(\mathbf{m}, v)$ and its solution $\psi$ and a fixed $\mathbf{z} \le \mathbf{m}$, we define $T = \{j \mid z_j = m_j\}$, $P = \{j \mid 0 < z_j < m_j\}$ and $F = \{j \mid z_j = 0\}$. It is clear that $T$, $P$, $F$ are mutually exclusive and $N = T \cup P \cup F$. Also, in fact, $\mathbf{z}^* = \mathbf{m}^{T^c}$.

(i) For each $\mathbf{y} \le \mathbf{z}$, each player $j \in P \cup F = T^c$ raises his action level from $y_j$ to $m_j$, then a new action vector $\mathbf{y} \vee \mathbf{m}^{T^c}$ is formed and $v(\mathbf{y} \vee \mathbf{m}^{T^c})$ is the new payoff.

(ii) Then, each player $j \in T^c$ takes his surplus of the value for arising his action from $\sigma_{y_j}$ to $\sigma_{m_j}$ in the game $(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v))$, say

$$(A.3) \qquad [\psi_{m_j,j}(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v) - \psi_{y_j,j}(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v)],$$

from the new payoff $v(\mathbf{y} \vee \mathbf{m}^{T^c})$ and leave the rest to the players in $T \cup P$. Then a "reduced function" with respect to $\mathbf{z}$ and $\psi$, say

$$v_{\mathbf{z}}^{\psi}(\mathbf{y}) = v(\mathbf{y} \vee \mathbf{m}^{T^c}) - \sum_{j \in T^c} [\psi_{m_j,j}(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v) - \psi_{y_j,j}(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v)]$$

is constructed for $\mathbf{y} \in \Gamma(\mathbf{z})$.

If every reduced function is a reduced game and (4.2) holds for each $j$ with $z_j \ne 0$, i.e. $j \in T \cup P = S(\mathbf{z})$ and each $i \le z_j$, that is, the "new value" of player

$j$ in the reduced game $\psi_{i,j}(\Gamma(\mathbf{z}), v^{\psi}_{\mathbf{z}})$ equals the "original value" of player $j$ in the original game $\psi_{i,j}(\Gamma(\mathbf{m}), v)$, then $\psi$ is said to be consistent.

If $\psi$ is not consistent, then the players who have bigger values in the reduced game won't respect the value in the original game. We assume that players want the re-calculation because they are not satisfied with $\psi_{i,j}(\Gamma(\mathbf{m}), v)$ where $0 < i \leq z_j$, or say, they doubt the value is unfair.

Please **note** that even in Hart and Mas-Colell's traditional game case, the **reduced function** is not necessarily a **reduced game**, some authors impose (4.1.2) to fix the problem.

Intuitively, our reduced game allow a player to recalculate **all** or **part** or **none** of his values of actions. Our reduced game and consistency are extensions of the reduced game and the consistency defined in [2], [8] and [9]. A detailed interpretation is given in Appendix.

The following theorem is one of our main results in this article.

**Theorem 4.1.** The H&R Shapley value $\phi^w$ is consistent.

*Proof.* Given a multi-choice cooperative game $(\mathbf{m}, v)$ and its Shapley value $\phi^w$. Given $\mathbf{z} \in \Gamma(\mathbf{m})$, the reduced function of $v$ with respect to $\mathbf{z}$ and the Shapley value $\phi^w$ is :$v^{\phi^w}_{\mathbf{z}} : \Gamma(\mathbf{m}) \to R$ such that

$$v^{\phi^w}_{\mathbf{z}}(\mathbf{y}) = v(\mathbf{y} \vee \mathbf{z}^*) - \sum_{z^*_j \neq 0} \left[ \phi^w_{m_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) - \phi^w_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) \right].$$

Suppose $S(\mathbf{z}^*) = \{i_1, \ldots, i_s\}$ and notice that $z^*_j$ is either 0 or $m_j$. For $\mathbf{y} \leq \mathbf{z}$, we see that $\mathbf{y} \vee \mathbf{z}^* = (y_1, ..., m_{i_1}, y_{i_1+1}, \ldots, m_{i_s}, y_{i_s+1}, \ldots, y_n)$. Write $\mathbf{y} \vee \mathbf{z}^* = \mathbf{b} = (b_1, ..., b_n)$, then $b_j = y_j$ for $j \neq i_1, \ldots, i_s$ and $b_{i_r} = m_{i_r}$ for $r = 1, \ldots, s$. Since the H&R Shapley value is efficiency , we have

$$\begin{aligned} v^{\phi^w}_{\mathbf{z}}(\mathbf{y}) &= \sum_{b_j \neq 0} \phi^w_{b_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) - \sum_{z^*_j \neq 0} \phi^w_{m_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) \\ &\quad + \sum_{z^*_j \neq 0} \phi^w_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) \\ &= \sum_{\substack{y_j \neq 0 \\ j \notin \{i_1, \ldots, i_s\}}} \phi^w_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) + \sum_{j \in \{i_1, \ldots, i_s\}} \phi^w_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) \\ &= \sum_{y_j \neq 0} \phi^w_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*), v). \end{aligned} \tag{4.3}$$

Especially

$$v^{\phi^w}_{\mathbf{z}}(\mathbf{0}) = 0. \text{ (because each } y_j = 0)$$

So every reduced function $v_{\mathbf{z}}^{\phi^w}$ is a reduced game.

Now by Theorem 3.2: $\phi^w_{y_j,j} = H_{y_j,j}P_w$, for each $j$, plug into the last expression in (4.3), we have

$$v_{\mathbf{z}}^{\phi^w}(\mathbf{y}) = \sum_{y_j \neq 0} H_{y_j,j}P_w(\ (\mathbf{y} \vee \mathbf{z}^*), v).$$

But $v_{\mathbf{z}}^{\phi^w}(\mathbf{y}) = \sum\limits_{y_j \neq 0} H_{y_j,j}P_w(\ \mathbf{y}, v_{\mathbf{z}}^{\phi^w})$, where $P_w(\mathbf{y}, v_{\mathbf{z}}^{\phi^w})$ is the $w$-potential function. So one obtain

$$\sum_{y_j \neq 0} H_{y_j,j}P_w((\mathbf{y} \vee \mathbf{z}^*), v) = \sum_{y_j \neq 0} H_{y_j,j}P_w(\mathbf{y}, v_{\mathbf{z}}^{\phi^w}). \tag{4.4}$$

Next we assign to each $(\mathbf{y}, v_{\mathbf{z}}^{\phi^w})$ a value $\bar{P}_w(\mathbf{y}, v_{\mathbf{z}}^{\phi^w}) = P_w(\ (\mathbf{y} \vee \mathbf{z}^*), v)$. Therefore equation (4.4) is equivalent to

$$\sum_{y_j \neq 0} H_{y_j,j}\bar{P}_w(\ \mathbf{y}, v_{\mathbf{z}}^{\phi^w}) = \sum_{y_j \neq 0} H_{y_j,j}P_w(\ \mathbf{y}, v_{\mathbf{z}}^{\phi^w}).$$

Suppose $P_w((\mathbf{0} \vee \mathbf{z}^*), v) = c$. Then by Lemma 3.1 we have $P_w((\mathbf{y} \vee \mathbf{z}^*), v) = \bar{P}_w(\mathbf{y}, v_{\mathbf{z}}^{\phi^w}) = P_w(\mathbf{y}, v_{\mathbf{z}}^{\phi^w}) + c$. Take $\mathbf{y} = \mathbf{z}$ then $\mathbf{y} \vee \mathbf{z}^* = \mathbf{m}$ and hence

$$\phi^w_{i,j}(\mathbf{z}, v_{\mathbf{z}}^{\phi^w}) = H_{i,j}P_w(\mathbf{z}, v_{\mathbf{z}}^{\phi}) = H_{i,j}P_w(\mathbf{m}, v) = \phi^w_{i,j}(\mathbf{m}, v) \tag{4.5}$$

for all $i \leq z_j$ and all $j \in S(\mathbf{z})$. This proves $(4.2)$. The proof is complete. ∎

**Note 4.2.** Let $\mathbf{x}$ be the minimal carrier of $v$. Since the Shapley value $\phi^w$ is redundant free( see Theorem 2.0), we have

$$\phi^w_{i,j}(\mathbf{m}, v) = \phi^w_{x_j,j}(\mathbf{x}, v), \text{ for all } i \geq x_j \tag{4.6}$$

by (4.5)and (4.6), we obtain that for $z_j \geq x_j$

$$\phi^w_{i,j}(\mathbf{z}, v_{\mathbf{z}}^{\phi^w}) = \phi^w_{x_j,j}(\mathbf{x}, v) \text{ for all } i \text{ with } x_j \leq i \leq z_j$$

**Note 4.3.** Given $(\mathbf{m}, v), (\mathbf{m}, \bar{v}) \in G$ and two scalars $a, b$, since

$$\phi^w_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*), av + b\bar{v}) = a\ \phi^w_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*), v) + b\ \phi^w_{y_j,j}((\mathbf{y} \vee \mathbf{z}^*), \bar{v}),$$

it follows from (4.3) that

$$\phi^w((av + b\bar{v})_{\mathbf{z}}^{\phi^w}) = a\ \phi^w(v_{\mathbf{z}}^{\phi^w}) + b\ \phi^w(\bar{v}_{\mathbf{z}}^{\phi^w}). \tag{4.7}$$

In the proof of Theorem 4.1, the following results are obtained.

**Proposition 4.1.** If a solution $\psi$ is efficient for a game $(\mathbf{m}, v)$ and all its sub-games, then every reduced function $v_z^{\psi}$ is a reduced game. Furthermore,

$(a)$ $v_{\mathbf{z}}^{\psi}(\mathbf{y}) = \sum_{y_j \neq 0} \psi_{y_j,j}(\ (\mathbf{y} \vee \mathbf{z}^*), v)$, and

$(b)$ for any $\mathbf{z} \leq \mathbf{z}_0$ with $\mathbf{z}^* = \mathbf{z}_0^*, v_{\mathbf{z}}^{\psi}(\mathbf{y}) = v_{\mathbf{z}_0}^{\psi}(\mathbf{y}), \forall \mathbf{y} \leq \mathbf{z}$.

*Proof.* Since $\psi$ is efficient, by repeating the process in (4.3), where $\phi^w$ is replaced by $\psi$, one obtain that

$$v_{\mathbf{z}}^{\psi}(\mathbf{y}) = \sum_{y_j \neq 0} \psi_{y_j,j}(\ (\mathbf{y} \vee \mathbf{z}^*), v).$$

Especially

$$v_{\mathbf{z}}^{\psi}(\mathbf{0}) = 0. \text{ (because each } y_j = 0)$$

Hence, $v_{\mathbf{z}}^{\psi}$ is a reduced game. Since $\mathbf{y} \vee \mathbf{z}^* = \mathbf{y} \vee \mathbf{z}_0^*$, (b) follows from the equality in (a) immediately. ■

## 5. *w*-Proportional for Multi-choice Two-person Games

In this section, we will characterize the H&R Shapley value. This generalize Hart and Mas-Colell's Theorem 5.1 in [2]. Given an $n$-person multi-choice cooperative game $(\Gamma(\mathbf{m}), v)$.

Let $(\mathbf{0}|x_j = k)$ be an action vector where player $j$ takes action $\sigma_k$ and all the other players take action $\sigma_0$. Also, let $(\mathbf{0}|x_j = k, x_\ell = r)$ be an action vector where player $j$ takes action $\sigma_k$, player $\ell$ takes action $\sigma_r$ and all the other players take action $\sigma_0$.

Following [5], we have the following definition.

**Definition 5.1.** Given $w(0) = 0, w(1), \cdots$, a solution function $\psi$ is said to be *$w$-proportional for multi-choice two-person games* if for any two-person game $(\Gamma(\mathbf{m}^{\{j,\ell\}}), v)$ with $\mathbf{m}^{\{j,\ell\}} = (0, \cdots, m_j, 0, \cdots, m_\ell, 0, \cdots, 0), m_j, m_\ell > 0, \psi$ satisfies the following

$$\psi_{k,j}(\mathbf{m}^{\{j,\ell\}}, v) = \sum_{t=1}^{k} \Bigg( d_j v((\mathbf{0}|x_j = t, x_\ell = 0)) + \left[ \sum_{r=1}^{m_\ell} [\frac{w(t)}{w(t) + w(r)}] \cdot d_{j\ell} v((\mathbf{0}|x_j = t, x_\ell = r)) \right] \Bigg) \tag{5.1}$$

As a matter of fact, (5.1) can be written as

$$\psi_{k,j}(\mathbf{m}^{\{j,\ell\}}, v) = v((\mathbf{0}|x_j = k)) + \sum_{t=1}^{k} \left[ \sum_{r=1}^{m_\ell} [\frac{w(t)}{w(t) + w(r)}] \cdot d_{j\ell} v((\mathbf{0}|x_j = t, x_\ell = r)) \right]$$

For player $\ell$, we have a formula of $\psi_{r,\ell}(\mathbf{m}^{\{j,\ell\}}, v)$ similar to (5.1) which is omitted.

It is easy to see that (5.1) is an extension of the definition of standard for two-person games in [2].

We are now ready to characterize the *H&R Shapley value*. As we mention in Note 4.1, assuming that $v^{\psi}_{\mathbf{z}}(\mathbf{0}) = 0$ for each $\mathbf{z} \leq \mathbf{m}$ with $\mathbf{z} \neq \mathbf{0}$ is not much less than assuming the efficiency of $\psi$. Also, please note that in Definition 5.1, $m_j > 0$ and $m_\ell > 0$ is required, hence, *$w$-proportional for multi-choice two-person games* does not guarantee the efficiency of $\psi$ of an one-person game, therefore we prefer to use the following Definition for the characterization.

**Definition 5.2.** A solution function $\psi$ defined on $G$ is said to be *simple Pareto optimal* if and only if for the one-person-one-action game $(\{h\}, (1), v)$ such that $v(\mathbf{0}) = 0$ and $v((1)) = 1$, we have $\psi_{1,h}(\{h\}, (1), v) = v((1)) = 1$.

When we need to emphasis that player $h$ takes action $\sigma_1$, we denote the 1-dimensional vector $(1)$ by $(1_h)$.

**Definition 5.3.** ([6]) Player $j$ in the game $(\mathbf{m}, v)$ is called a non-essential player if

$$v(\mathbf{x}) = v((\mathbf{x} \mid x_j = 0)) + v((\mathbf{0} \mid x_j))$$

for all $\mathbf{x} \in \Gamma(\mathbf{m})$ where $(\mathbf{0} \mid x_j) = (0, \ldots, 0, x_j, 0, \ldots, 0)$. Extending a game $v$ to $\bar{v}$ by allowing an additional non-essential player to participate in the game $v$, then $\bar{v}$ is called non-essential extension game of $v$.

The following theorem generalizes proposition 4.5 in [2].

**Theorem 5.1.** Let $\psi$ be a solution function. If $\psi$ is (i) simple Pareto optimal, (ii) $w$-proportional for multi-choice two-person games and (iii) partially consistent, then $\psi$ is efficient. Furthermore, $\psi$ is consistent,

*Proof.* We shall prove for all $(N, \mathbf{m}, v) \in G$, write $N = \{1, 2, \ldots, n\}$ and $\mathbf{m} = (m_1, \ldots, m_n)$ the following equation holds.

$$\sum_{j \in S(\mathbf{m})} \psi_{m_j,j}(N, \mathbf{m}, v) = v(\mathbf{m}) \tag{5.2}$$

Given any one-person $(m+1)$-choice game $(\{j\}, (m), v) \in G$, consider its non-essential extension game with non-essential player $h$, say $(\{j, h\}, (m, 1), \bar{v})$ such that $\bar{v}((0,0)) = v((0)) = 0$, $\bar{v}((x_j, 0)) = v((x_j))$. Assign $\bar{v}((0,1)) = 1$ and $\bar{v}((x_j, 1)) = v((x_j)) + \bar{v}((0,1)) = v((x_j)) + 1$, then $(\{j, h\}, (m, 1), \bar{v})$ is well-defined.

Then by (ii), $w$-proportional for multi-choice two-person games, we have for any fixed $0 < y \leq m$,

$$\begin{aligned}
&\psi_{r,j}(\{j,h\},(y,1),\bar{v}) \\
&= \bar{v}((r,0)) + \sum_{t=1}^{r}\left[\frac{w(t)}{w(t)+w(1)}\right]\cdot\Big[\bar{v}((t,1)) - \bar{v}((t,0)) - \bar{v}((t-1,1)) + \bar{v}((t-1,0))\Big] \\
&= v((r)) + \sum_{t=1}^{r}\left[\frac{w(t)}{w(t)+w(1)}\right]\cdot\Big[[v((t))+1] - v((t)) - [v((t-1))+1] + v((t-1))\Big] \\
&= v((r)),\ r = 1,\ldots,y,
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
&\psi_{1,h}(\{j,h\},(y,1),\bar{v}) \\
&= \bar{v}((0,1)) + \sum_{t=1}^{y}\left[\frac{w(1)}{w(1)+w(t)}\right]\cdot\Big[\bar{v}((t,1)) - \bar{v}((t,0)) - \bar{v}((t-1,1)) + \bar{v}((t-1,0))\Big] \\
&= 1 + \sum_{t=1}^{y}\left[\frac{w(1)}{w(1)+w(t)}\right]\cdot\Big[[v((t))+1] - v((t)) - [v((t-1))+1] + v((t-1))\Big] \\
&= 1,
\end{aligned} \tag{5.4}$$

Let $\mathbf{z} = (m,0)$, then $\mathbf{z}^* = (0,1)$ and $S(\mathbf{z}^*) = \{h\}$. For $\mathbf{y} = (y, y_h) \leq \mathbf{z}$, we have $0 \leq y \leq m, y_h = 0$ and $\mathbf{y} \vee \mathbf{z}^* = (y,1)$.

Consider the reduced function $(\mathbf{z}, \bar{v}^{\psi}_{\mathbf{z}})(\mathbf{y})$ and notice that the one-person-one-action sub-game of $(\{j,h\},(y,1),\bar{v})$, say $(\{h\},(1_h),\bar{v})$ satisfies $((1_h),\bar{v})((0)) = ((m,1),\bar{v})((0,0)) = 0$ and $((1_h),\bar{v})((1_h)) = ((m,1),\bar{v})((0,1)) = 1$. Then by (i): simple Pareto optimal, we have $\psi_{1,h}(\{h\},(1_h),\bar{v}) = 1$.

For $y = 0$, we have

$$\begin{aligned}
&((m,0),\bar{v}^{\psi}_{\mathbf{z}})((y,0)) = ((m,1),\bar{v})((0,0)\vee(0,1)) - \psi_{1,h}((0,1),\bar{v}) \\
&= 1 - \psi_{1,h}(\{h\},(1_h),\bar{v}) = 0.
\end{aligned}$$

Therefore, $(\mathbf{z}, \bar{v}^{\psi}_{\mathbf{z}})$ is a reduced game. Next, with no risk of confusion, we identify $(y,0) \equiv (y)$, then by equation (5.4)

$$\bar{v}^{\psi}_{\mathbf{z}}((y)) = \bar{v}^{\psi}_{\mathbf{z}}((y,0))) = \bar{v}((y,1)) - \psi_{1,h}(\{j,h\},(y,1),\bar{v}) = v((y)) + 1 - 1 = v((y))$$

Therefore $(\{j\},(m),v) \equiv (\{j\},(m),\bar{v}^{\psi}_{\mathbf{z}})$ and hence

$$\psi_{r,j}(\{j\},(m),v) = \psi_{r,j}(\{j\},(m),\bar{v}^{\psi}_{\mathbf{z}}),\ \forall 0 \leq r \leq m \tag{5.5}$$

Next

$$\begin{aligned}
&\psi_{r,j}(\{j\},(m),\bar{v}^{\psi}_{\mathbf{z}}) = \psi_{r,j}((m,0),\bar{v}^{\psi}_{\mathbf{z}}) \\
&= \psi_{r,j}(\{j,h\},(m,1),\bar{v}) \qquad \text{(by partial consistency)} \\
&= v((r)) \qquad \text{(by (5.3))}
\end{aligned} \tag{5.6}$$

It follows frow (5.5) and (5.6) that

$$\psi_{r,j}(\{j\},(m),v) = v((r)),\ \forall 1 \leq r \leq m$$

Therefore (5.2) holds for all $|N| = 1$. Furthermore for any game $(\mathbf{m}, v)$ and any $\mathbf{z} \in \Gamma(\mathbf{m})$ with $|S(\mathbf{z}^*)| = 1$, say $\mathbf{z}^* = (0,\ldots,0,m_j,0,\ldots,0)$. We see that

$$v^{\psi}_{\mathbf{z}}(\mathbf{0}) = v(0,\ldots,0,m_j,0,\ldots,0) - \psi_{m_j,j}(0,\ldots,0,m_j,0,\ldots,0),v) = 0$$

because of efficiency of one player games. Therefore any reduced function $v^{\psi}_{\mathbf{z}}$ with $|S(\mathbf{z}^*)| = 1$ is a well-defined reduced game.

As a matter of fact (5.2) holds for $|N| = 2$ by (ii), therefore, any reduced function $v^{\psi}_{\mathbf{z}}$ with $|S(\mathbf{z}^*)| = 2$ is a well-defined reduced game. Let $n \geq 3$, and assume (5.2) holds for all games with less than $n$ players and hence any reduced function $v^{\psi}_{\mathbf{z}}$ with $|S(\mathbf{z}^*)| < n$ is a well-defined reduced game. For a game $(N, \mathbf{m}, v)$ with $|N| = n$. Consider a special action vector $\mathbf{m}_{-j} = (m_1,\ldots,m_{j-1},m_{j+1},m_{j+2},\ldots,m_n) \equiv (\mathbf{m}|m_j = 0)$. And notice that the game $(N-\{j\}, \mathbf{m}_{-j}, v^{\psi}_{\mathbf{m}_{-j}})$ contains only $n-1$ players. Therefore by assumption of efficiency of $n-1$ players games and by partial consistency

$$\begin{aligned} v^{\psi}_{(\mathbf{m}|m_j=0)}((\mathbf{m}|m_j = 0)) &= \sum_{\ell \in N-\{j\}} \psi_{m_\ell,\ell}((\mathbf{m}|m_j = 0), v^{\psi}_{(\mathbf{m}|m_j=0)}) \\ &= \sum_{\ell \in N-\{j\}} \psi_{m_\ell,\ell}(\mathbf{m}, v) \end{aligned}$$

Also since $(\mathbf{m}|m_j = 0) \vee (\mathbf{m}|m_j = 0)^* = \mathbf{m}$, we have

$$v^{\psi}_{(\mathbf{m}|m_j=0)}((\mathbf{m}|m_j = 0)) = v(\mathbf{m}) - \psi_{m_j,j}(\mathbf{m}, v)$$

We see that

$$v(\mathbf{m}) = \sum_{\ell \in N-\{j\}} \psi_{m_\ell,\ell}(\mathbf{m}, v) + \psi_{m_j,j}(\mathbf{m}, v) = \sum_{j \in N} \psi_{m_j,j}(\mathbf{m}, v).$$

Therefore efficiency of $n$ players holds. We have proved the efficiency of the solution for any multi-choice cooperative game.

Finally, since $\psi$ is efficient, by Proposition 4.1 every reduced function $v^{\psi}_{\mathbf{z}}$ is a reduced game. Then by (iii) partially consistent $\psi$ is consistent, and the proof is complete. ■

The following Lemma 5.1 will be used in Theorem 5.2

**Lemma 5.1.** Given $\mathbf{x} = (x_1, \cdots, x_n)$ with each component $> 0$ and fixed $k, 1 \leq k \leq n$. Choose $c_{j_\ell} \in \{0, 1, \cdots, x_{j_\ell}\}, j_\ell \neq k, \ell = 1, \cdots, s$. Then, for any $\mathbf{y} \leq \mathbf{x} - \sum\limits_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell} - \mathbf{e}_k$,

$$v^{\psi}_{\mathbf{x}-\sum_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell}-\mathbf{e}_k}(\mathbf{y}) = v^{\psi}_{\mathbf{x}-\mathbf{e}_k}(\mathbf{y}),$$

where $v^{\psi}_{\mathbf{x}-\sum_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell}-\mathbf{e}_k}$ is the reduced function of $(\mathbf{x}-\sum\limits_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell},\ v)$ with respect to $(\mathbf{x}-\sum\limits_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell})-\mathbf{e}_k$ and $\psi$, and $v^{\psi}_{\mathbf{x}-\mathbf{e}_k}$ is the reduced function of $(\mathbf{x}, v)$ with respect to $\mathbf{x} - \mathbf{e}_k$ and $\psi$.

*Proof.* For the game $(\mathbf{x}, v)$, consider the action vector $(\mathbf{x} - \mathbf{e}_k)$, we have

$$(\mathbf{x} - \mathbf{e}_k)^*_j = \begin{cases} x_k & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases}$$

and for any $\mathbf{y} \leq \mathbf{x} - \mathbf{e}_k, \mathbf{y} \vee (\mathbf{x} - \mathbf{e}_k)^* = (y_1, \ldots, y_{k-1}, x_k, y_{k+1}, \ldots, y_n)$, we have

$$\begin{aligned} v^{\psi}_{\mathbf{x}-\mathbf{e}_k}(\mathbf{y}) &= v((y_1, ..., y_{k-1}, x_k, y_{k+1}, ..., y_n)) \\ &- \psi_{x_k,k}((y_1, ..., y_{k-1}, x_k, y_{k+1}, ..., y_n), v) \\ &+ \psi_{y_k,k}((y_1, ..., y_{k-1}, x_k, y_{k+1}, ..., y_n), v). \end{aligned}$$

Similarly, for the game $((\mathbf{x} - \sum\limits_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell}), v)$, since

$$((\mathbf{x} - \sum_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell}) - \mathbf{e}_k)^*_j = \begin{cases} x_k & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases}$$

and $\mathbf{y} \vee ((\mathbf{x} - \sum\limits_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell}) - \mathbf{e}_k)^* = (y_1, \ldots, y_{k-1}, x_k, y_{k+1}, \ldots, y_n)$, for all $\mathbf{y} \leq \mathbf{x} - \sum\limits_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell} - \mathbf{e}_k$, again we get

$$\begin{aligned} v^{\psi}_{\mathbf{x}-\sum_{\ell=1}^{s} c_{j_\ell}\mathbf{e}_{j_\ell}-\mathbf{e}_k}(\mathbf{y}) &= v((y_1, ..., y_{k-1}, x_k, y_{k+1}, ..., y_n)) \\ &- \psi_{x_k,k}((y_1, ..., y_{k-1}, x_k, y_{k+1}, ..., y_n), v) \\ &+ \psi_{y_k,k}((y_1, ..., y_{k-1}, x_k, y_{k+1}, ..., y_n), v). \end{aligned}$$

The result follows. ■

The above result will be applied for the case $c_{j\ell} \in \{0,1\}, s=2$ in Theorem 5.2.

The following Proposition plays a key role in the proof of Theorem 5.2.

**Proposition 5.1.** The weight potential can be expressed recursively as

(5.7)

$$P_w(\mathbf{x}, v) = \frac{1}{w(x_j)}\Big[\phi^w_{x_j,j}(\mathbf{x}, v) - \phi^w_{x_j-1,j}(\mathbf{x} - \mathbf{e}_j, v)\Big] + P_w((\mathbf{x} - \mathbf{e}_j), v), \text{ for all } j,$$

where $\phi^w$ is the H&R Shapely value.

*Proof.* In theorem 3.2 the weight potential satisfies

$$(5.8)\quad \phi^w_{x_j,j}(\mathbf{x}, v) = \sum_{r=1}^{x_j} w(r) \cdot \Big[P_w(\ (\mathbf{x}|x_j = r), v) - P_w(\ (\mathbf{x}|x_j = r-1), v)\Big],$$

for all player $j$.

Apply (5.8) for $\phi^w_{x_j,j}(\mathbf{x}, v)$ and $\phi^w_{x_j-1,j}(\mathbf{x} - \mathbf{e}_j, v)$, we have

$$\begin{aligned}
&\phi^w_{x_j,j}(\mathbf{x}, v) - \phi^w_{x_j-1,j}(\mathbf{x} - \mathbf{e}_j, v)\\
=&\sum_{r=1}^{x_j} w(r) \cdot \Big[P_w(\ (\mathbf{x}|x_j = r), v) - P_w(\ (\mathbf{x}|x_j = r-1), v)\Big]\\
&- \sum_{r=1}^{x_j-1} w(r) \cdot \Big[P_w(\ (\mathbf{x}|x_j = r), v) - P_w(\ (\mathbf{x}|x_j = r-1), v)\Big]\\
=&P_w(\mathbf{x}, v) - P_w(\ (\mathbf{x} - \mathbf{e}_j), v).
\end{aligned}$$

Hence, $P_w(\mathbf{x}, v) = \frac{1}{w(x_j)}\Big[\phi^w_{x_j,j}(\mathbf{x}, v) - \phi^w_{x_j-1,j}(\mathbf{x} - \mathbf{e}_j, v)\Big] + P_w(\ (\mathbf{x} - \mathbf{e}_j), v)$, for all $j$. So the weight potential can be computed recursively by means of solution $\phi^w$. The proof is completed. ■

The following characterization of the *H&R Shapley value* is an extension of Hart and Mas-Colell's [2] axiomatization of the traditional Shapley value. Moreover, it shows more than what Hsiao tried to do at the end of [5].

**Theorem 5.2.** Given a weight function $w$, let $\psi$ be a solution function for $G$. Then: (i) $\psi$ is simple Pareto optimal, (ii) $\psi$ is $w$-proportional for multi-choice two-person games, and (iii) $\psi$ is partially consistent, if and only if $\psi$ is the multi-choice Shapley value.

*Proof.* One direction is immediate, see Theorem 4.1. For the other direction, by Theorem 5.1, solution $\psi$ is efficient and consistent. We shall show that $\psi$ admits a potential.

In views of Proposition 5.1, define a function $P : G \to R$ which associated each game $(N, \mathbf{x}, v)$ a real number

$$(5.9) \quad P(\mathbf{x}, v) = \frac{1}{w(x_j)}\left[\psi_{x_j,j}(\mathbf{x}, v) - \psi_{x_j-1,j}(\mathbf{x}-\mathbf{e}_j, v)\right] + P(\mathbf{x}-\mathbf{e}_j, v), \forall j \in N.$$

If $|N| = 1$, the second term of (5.9) vanishes. We prove that the function $P$ is well defined and satisfies

$$(5.10) \qquad H_{x_j,j}P(\mathbf{x}, v) = \psi_{x_j,j}(\mathbf{x}, v)$$

by mathematical induction on $|\mathbf{x}|$, where $|\mathbf{x}| = \sum x_j$. First notice that for one person game $(\{j_0\}, (r), v)$,

$$(5.11) \qquad P((r), v) = \frac{1}{w(r)}[\psi_{r,j_0}((r), v) - \psi_{r-1,j_0}((r-1), v)] = P_w(\ (r), v),$$

where $P_w$ is the weight potential.

Next since by assumption $\psi$ is $w$-proportional for two-person multi-choice games, it is the H&R Shapley value for two-person games. Therefore $P(\mathbf{x}, v) = P_w(\mathbf{x}, v)$ for one-player and two-player games which proves (5.9) and (5.10) for $|N| = 1, 2$, especially (5.9) and (5.10) hold for $|\mathbf{x}| = 1, 2$. Suppose that (5.9)and (5.10) hold for all $|\mathbf{x}| \leq m-1, m \geq 3$. Consider a game $(\mathbf{x}, v)$, where $\mathbf{x} = (x_1, \ldots, x_n)$ (note: each $x_\ell \neq 0$) with $|\mathbf{x}| = m$. We compute the following

$$\begin{aligned}
&\frac{1}{w(x_\ell)}\left[\psi_{x_\ell,\ell}(\mathbf{x}, v) - \psi_{x_\ell-1,\ell}(\mathbf{x}-\mathbf{e}_\ell, v)\right] \\
&- \frac{1}{w(x_j)}\left[\psi_{x_j,j}(\mathbf{x}, v) - \psi_{x_j-1,j}(\mathbf{x}-\mathbf{e}_j, v)\right] \\
=&\frac{1}{w(x_\ell)}\left[\psi_{x_\ell,\ell}(\mathbf{x}-\mathbf{e}_k, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - \psi_{x_\ell-1,\ell}((\mathbf{x}-\mathbf{e}_\ell)-\mathbf{e}_k, v^{\psi}_{(\mathbf{x}-\mathbf{e}_\ell)-\mathbf{e}_k})\right] \text{ (by consistency)} \\
&- \frac{1}{w(x_j)}\left[\psi_{x_j,j}(\mathbf{x}-\mathbf{e}_k, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - \psi_{x_j-1,j}((\mathbf{x}-\mathbf{e}_j)-\mathbf{e}_k, v^{\psi}_{(\mathbf{x}-\mathbf{e}_j)-\mathbf{e}_k})\right] \\
=&\frac{1}{w(x_\ell)}\left[\psi_{x_\ell,\ell}(\mathbf{x}-\mathbf{e}_k, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - \psi_{x_\ell-1,\ell}(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_\ell, v^{\psi}_{\mathbf{x}-\mathbf{e}_k})\right] \quad \text{(by lemma 5.1)} \\
&- \frac{1}{w(x_j)}\left[\psi_{x_j,j}(\mathbf{x}-\mathbf{e}_k, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - \psi_{x_j-1,j}(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_j, v^{\psi}_{\mathbf{x}-\mathbf{e}_k})\right]
\end{aligned}$$

$$
\begin{aligned}
=&\Big[P(\mathbf{x}-\mathbf{e}_k, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - P(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_\ell, v^{\psi}_{\mathbf{x}-\mathbf{e}_k})\Big] \quad \text{((5.9) holds for } |\mathbf{x}-\mathbf{e}_k| = m-1)\\
&-\Big[P(\mathbf{x}-\mathbf{e}_k, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - P(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_j, v^{\psi}_{\mathbf{x}-\mathbf{e}_k})\Big]\\
=&P(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_j, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - P(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_\ell, v^{\psi}_{\mathbf{x}-\mathbf{e}_k})\\
=&\Big[P(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_j, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - P(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_j-\mathbf{e}_\ell, v^{\psi}_{\mathbf{x}-\mathbf{e}_k})\Big] \quad \text{(note } m\geq 3)\\
&-\Big[P(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_\ell, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - P(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_\ell-\mathbf{e}_j, v^{\psi}_{\mathbf{x}-\mathbf{e}_k})\Big]\\
=&\frac{1}{w(x_\ell)}\Big[\psi_{x_\ell,\ell}(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_j, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - \psi_{x_\ell-1,\ell}(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_j-\mathbf{e}_\ell, v^{\psi}_{\mathbf{x}-\mathbf{e}_k})\Big] \text{((5.9) holds for } (m\text{–}2))\\
&-\frac{1}{w(x_j)}\Big[\psi_{x_j,j}(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_\ell, v^{\psi}_{\mathbf{x}-\mathbf{e}_k}) - \psi_{x_j-1,j}(\mathbf{x}-\mathbf{e}_k-\mathbf{e}_\ell-\mathbf{e}_j, v^{\psi}_{\mathbf{x}-\mathbf{e}_k})\Big]\\
=&\frac{1}{w(x_\ell)}\Big[\psi_{x_\ell,\ell}\big((\mathbf{x}-\mathbf{e}_j)-\mathbf{e}_k, v^{\psi}_{(\mathbf{x}-\mathbf{e}_j)-\mathbf{e}_k}\big)\\
&-\psi_{x_\ell-1,\ell}\big(\mathbf{x}-\mathbf{e}_j-\mathbf{e}_k-\mathbf{e}_\ell, v^{\psi}_{(\mathbf{x}-\mathbf{e}_j-\mathbf{e}_\ell)-\mathbf{e}_k}\big)\Big] \quad \text{(by Lemma 5.1)}\\
&-\frac{1}{w(x_j)}\Big[\psi_{x_j,j}\big((\mathbf{x}-\mathbf{e}_\ell)-\mathbf{e}_k, v^{\psi}_{(\mathbf{x}-\mathbf{e}_\ell)-\mathbf{e}_k}\big) - \psi_{x_j-1,j}\big(\mathbf{x}-\mathbf{e}_\ell-\mathbf{e}_k-\mathbf{e}_j, v^{\psi}_{(\mathbf{x}-\mathbf{e}_\ell-\mathbf{e}_j)-\mathbf{e}_k}\big)\Big]\\
=&\frac{1}{w(x_\ell)}\Big[\psi_{x_\ell,\ell}(\mathbf{x}-\mathbf{e}_j, v) - \psi_{x_\ell-1,\ell}(\mathbf{x}-\mathbf{e}_j-\mathbf{e}_\ell, v)\Big] \quad \text{(by consistency)}\\
&-\frac{1}{w(x_j)}\Big[\psi_{x_j,j}(\mathbf{x}-\mathbf{e}_\ell, v) - \psi_{x_j-1,j}(\mathbf{x}-\mathbf{e}_\ell-\mathbf{e}_j, v)\Big]\\
=&\Big[P(\mathbf{x}-\mathbf{e}_j, v) - P\big((\mathbf{x}-\mathbf{e}_j)-\mathbf{e}_\ell, v\big)\Big] \quad \text{((5.9) holds for } m-1)\\
&-\Big[P(\mathbf{x}-\mathbf{e}_\ell, v) - P\big((\mathbf{x}-\mathbf{e}_\ell)-\mathbf{e}_j, v\big)\Big]\\
=&P(\mathbf{x}-\mathbf{e}_j, v) - P(\mathbf{x}-\mathbf{e}_\ell, v).
\end{aligned}
$$

Therefore, for all $j, \ell$ in $N$,

$$
\begin{aligned}
P(\mathbf{x}, v) =& \frac{1}{w(x_j)}\Big[\psi_{x_j,j}(\mathbf{x}, v) - \psi_{x_j-1,j}(\mathbf{x}-\mathbf{e}_j, v)\Big] + P((\mathbf{x}-\mathbf{e}_j), v)\\
=& \frac{1}{w(x_\ell)}\Big[\psi_{x_\ell,\ell}(\mathbf{x}, v) - \psi_{x_\ell-1,\ell}(\mathbf{x}-\mathbf{e}_\ell, v)\Big] + P((\mathbf{x}-\mathbf{e}_\ell), v).
\end{aligned}
$$

We have proved that $P(\mathbf{x}, v)$ defined by (5.9) is well-defined. Next for $|\mathbf{x}| = m$,

$$
\begin{aligned}
&H_{x_j,j}P(\mathbf{x}, v)\\
&= \sum_{r=1}^{x_j} w(r)\cdot\Big[P((\mathbf{x}|x_j = r), v) - P((\mathbf{x}|x_j = r-1), v)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{r=1}^{x_j-1} w(r) \cdot \Big[ P(\ (\mathbf{x}|x_j = r), v) - P(\ (\mathbf{x}|x_j = r-1), v) \Big] \\
&\quad + w(x_j)[P(\mathbf{x}, v) - P(\mathbf{x}-\mathbf{e}_j, v)] \\
&= H_{x_j-1,j} P(\mathbf{x}-\mathbf{e}_j, v) \\
&\quad + \Big[ \psi_{x_j,j}(\mathbf{x}, v) - \psi_{x_j-1,j}(\mathbf{x}-\mathbf{e}_j, v) \Big] \quad \text{(by (5.9))} \\
&= \psi_{x_j-1,j}(\mathbf{x}-\mathbf{e}_j, v) \quad \text{(by mathematical induction, (5.10) holds for } |\mathbf{x}| = m-1) \\
&\quad + [\psi_{x_j,j}(\mathbf{x}, v) - \psi_{x_j-1,j}(\mathbf{x}-\mathbf{e}_j, v)] \\
&= \psi_{x_j,j}(\mathbf{x}, v).
\end{aligned}
$$

We have proved that (5.10) holds for $|\mathbf{x}| = m$. Now by efficiency of $\psi$, we have

$$
\sum_{j \in S(\mathbf{m})} H_{m_j,j} P(\mathbf{m}, v) = \sum_{j \in S(\mathbf{m})} \psi_{m_j,j}(\mathbf{m}, v) = v(\mathbf{m}, v).
$$

We see that $P$ is the $w$-potential, therefore $\psi$ is the H&R Shapely value. The proof is complete. ■

As we mentioned in Note 4.1, there are some other ways to characterize the H&R Shapley value, here we state the following Corollaries. If we define our reduced game by (4.1.2) instead of (4.1.1), then we have the following corollary.

**Corollary 5.1.** Given a weight function $w$, let $\psi$ be a solution function for $G$. If the reduced game is defined by (4.1.2), then:
(i) $\psi$ is $w$-proportional for multi-choice two-person games, and (ii)$\psi$ is partially consistent, if and only if $\psi$ is the multi-choice Shapley value.

If we allow $m_j = 0$ or $m_\ell = 0$ in the definition of *is $w$-proportional for multi-choice two-person games* , then we also have the following corollary.

**Corollary 5.2.** Given a weight function $w$, let $\psi$ be a solution function for $G$. If the definition of *$w$-proportional for multi-choice two-person games* allows $m_j = 0$ or $m_\ell = 0$, then:

(i) $\psi$ is $w$-proportional for multi-choice two-person games, and
(ii) $\psi$ is partially consistent, if and only if $\psi$ is the multi-choice Shapley value.

**Closing Remark.** It seems that Corollary 5.1 and Corollary 5.2 need only two assumptions to characterize the *H&R Shapley value* , but, in fact, in Corollary 5.1 the "consistency" combines at least $[\Pi_{j=1}^{n} m_j] - 1$ assumptions into one. Moreover, if we allow $m_j = 0$ or $m_\ell = 0$ in the definition of $w$-*proportional for multi-choice two-person games,* then we are, in fact, assuming the efficiency of $\psi$ for all one-person games and all two-person games. We have no comments on studying the multi-choice game by combining many assumptions into one. However, we believe that the explicit formulas of the *H&R Shapley value* ($\star$), the $w$-potential function (3.4) and the reduced games (4.1.1) are essentially enough to study the values related to *H&R Shapley value*. Once the formulas are given explicitly, readers with different culture backgrounds or different cognitive schemas might get different kinds of properties from the explicit formulas. Then, the proofs of their properties are simply combinatorial calculations of the explicit formulas.

## APPENDIX

### Motivations of Investigating the Multi-choice Games

**Example 1.** (**Disease Control Game**). Suppose in a small town, the mayor announces " the current phase of pandemic alert is 3 for enteroviruses". Suppose there are three disease control agents (players) who can make money by helping a kindergarten in the town to keep away from enteroviruses. The agents may be cooperative with one another, form a coalition, to work together in order to (a) make more money, or to (b) do the job more efficiently.

(a) From agents' viewpoint, they want to make more money and they are concern with how to share the money. (b) From the owner of the kindergarten's viewpoint, he doesn't care too much about how the agents share the money, he is concern with how the agents can efficiently fight against enteroviruses.

**Idea 1. The traditional cooperative game.** From the viewpoint of how the agents should share the money they make, we may model the above game with the traditional cooperative game. Let $N = \{1, 2, 3\}$ be the set of three agents, a subset $T$ of $N$ is called a coalition, $N$ is called the grand coalition. Since the powerset $2^N \simeq \Gamma(\mathbf{e}^N)$, without risk of confusion, we can use binary vectors to denote the coalitions, then the traditional cooperative game in characteristic function form is $v : \Gamma(\mathbf{e}^N) \to R$ such that $v(\mathbf{e}^\emptyset) = v(\mathbf{0}) = 0$. Here, $v(\mathbf{e}^S)$ is the payoff of the coalition $S \subseteq N$.

Also, in the above traditional cooperative game, each player has two choices, namely $\sigma_0$-"not to participate in a coalition" and $\sigma_1$-" to participate in a coalition". Player $j$ is in coalition $S$ if and only if $e_j^S = 1$.

**Idea 2. The traditional solutions**. Suppose the grand coalition $N$ is formed and the three agents together get a payoff $v(\mathbf{e}^N)$ (money), how should they share the money (utility) with fairness ? There are many kinds of solutions, in this article, we focus on the remarkable solution, the Shapley **value** which satisfies three axioms, say symmetric axiom, carrier axiom and additivity axiom. Some authors separate the carrier axiom into efficient axiom and dummy axiom.

Please note that, from the viewpoint of sharing the payoff (money), the Shapley value makes sense only when the grand coalition $N$ is formed. Also, please note that the very original definition of a cooperative game in characteristic function form was assumed to be super-additive, please see Shapley's original paper, Chapter 2 in [10]. The super-additivity makes the formation of the grand coalition reasonable.

We do not know when and which author first took the super-additivity away from the definition of a traditional cooperative game. Without the super-additivity, it is not very reasonable to assume that the grand coalition is formed. Some authors assume that the grand coalition is formed compulsorily by law. However, from political sciences' viewpoint, the grand coalition is seldom formed in a voting game. Also, in studying a voting game, the Shapley **value** should be regarded as the Shapley **power index**. The Shapley value is applicable to political sciences.

For the disease control game, from management sciences's viewpoint, if we were employers of the three agents, we will regard the payoff $v(\mathbf{e}^S)$ as the probability that $S$ will be success in disease control mission. At least, we will regard it as a simple game where $v(\mathbf{e}^S) = 0$ (or 1) means $S$ fail (or success) in the disease control mission. We are concerned about how they deal with the disease, rather than how they share our payments. We will find the Shapley **power index** of each player and won't hire any agent with very low power index. Moreover, since the Shapley value is transferable utility invariant, it is also easy for us to decide how much we should pay the players, according to their Shapley power indices. The Shapley value is applicable to management sciences.

The traditional Shapley value was found to have applications in many fields of social sciences, even accountings and military sciences. Nevertheless, in a benefit sharing problem or a cost allocation problem, the Shapley value was found to be **fair** or say **stable** in many sense, for example the traditional Shapley value is also dummy free of players, one can not get more or pay less by inviting a dummy player to join the game. If a solution is proposed to share the payoff $v(\mathbf{e}^N)$ and player $j$ may get more by inviting a dummy player to join the game, the solution will be unstable or say controversial.

**Idea 3. The weak points of the traditional cooperative games.** If an university pay a professors according whether he teach more than 6 hours per week or not. A full professor or an assistant professor gets the same pay as the others' as long as he teach more than 6 hours per week. Research performance and teaching performance are not taken into account, then why should an assistant professor work hard to get a promotion? A main weak point of the model of the traditional cooperative game is that a player might take action $\sigma_1$-to participate in a coalition, but work lazily in the coalition. The coalition $S$ does not distinguish lazy players and diligent players. Therefore, some authors consider the traditional weighted Shapley value, they give weights to the players.

Hsiao and Raghavan thought that a player should be paid according to what he does rather than according to who he is. Therefore, in [3], Hsiao and Raghavan extended the traditional Shapley value to the multi-choice Shapley value.

**A multi-choice game.** We now go back to the disease control game, since other persons may become infected with enteroviruses by direct contact with secretions from an infected person or by contact with contaminated surfaces or objects, such as a drinking glass or telephone. Suppose, we can reasonably simplify the choices of the agents as:

$(a)$ $\sigma_0$-do nothing,

$(b)$ $\sigma_1$-take the temperature of anyone who wants to enter the kindergarten, don't let him(her) in if s/he has a fever.

$(c)$ $\sigma_2$-clean up the kindergarten every two hours and don't let anybody with a fever get in the kindergarten.

We now can model the disease control game as follows. Let $N = \{1, 2, 3\}$ be the set of three agents. Let the action vector $\mathbf{2} = (2, 2, 2)$, then a multi-choice cooperative game in characteristic function form is $v : \Gamma(\mathbf{2}) \to R$ such that $v(\mathbf{0}) = 0$ which means nobody works and the payoff of the action vector $\mathbf{0}$ is zero. Also, $v(\mathbf{x})$ is the payoff of the action vector $\mathbf{x} \in \Gamma(\mathbf{2})$.

**Idea 4. The solutions of the multi-choice games**. Similar to the solution of the traditional games, now, suppose the agents choose action vector $\mathbf{2}$ and get a payoff $v(\mathbf{2})$, how should they share the payoff? Of course, fairness is the players' main concern. Hsiao and Ragvan suggest the H&R Shapley value which satisfies four axioms that are analogous to Shapley's axioms in the traditional game. They give weights (discriminations) to the choices instead of the players.

As a matter of fact, in this disease control game, the prior value (power index) of $\sigma_0$ is assigned to be $w(0) = 0$, and the prior power indeces $w(1)$ and $w(2)$ are given by public health professionals. Normally, a prior power index of a profession action is given by professionals in that field.

Similar to the traditional Shapley value, the H&R Shapley value makes sense only when the players (jointly) take an action which is a carrier action vector. Please note that if $\boldsymbol{c} = (c_1, c_2, c_3)$ is the minimal carrier action vector in the disease control game, then $\mathbf{x} \geq \boldsymbol{c}$ is also a carrier action vector. Then, from the point of view of sharing the payoff $v(\boldsymbol{c})$ among the payers, the H&R Shapely **value** suggests that player $j$ gets the **value** $\phi^w_{c_j,j}(v)$ in the disease control game while the minimal carrier action $\boldsymbol{c}$ is taken. Readers might ask then why don't we just define the H&R

Shapley value as an $n$-dimensional vector$(\phi^w_{c_1,1}(v), \phi^w_{c_2,2}(v), \ldots, \phi^w_{c_n,n}(v))$ for an $n$-person multi-choice game? However, we want the H&R Shapley value be used as not only a **value** but also a **power index**. Therefore, we decided to define the *H&R Shapley value* as a matrix and slightly extended to the matrix-type table in this article. ¿From management's viewpoint, if we were the employer of the three agents, we do not concern too much about how the players share the payoff. We want the whole picture of the players' power indices which stand for their influences in the disease control game. If we were the commander at the war against viruses or whatever, we will not hire an agent who has very little influence in the game.

**Idea 5. Hart and Mas-Colell's Consistency**. We now go back to the problem of sharing utility. Given a traditional cooperative game $(\Gamma(\mathbf{e}^N), v)$ and its traditional Shapley value $\phi$

(i) Suppose a sub-group of players of $N$, say $T$ are not satisfied with their Shapley values, or say they doubt that their Shapley values $\phi_{1,j}(\Gamma(\mathbf{e}^N), v)$ are not fairly distributed among them(**without satisfaction**). Conversely, suppose each player $j \in T^c$ is satisfied with his Shapley value (**with full satisfaction**).

(ii) Then, from coalition' viewpoint (actions' viewpoints), for each $S \subseteq T$ (for each action vector $\mathbf{e}^S \in \Gamma(\mathbf{e}^T)$), each player $j \in T^c$(**with full satisfaction**) helps the players in $S$ by joining the coalition (by raising his action level from $\sigma_0$ to $\sigma_1$) to form a bigger coalition $S \cup T^c$ (action vector $\mathbf{e}^S \vee \mathbf{e}^{T^c} = \mathbf{e}^{S \cup T^c}$) and get a new payoff $v(\mathbf{e}^S \vee \mathbf{e}^{T^c})$.

(iii) Finally, each player $j$ (with full satisfaction) takes his surplus of the Shapley value for arising his actions from $\sigma_0$ to $\sigma_1$ in the game $(\Gamma(\mathbf{e}^S \vee \mathbf{e}^{T^c}), v))$, say

$$(A.1) \quad [\phi_{1,j}(\Gamma(\mathbf{e}^S \vee \mathbf{e}^{T^c}), v) - \phi_{0,j}(\Gamma(\mathbf{e}^S \vee \mathbf{e}^{T^c}), v)] = \phi_{1,j}(\Gamma(\mathbf{e}^S \vee \mathbf{e}^{T^c}), v)],$$

from the new payoff $v(\mathbf{e}^S \vee \mathbf{e}^{T^c})$ and leave the rest to the players in $S$. Then a "reduced game" with respect to $\mathbf{e}^T$ and $\phi$, say

$$v^{\phi}_{\mathbf{e}^T}(\mathbf{e}^S) = v(\mathbf{e}^S \vee \mathbf{e}^{T^c}) - \sum_{j \in T^c} \phi_{1,j}(\Gamma(\mathbf{e}^S \vee \mathbf{e}^{T^c}), v)$$

is constructed for $\mathbf{e}^S \in \Gamma(\mathbf{e}^T)$.

Hart and Mas-Colell found an **elegant** property that the "new Shapley value" of player $j \in T$ in the reduced game $\phi_{1,j}(\Gamma(\mathbf{e}^T), v^{\phi}_{\mathbf{e}^T})$ equals the "original Shapley value" of player $j$ in the original game $\phi_{1,j}(\Gamma(\mathbf{e}^N), v)$. They call this property the consistency. If the new Shapley value for player $j$ were greater than the original Shapley value, then it is reasonable that he won't respect the original Shapley value.

Please **note** that in Hart and Mas-Colell's definition of reduced game if we replace the Shapley $\phi$ by another solution $\psi$ then $v^{\psi}_{\mathbf{e}^T}(\mathbf{0})$=$v(\mathbf{e}^{T^c})-\sum_{j\in S}\psi_{1,j}(\Gamma(\mathbf{e}^{T^c}),v)$ is not necessary zero, therefore we must impose $2^{|N|}-1$ additional assumptions, say $v^{\psi}_{\mathbf{e}^T}(\mathbf{0})=0$ for each $T\subseteq N$except $T=\emptyset$, to the definition of Hart and Mas-Colel1's consistency.

**Idea 6. A straight analogy to Hart and Mas-Colell's consistency**. Only "**without satisfaction**" or "**with full satisfaction** " is allowed. Suppose the H&R Shapley value will be used as the solution for a multi-choice cooperative game.

(i) For a multi-choice game $(\Gamma(\mathbf{m}),v)$ and its H&R Shapley value $\phi^w$, suppose a sub-group of players of $N$, say $T$ are not satisfied with their Shapley values, or say they doubt that their Shapley values are not fairly calculated among them(**without satisfaction**). Conversely, suppose each player $j\in T^c$ is satisfied with his Shapley value(**with full satisfaction**).

(ii) Then, from actions' viewpoints, for each $\mathbf{x}\leq\mathbf{m}^T$ where $m_j=0$ if $j\notin T$, each player in $j\in T^c$ helps the players in $T$ by raising his action level from $\sigma_0$ to $\sigma_{m_j}$, to form an new action vector $\mathbf{x}\vee\mathbf{m}^{T^c}$ and get a new payoff $v(\mathbf{x}\vee\mathbf{m}^{T^c})$.

(iii) Finally, each player(with full satisfaction) $j$ takes his surplus of the Shapley value for arising his actions from $\sigma_0$ to $\sigma_{m_j}$ in the game $(\Gamma(\mathbf{x}\vee\mathbf{m}^{T^c}),v))$, say

$$(A.2)\quad [\phi^w_{m_j,j}(\Gamma(\mathbf{x}\vee\mathbf{m}^{T^c}),v)-\phi^w_{0,j}(\Gamma(\mathbf{x}\vee\mathbf{m}^{T^c}),v)]=\phi^w_{1,j}(\Gamma(\mathbf{x}\vee\mathbf{m}^{T^c}),v)],$$

from the new payoff $v(\mathbf{x}\vee\mathbf{m}^{T^c})$ and leave the rest to the players in $T$. Then a "reduced game" with respect to $\mathbf{m}^T$ and $\phi^w$, say

$$v^{\phi^w}_{\mathbf{m}^T}(\mathbf{x})=v(\mathbf{x}\vee\mathbf{m}^{T^c})-\sum_{j\in T^c}\phi^w_{m_j,j}(\Gamma(\mathbf{x}\vee\mathbf{m}^{T^c}),v)$$

is constructed for $\mathbf{x}\in\Gamma(\mathbf{m}^T)$.

If for each $i\leq m_j$ and $j\in T$, the "new Shapley value" of player $j\in T$ in the reduced game $\phi^w_{i,j}(\Gamma(\mathbf{m}^T),v^{\phi^w}_{\mathbf{m}^T})$ equals the "original Shapley value" of player $j$ in the original game $\phi^w_{i,j}(\Gamma(\mathbf{m}),v)$, then $\phi^w$ is said to be consistent.

Please **note** that in the above reduced game if we replace the H&R Shapley value $\phi^w$ by another solution $\psi$ then $v^{\psi}_{\mathbf{e}^T}(\mathbf{0})$=$v(\mathbf{m}^{T^c})-\sum_{j\in T^c}\psi_{m_j,j}(\Gamma(\mathbf{m}^{T^c}),v)$ is not necessary zero.

However, since 1994, Hsiao tried to extend the consistency to even more general definition as the following.

**Idea 7. More general analogy to Hart and Mas-Colell's consistency**. A player "**without satisfaction**" or "**with partial satisfaction** " or "**with full satisfaction** " is allowed.

(i) For a multi-choice game $(\Gamma(\mathbf{m}), v)$ and its H&R Shapley value $\phi^w$, suppose a sub-group of players of $N$, say $T$ are not satisfied with their Shapley values, i.e. each player $j \in T$ is satisfied with none of $\phi^w_{i,j}(\Gamma(\mathbf{m}), v)$ where $0 < i \leq z_j = m_j$ **(without satisfaction)**. Suppose $T^c = P \cup F$ with $P \cap F = \emptyset$, such that player $j \in P$ is satisfied with none of his Shapley value $\phi^w_{i,j}(\Gamma(\mathbf{m}), v)$ where $0 < i \leq z_j$ and he is satisfied with his Shapley value $\phi^w_{i,j}(\Gamma(\mathbf{m}), v)$ where $z_j < i \leq m_j$**(with partial satisfaction)**. Also, suppose player $j \in F$ is satisfied with all his Shapley value $\phi^w_{i,j}(\Gamma(\mathbf{m}), v)$ where $z_j = 0 < i \leq m_j$**(with full satisfaction)**. In conclusion, player $j \in N$ is not satisfied with $\phi^w_{i,j}(\Gamma(\mathbf{m}), v)$ where $0 < i \leq z_j$. Also, in fact $\mathbf{z}^* = \mathbf{m}^{T^c}$.

(ii) Then, from actions' viewpoints, for each $\mathbf{y} \in \Gamma(\mathbf{m}^T)$ where $m_j = 0$ if $j \notin T$, each player in $j \in T^c$ helps the players in $T$ by raising his action level from $\sigma_{y_j}$ to $\sigma_{m_j}$), to form a new action vector $\mathbf{y} \vee \mathbf{m}^{T^c}$ and get a new payoff $v(\mathbf{y} \vee \mathbf{m}^{T^c})$.

(iii) Finally, each player(with partial or full satisfaction) $j$ takes his surplus of the Shapley value for arising his actions from $\sigma_{y_j}$ to $\sigma_{m_j}$ in the game $(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v))$, say

$$(A.3) \qquad [\phi^w_{m_j,j}(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v) - \phi^w_{y_j,j}(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v)],$$

from the new payoff $v(\mathbf{y} \vee \mathbf{m}^{T^c})$ and leave the rest to the players in $T \cup P$. Then a "reduced game" with respect to $\mathbf{z}$ and $\phi^w$, say

$$v^{\phi^w}_{\mathbf{z}}(\mathbf{y}) = v(\mathbf{y} \vee \mathbf{m}^{T^c}) - \sum_{j \in T^c} [\phi^w_{m_j,j}(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v) - \phi^w_{y_j,j}(\Gamma(\mathbf{y} \vee \mathbf{m}^{T^c}), v)]$$

is constructed for $\mathbf{y} \in \Gamma(\mathbf{z})$.

If for each $j$ with $z_j \neq 0$, i.e. $j \in T \cup P$ and each $i \leq z_j$, the "new Shapley value" of player $j$ in the reduced game $\phi^w_{i,j}(\Gamma(\mathbf{z}), v^{\phi^w}_{\mathbf{z}})$ equals the "original Shapley value" of player $j$ in the original game $\phi^w_{i,j}(\Gamma(\mathbf{m}), v)$, then $\phi^w$ is said to be consistent.

Please **note** that in the above reduced game if we replace the H&R Shapley value $\phi^w$ by another solution $\psi$ then $v^{\psi}_{\mathbf{m}^T}(\mathbf{0})$=$v(\mathbf{m}^{T^c}) - \sum_{j \in T^c} \psi_{m_j,j}(\Gamma(\mathbf{m}^{T^c}), v)$ is not necessary zero. We must impose some additional equations or make a different definition to characterize the H&R Shapley value.

**Idea 8.** (**The insights of the multi-choice games**). After Hsiao and Raghavan developed the multi-choice games in [3], many authors published their new results in the field of the multi-choice games such as multi-choice cores, multi-choice stable sets, multi-choice Weber sets, convex multi-choice games, multi-choice NTU games, multi-choice voting game, ...,etc.

However, what interests us more is the following. If we regard the action vector $\mathbf{x}$ as a status of an international coalition to fight against an international pandemic, each country may take different action according to the WHO (World Health Organization) phase of pandemic alert and her own pandemic situation. We regard $v(\mathbf{x})$ as the probability that the international action vector $\mathbf{x}$ is able to overcome the worldwide pandemic. Since a pandemic is a dynamic process, then $v(\mathbf{x})$ is a fuzzy number. Also, as the pandemic goes better or worse, each nation may change her action. Therefore, we are studying a dynamic process of action vector (status) formation with fuzzy payoff. The WHO may observe the dynamic process (by simulation) and help or warn some countries to raise their action levels before it is too late and the whole world suffers from the pandemic.

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