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THREE-SPACE PROBLEM FOR SOME APPROXIMATION PROPERTIES

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Abstract. Suppose that M is a closed subspace of a Banach space X such that $M^\perp$ is complemented in the dual space X^*, where $M^\perp = \{x^* \in X^* : x^*(m) = 0 \text{ for all } m \in M\}$. Godefroy and Saphar [4] study the three-space problem for the approximation properties on (X, M). In this paper, we extend some of their results and solve the three-space problem for the weak bounded approximation property on (X, M), which was introduced in Lima and Oja [10].

1. Introduction and Main Results

Let P be a property defined for Banach spaces and let M be a closed subspace of a Banach space X. Then we say that P is a *three-space property* on (X, M) if two of X, M, and the quotient space X/M have P, then the third must also have P. The *three-space problem* for P on (X, M) is whether P is a *three-space property* on (X, M). In this paper, we solve the three-space problem for some approximation properties on (X, M) for the case when $M^\perp$ is complemented in the dual space X^*. The results extend [4, Theorem 2.4] and solve the three-space problem for the *weak bounded approximation property* of Banach spaces on (X, M), which is a variant of the *bounded approximation property* (see Section 2), for the case when $M^\perp$ is complemented in the dual space X^*.

Throughout this paper, X and Y are Banach spaces. We denote by τ the *topology of compact convergence* on $\mathcal{B}(X, Y)$, the space of bounded linear operators from X into Y, which is strictly weaker than the operator norm topology; for a net (T_α) and T in $\mathcal{B}(X, Y)$,

$$T_\alpha \xrightarrow{\tau} T \quad \text{if and only if} \quad \sup_{x \in K} \|T_\alpha x - Tx\| \longrightarrow 0$$

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for each compact $K \subset X$; for $\mathcal{S} \subset \mathcal{B}(X,Y)$ and $T \in \mathcal{B}(X,Y)$,

$$T \in \overline{\mathcal{S}}^{\tau} \text{ if and only if for each compact } K \subset X \text{ and } \varepsilon > 0,$$
$$\text{there exists a } R \in \mathcal{S} \text{ so that } \sup_{x \in K} \|Rx - Tx\| < \varepsilon.$$

We denote by $\mathcal{A}(X,Y)$ a subspace of $\mathcal{B}(X,Y)$ such that $\mathcal{A}$ satisfies the ideal property and that the adjoint $T^* \in \mathcal{A}(Y^*,X^*)$ for all $T \in \mathcal{A}(X,Y)$. For example, $\mathcal{B}(X,Y)$, $\mathcal{W}(X,Y)$, $\mathcal{K}(X,Y)$, $\mathcal{F}(X,Y)$, respectively, the spaces of bounded, weakly compact, compact, and finite rank linear operators between X and Y satisfy the above properties. Let $\mathcal{A}(X,Y;\lambda) = \{T \in \mathcal{A}(X,Y) : \|T\| \le \lambda\}$, $\mathcal{A}^*(X,Y) = \{T^* : T \in \mathcal{A}(X,Y)\}$, and $\mathcal{A}^*(X,Y;\lambda) = \{T^* \in \mathcal{A}^*(X,Y) : \|T\| \le \lambda\}$.

We now define the following; X is said to have the *$\mathcal{A}$-approximation property* ($\mathcal{A}$-AP) if for every Banach space Y $\mathcal{A}(X,Y) \subset \overline{\mathcal{F}(X,Y)}^{\tau}$, for $\lambda \ge 1$ X is said to have the *$\mathcal{A}$-λ-bounded approximation property* ($\mathcal{A}$-λ-BAP) if for every Banach space Y $\mathcal{A}(X,Y;1) \subset \overline{\mathcal{F}(X,Y;\lambda)}^{\tau}$, X is said to have the *$\mathcal{A}$-bounded approximation property* ($\mathcal{A}$-BAP) if for every Banach space Y and $T \in \mathcal{A}(X,Y)$ there exists a $\lambda_T > 0$ so that $T \in \overline{\mathcal{F}(X,Y;\lambda_T)}^{\tau}$. We clearly have the following implications;

$$\mathcal{A} - \lambda - BAP \Longrightarrow \mathcal{A} - BAP \Longrightarrow \mathcal{A} - AP.$$

In Section 2, we will see that the well known approximation properties (AP, BAP, λ-BAP) and their recent variants are contained in the above definitions.

If $M^{\perp}$ is complemented in X^*, then there exists a projection $P : X^* \longrightarrow M^{\perp}$ onto $M^{\perp}$. Define a map $U : M^* \longrightarrow X^*$ by

$$Um^* = x^* - Px^*$$

where $x^* \in X^*$ with $x^* = m^*$ on M. Then we see that U is a well defined bounded operator and

$$(Um^*)m = m^*m$$

for all $m^* \in M^*$ and $m \in M$.

We are now ready to state the main results of this paper, which are proved in Section 4.

Theorem 1.1. Suppose that M is a closed subspace of X and $M^{\perp}$ is complemented in X^*.

(a) If X has the $\mathcal{A}$-AP, then M has the $\mathcal{A}$-AP.

(b) If X has the $\mathcal{A}$-BAP, then M has the $\mathcal{A}$-BAP.

(c) If X has the $\mathcal{A}$-λ-BAP, then M has the $\mathcal{A}$-$\lambda\|U\|$-BAP.

Theorem 1.2. Suppose that M is a closed subspace of X and $M^\perp$ is complemented in X^*.

(a) If M has the $\mathcal{A}$-BAP and X/M has the AP, then X has the $\mathcal{A}$-AP.

(b) If M has the $\mathcal{A}$-BAP and X/M has the BAP, then X has the $\mathcal{A}$-BAP.

(c) If M has the $\mathcal{A}$-μ-BAP and X/M has the λ-BAP, then X has the $\mathcal{A}$-$\mu\|U\| + (\mu\|U\|+1)\lambda$-BAP.

In Section 4, if $\mathcal{A} = \mathcal{W}$ in Theorem 1.2, then the assumptions of X/M are weakened.

Theorem 1.3. There exist a Banach space Y and a closed subspace M of Y so that $M^\perp$ is complemented in Y^*, Y has the 1-BAP, M has the BAP, but $\mathcal{K}(Y/M, Y/M) \not\subset \overline{\mathcal{F}(Y/M, Y/M)}^\tau$.

2. The Approximation Properties and the Weak Bounded Approximation Property

X is said to have the *approximation property* (AP) if $I_X \in \overline{\mathcal{F}(X,X)}^\tau$, where I_X is the identity operator on X. For $\lambda \geq 1$, X is said to have the λ-*bounded approximation property* (λ-BAP) if $I_X \in \overline{\mathcal{F}(X,X;\lambda)}^\tau$. We also say that X has the *bounded approximation property* (BAP) if X has the λ-BAP for some $\lambda \geq 1$. Grothendieck [3] systematically investigated the AP and showed the following. For a concrete proof one may see Casazza [1, Proposition 2.4] or Choi and Kim [2, Lemma 3.1].

Fact. $(\mathcal{B}(X,Y),\tau)^*$ consists of all functionals f of the form $f(T) = \sum_n y_n^*(Tx_n)$, where $(x_n) \subset X, (y_n^*) \subset Y^*$, and $\sum_n \|x_n\|\|y_n^*\| < \infty$.

Simple calculations show that X has the AP if and only if X has the $\mathcal{B}$-AP, and X has the λ-BAP if and only if X has the $\mathcal{B}$-λ-BAP (cf. [1, Theorems 2.5 and 3.14]). Also if X has the BAP, clearly X has the $\mathcal{B}$-BAP, and if X has the $\mathcal{B}$-BAP, then X has the BAP since for some $\lambda_{I_X} > 0$ $I_X \in \overline{\mathcal{F}(X,X;\lambda_{I_X})}^\tau$.

We now introduce the recent variant of the BAP. In [10], the authors introduced and investigated the following properties; for $\lambda \geq 1$, X is said to have the *weak λ-bounded approximation property* (weak λ-BAP) if for every Banach space Y and $T \in \mathcal{W}(X,Y)$, there exists a net $(S_\alpha) \subset \mathcal{F}(X,X)$ with $\sup_\alpha \|TS_\alpha\| \leq \lambda\|T\|$ such that

$$S_\alpha \xrightarrow{\tau} I_X.$$

We say that X has the *weak bounded approximation property* (weak BAP) if X has the weak λ-BAP for some $\lambda \geq 1$. In [10], interesting criteria of the above properties are established. Using the criteria, we will show that weak λ-BAP and $\mathcal{W}$-λ-BAP are equivalent.

Recall that for every Banach space Z and W $(Z\widehat{\otimes}_\pi W)^* = \mathcal{B}(Z, W^*)$ and for a net (T_α) and T in $\mathcal{B}(Z, W^*)$

$$T_\alpha \overset{weak^*}{\longrightarrow} T \text{ if and only if } \sum_n (T_\alpha z_n) w_n \longrightarrow \sum_n (T z_n) w_n$$

for every $(z_n) \subset Z$ and $(w_n) \subset W$ with $\sum_n \|z_n\|\|w_n\| < \infty$ (cf. Ryan [13, p. 24]).

Proposition 2.1. *The following are equivalent.*

(a) *X has the weak λ-BAP.*

(b) *X has the $\mathcal{W}$-λ-BAP.*

(c) *For every reflexive Banach space Y, $\mathcal{W}(X, Y; 1) \subset \overline{\mathcal{F}(X, Y; \lambda)}^{\tau}$.*

(d) *For every reflexive Banach space Y, the trace mapping $V : X\widehat{\otimes}_\pi Y^* \longrightarrow \mathcal{F}(X, Y)^*$ satisfies $\|u\|_\pi \leq \lambda\|Vu\| \leq \lambda\|u\|_\pi$ for every $u \in X\widehat{\otimes}_\pi Y^*$.*

Proof. We show (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d) $\Longrightarrow$(a). But (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(c) are clear.

(c)$\Longrightarrow$(d) From the definitions of the projective tensor product and the trace mapping we see $\|Vu\| \leq \|u\|_\pi$. Let Y be a reflexive Banach space and let $u = \sum_n x_n \otimes y_n^* \in X\widehat{\otimes}_\pi Y^*$. Since Y is reflexive, $(X\widehat{\otimes}_\pi Y^*)^* = \mathcal{B}(X, Y^{**}) = \mathcal{W}(X, Y^{**}) = \mathcal{W}(X, Q_Y(Y))$, where $Q_Y : Y \longrightarrow Y^{**}$ is the natural map. Then there exists $T \in \mathcal{W}(X, Q_Y(Y); 1)$ so that

$$\|u\|_\pi = \sum_n (Tx_n) y_n^* = \sum_n y_n^*(Q_Y^{-1} T x_n).$$

By the assumption $Q_Y^{-1}T \in \overline{\mathcal{F}(X, Y; \lambda)}^{\tau}$. Let $\sup_{S\in\mathcal{F}(X,Y;\lambda)} |\sum_n y_n^*(Sx_n)| = t$. Then for every $S \in \mathcal{F}(X, Y; \lambda)$, $|\sum_n y_n^*((1/t)Sx_n)| \leq 1$. So for every $R \in \mathcal{F}(X, Y; (1/t)\lambda)$, $|\sum_n y_n^*(Rx_n)| \leq 1$. Since $(1/t)Q_Y^{-1}T \in \overline{\mathcal{F}(X, Y; (1/t)\lambda)}^{\tau}$, by **Fact**

$$\frac{1}{t}\sum_n (Tx_n) y_n^* = \frac{1}{t}\sum_n y_n^*(Q_Y^{-1}Tx_n) = \sum_n y_n^*(\frac{1}{t}Q_Y^{-1}Tx_n) \leq 1.$$

Hence we have

$$\begin{aligned}\|u\|_\pi &= \sum_n (Tx_n)y_n^* \\ &\le t \\ &= \sup_{S\in\mathcal{F}(X,Y;\lambda)} \Big|\sum_n y_n^*(Sx_n)\Big| \\ &= \lambda \sup_{R\in\mathcal{F}(X,Y;1)} \Big|\sum_n y_n^*(Rx_n)\Big| \\ &= \lambda \sup_{R\in\mathcal{F}(X,Y;1)} |(Vu)(R)| \\ &= \lambda\|Vu\|.\end{aligned}$$

(d)$\Longrightarrow$(a) See the proof of [10, Theorem 3.2]. ■

If X has the λ-BAP, then X has the weak λ-BAP, and in [10], the authors conjectured that the converse does not hold in general. Recently, Oja [12, Theorem 3.6] showed that X has the weak λ-BAP if and only if X has the $\mathcal{K}$-λ-BAP. More recently, Kim [6] showed that X has the $\mathcal{K}$-AP if and only if X has the AP ($\mathcal{B}$-AP).

The following lemma is in Lima, Nygaard, and Oja [9, Theorem 2.2].

Lemma 2.2. *If $T \in \mathcal{W}(X,Y)$ (resp. $\mathcal{K}(X,Y)$), then there exist a reflexive Banach space Z, $S \in \mathcal{W}(X,Z)$ (resp. $\mathcal{K}(X,Z)$), and $J \in \mathcal{W}(Z,Y)$ (resp. $\mathcal{K}(Z,Y)$) such that $\|J\| = 1$, $T = JS$, and $\|S\| = \|T\|$.*

Proposition 2.3.

(a) *X has the $\mathcal{W}$-BAP if and only if for every reflexive Banach space Y and $T \in \mathcal{W}(X,Y)$ there exists $\lambda_T > 0$ so that $T \in \overline{\mathcal{F}(X,Y;\lambda_T)}^{\tau}$.*

(b) *X has the $\mathcal{K}$-BAP if and only if for every reflexive Banach space Y and $T \in \mathcal{K}(X,Y)$ there exists $\lambda_T > 0$ so that $T \in \overline{\mathcal{F}(X,Y;\lambda_T)}^{\tau}$.*

Proof. Since the proofs of (a) and (b) are the same, we only prove (a). Also we only need to prove the "if" part. Let Y be a Banach space and $T \in \mathcal{W}(X,Y)$. By Lemma 2.2 there exist a reflexive Banach space Z, $S \in \mathcal{W}(X,Z)$, and $J \in \mathcal{W}(Z,Y)$ such that $\|J\| = 1$, $T = JS$, and $\|S\| = \|T\|$. Then by the assumption there exist a $\lambda_S > 0$ and a net $(S_\alpha) \subset \mathcal{F}(X,Z;\lambda_S)$ such that

$$S_\alpha \xrightarrow{\tau} S.$$

Consider $(JS_\alpha) \subset \mathcal{F}(X,Y;\lambda_S)$. Then clearly $JS_\alpha \xrightarrow{\tau} JS = T$. Hence X has the $\mathcal{W}$-BAP. ■

Using [8, Proposition 1], in Propositions 2.1(c) and 2.3(a)(b), every reflexive Banach space Y can be replaced by every separable reflexive Banach space Y.

In [12, Conjecture 3.7], the author conjectured that there exists a Banach space X which satisfies the $\mathcal{K}$-BAP (the *strong approximation property*) but fails the $\mathcal{K}$-λ-BAP ($\mathcal{W}$-λ-BAP) for any $\lambda \geq 1$. But we don't know whether $\mathcal{K}$-BAP implies $\mathcal{W}$-BAP.

3. Main Tools of Proofs of Theorems 1.1 and 1.2

In this section we establish some equivalent conditions and sufficient conditions, which are main tools of proofs of Theorems 1.1 and 1.2, to have the $\mathcal{A}$-approximation properties. The following lemma is deduced by [5, Proposition 3.1].

Lemma 3.1. *For every Banach space X, Y, and $\lambda > 0$, $\overline{\mathcal{F}(Y^*, X^*; \lambda)}^{weak^*} = \overline{\mathcal{F}^*(X, Y; \lambda)}^{weak^*}$.*

We introduce a topology induced by a subspace of $\mathcal{B}(X,Y)^\sharp$, the vector space of all linear functionals on $\mathcal{B}(X,Y)$. Let $\mathcal{Z}$ be the linear span of all linear functionals φ on $\mathcal{B}(X,Y)$ of the form

$$\varphi(T) = \sum_n y_n^*(Tx_n)$$

for $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$, which is also the form of elements of $\mathcal{Z}$. Then the *summable weak operator topology* (swo) on $\mathcal{B}(X,Y)$ is the topology induced by $\mathcal{Z}$. From Megginson [11, Proposition 2.4.4 and Theorem 2.4.11] we have that

$$T_\alpha \xrightarrow{swo} T \quad \text{if and only if} \quad \sum_n y_n^*(T_\alpha x_n) \longrightarrow \sum_n y_n^*(Tx_n)$$

for every $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$, swo is a locally convex vector topology, and the dual space of $\mathcal{B}(X,Y)$ with respect to swo is $\mathcal{Z}$. From [11, Corollary 2.2.29] and **Fact** in Section 2 we have

Lemma 3.2. *For every convex set $\mathcal{C}$ in $\mathcal{B}(X,Y)$ $\overline{\mathcal{C}}^\tau = \overline{\mathcal{C}}^{swo}$.*

We now establish simple equivalent conditions of the $\mathcal{A}$-approximation properties. Since the proofs of Propositions 3.3, 3.4, and 3.5 are the same, we only prove Proposition 3.4.

Proposition 3.3. *The following are equivalent.*

(a) X *has the* $\mathcal{A}$*-AP.*

(b) For every Banach space Y, $\mathcal{A}(X, Y^{**}) \subset \overline{\mathcal{F}(X, Y^{**})}^{\tau}$.

(c) For every Banach space Y, $\mathcal{A}(X, Y^{**}) \subset \overline{\mathcal{F}(X, Y^{**})}^{weak^*}$.

(d) For every Banach space Y, $\mathcal{A}(Y^*, X^*) \subset \overline{\mathcal{F}^*(X, Y)}^{weak^*}$.

Proposition 3.4. *The following are equivalent.*

(a) X *has the* $\mathcal{A}$*-*λ*-BAP.*

(b) For every Banach space Y, $\mathcal{A}(X, Y^{**}; 1) \subset \overline{\mathcal{F}(X, Y^{**}; \lambda)}^{\tau}$.

(c) For every Banach space Y, $\mathcal{A}(X, Y^{**}; 1) \subset \overline{\mathcal{F}(X, Y^{**}; \lambda)}^{weak^*}$.

(d) For every Banach space Y, $\mathcal{A}(Y^*, X^*; 1) \subset \overline{\mathcal{F}^*(X, Y; \lambda)}^{weak^*}$.

Proof. We show (a)$\Longrightarrow$(b)$\Longrightarrow$(c)$\Longrightarrow$(d)$\Longrightarrow$(a). But (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(c) are clear.

(c)$\Longrightarrow$(d) Let Y be a Banach space and $T \in \mathcal{A}(Y^*, X^*; 1)$. Then $T^*Q_X \in \mathcal{A}(X, Y^{**}; 1)$. By the assumption $T^*Q_X \in \overline{\mathcal{F}(X, Y^{**}; \lambda)}^{weak^*}$. Thus there exists a net $(S_\alpha) \subset \mathcal{F}(X, Y^{**}; \lambda)$ so that for every $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$

$$\sum_n (S_\alpha^* Q_{Y^*}(y_n^*))(x_n) = \sum_n (S_\alpha x_n)(y_n^*) \longrightarrow \sum_n (T^*Q_X x_n)(y_n^*) = \sum_n (Ty_n^*)(x_n).$$

Since $(S_\alpha^* Q_{Y^*}) \subset \mathcal{F}(Y^*, X^*; \lambda)$, by Lemma 3.1 $T \in \overline{\mathcal{F}(Y^*, X^*; \lambda)}^{weak^*} = \overline{\mathcal{F}^*(X, Y; \lambda)}^{weak^*}$.

(d)$\Longrightarrow$(a) Let Y be a Banach space and $T \in \mathcal{A}(X, Y; 1)$. Then by the assumption $T^* \in \overline{\mathcal{F}^*(X, Y; \lambda)}^{weak^*}$. From the definitions of swo and $weak^*$ we see $T \in \overline{\mathcal{F}(X, Y; \lambda)}^{swo}$. Hence X has the $\mathcal{A}$-λ-BAP from Lemma 3.2. ∎

Proposition 3.5. *The following are equivalent.*

(a) X *has the* $\mathcal{A}$*-BAP.*

(b) *For every Banach space* Y *and* $T \in \mathcal{A}(X, Y^{**})$*, there exists a* $\lambda_T > 0$ *so that* $T \in \overline{\mathcal{F}(X, Y^{**}; \lambda_T)}^{\tau}$.

(c) *For every Banach space* Y *and* $T \in \mathcal{A}(X, Y^{**})$*, there exists a* $\lambda_T > 0$ *so that* $T \in \overline{\mathcal{F}(X, Y^{**}; \lambda_T)}^{weak^*}$.

(d) *For every Banach space* Y *and* $T \in \mathcal{A}(Y^*, X^*)$*, there exists a* $\lambda_T > 0$ *so that* $T \in \overline{\mathcal{F}^*(X, Y; \lambda_T)}^{weak^*}$.

Next we establish some sufficient conditions to have the $\mathcal{A}$-approximation properties using Propositions 3.3, 3.4, and 3.5. Since the proofs of Propositions 3.6, 3.8, and 3.10 are the same, we only prove Proposition 3.8.

Proposition 3.6. *Let M be a closed subspace of X. Suppose that X/M has the $\mathcal{B}$-AP. If for every Banach space Y and $T \in \mathcal{A}(X,Y)$ there exist $\lambda_T > 0$ and a net (S_α) in $\mathcal{F}(Y^*, X^*; \lambda_T)$ so that $(S_\alpha y^*)m \longrightarrow y^*(Tm)$ for every $m \in M$ and $y^* \in Y^*$, then X has the $\mathcal{A}$-AP.*

Corollary 3.7. *Let M be a closed subspace of X. Suppose that X/M has the $\mathcal{W}$-AP. If for every reflexive Banach space Y and $T \in \mathcal{A}(X,Y)$ there exist $\lambda_T > 0$ and a net (S_α) in $\mathcal{F}(Y^*, X^*; \lambda_T)$ so that $(S_\alpha y^*)m \longrightarrow y^*(Tm)$ for every $m \in M$ and $y^* \in Y^*$, then for every reflexive Banach space Y $\mathcal{A}(X,Y) \subset \overline{\mathcal{F}(X,Y)}^{\tau}$.*

Proposition 3.8. *Let M be a closed subspace of X. Suppose that X/M has the $\mathcal{B}$-λ-BAP. If for every Banach space Y and $T \in \mathcal{A}(X,Y;1)$ there exists a net (S_α) in $\mathcal{F}(Y^*, X^*; \mu)$ so that $(S_\alpha y^*)m \longrightarrow y^*(Tm)$ for every $m \in M$ and $y^* \in Y^*$, then X has the $\mathcal{A}$-$\mu + (\mu+1)\lambda$-BAP.*

Proof. Let Y be a Banach space and $T \in \mathcal{A}(X,Y;1)$. Then by the assumption there exists a net (S_α) in $\mathcal{F}(Y^*, X^*; \mu)$ so that $(S_\alpha y^*)m \longrightarrow y^*(Tm)$ for every $m \in M$ and $y^* \in Y^*$. Since (S_α) is a bounded net in $\mathcal{B}(Y^*, X^*)$, by the Banach-Alaoglu theorem there exists a subnet (S_β) of (S_α) such that $S_\beta \overset{weak^*}{\longrightarrow} S$ for some $S \in \mathcal{B}(Y^*, X^*)$. Therefore $\|S\| \le \liminf_\beta \|S_\beta\| \le \mu$ and

$$\sum_n (S_\beta y_n^*)x_n \longrightarrow \sum_n (S y_n^*)(x_n)$$

for every $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ with $\sum_n \|x_n\|\|y_n^*\| < \infty$.

In particular, $(S_\beta y^*)m \longrightarrow (Sy^*)m$ for every $m \in M$ and $y^* \in Y^*$, hence $(Sy^*)m = y^*(Tm)$. Thus if we let $R = S - T^*$, then $R : Y^* \longrightarrow M^\perp$ is a well-defined bounded operator. Let $\varphi_M : (X/M)^* \longrightarrow M^\perp$ be the isometry defined by $(\varphi_M z^*)x = z^*\pi(x)$ for every $z^* \in (X/M)^*$ and $x \in X$, where $\pi : X \longrightarrow X/M$ is the quotient operator (cf. [11, Theorem 1.10.17]). Then $\varphi_M^{-1} R \in \mathcal{B}(Y^*, (X/M)^*; \mu + 1)$. Since X/M has the $\mathcal{B}$-λ-BAP, by Proposition 3.4(d) $\varphi_M^{-1} R \in \overline{\mathcal{F}^*(X/M, Y; (\mu+1)\lambda)}^{weak^*}$. So there exists a net $(T_\gamma^*) \subset \mathcal{F}^*(X/M, Y; (\mu+1)\lambda)$ so that

$$(\dagger) \qquad \sum_n (T_\gamma^* y_n^*)(x_n + M) \longrightarrow \sum_n (\varphi_M^{-1} R y_n^*)(x_n + M)$$

for every $(x_n + M) \subset X/M$ and $(y_n^*) \subset Y^*$ satisfying $\sum_n \|x_n + M\|\|y_n^*\| < \infty$.

Now consider the net $(I_{M^\perp}\varphi_M T_\gamma^*) \subset \mathcal{F}(Y^*, X^*; (\mu+1)\lambda)$, where $I_{M^\perp} : M^\perp \longrightarrow X^*$ is the inclusion map, and assume that $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ satisfy $\sum_n \|x_n\|\|y_n^*\| < \infty$. From (†) we have

$$\begin{aligned}&\sum_n (I_{M^\perp}\varphi_M T_\gamma^* y_n^*)x_n \\ &= \sum_n (T_\gamma^* y_n^*)(x_n + M) \longrightarrow \sum_n (\varphi_M^{-1} R y_n^*)(x_n + M) = \sum_n (R y_n^*)x_n.\end{aligned}$$

Let $U_\gamma = I_{M^\perp}\varphi_M T_\gamma^* \in \mathcal{F}(Y^*, X^*; (\mu+1)\lambda)$. Then we have shown that $S_\beta \overset{weak^*}{\longrightarrow} S$ and $U_\gamma \overset{weak^*}{\longrightarrow} R$. We can now find a net $(V_\delta) \subset \mathcal{F}(Y^*, X^*; \mu+(\mu+1)\lambda)$ such that

$$V_\delta \overset{weak^*}{\longrightarrow} S - R = T^*.$$

Therefore from Lemma 3.1

$$T^* \in \overline{\mathcal{F}(Y^*, X^*; \mu + (\mu+1)\lambda)}^{weak^*} = \overline{\mathcal{F}^*(X, Y; \mu + (\mu+1)\lambda)}^{weak^*}.$$

It from Lemma 3.2 follows that $T \in \overline{\mathcal{F}(X, Y; \mu + (\mu+1)\lambda)}^{swo} = \overline{\mathcal{F}(X, Y; \mu +(\mu+1)\lambda)}^{\tau}$. Hence X has the $\mathcal{A}$-$\mu + (\mu+1)\lambda$-BAP. ■

In the proof of Proposition 3.8, if Y is reflexive, then $\varphi_M^{-1} R \in \mathcal{W}(Y^*, (X/M)^*; \mu+1)$. Hence the assumption of X/M can be weakened.

Corollary 3.9. *Let M be a closed subspace of X. Suppose that X/M has the $\mathcal{W}$-λ-BAP. If for every reflexive Banach space Y and $T \in \mathcal{A}(X, Y; 1)$ there exists a net (S_α) in $\mathcal{F}(Y^*, X^*; \mu)$ so that $(S_\alpha y^*)m \longrightarrow y^*(Tm)$ for every $m \in M$ and $y^* \in Y^*$, then for every reflexive Banach space Y, $\mathcal{A}(X, Y; 1) \subset \overline{\mathcal{F}(X, Y; \mu + (\mu+1)\lambda)}^{\tau}$.*

Proposition 3.10. *Let M be a closed subspace of X. Suppose that X/M has the $\mathcal{B}$-BAP. If for every Banach space Y and $T \in \mathcal{A}(X, Y)$ there exists $\lambda_T > 0$ and a net (S_α) in $\mathcal{F}(Y^*, X^*; \lambda_T)$ so that $(S_\alpha y^*)m \longrightarrow y^*(Tm)$ for every $m \in M$ and $y^* \in Y^*$, then X has the $\mathcal{A}$-BAP.*

Corollary 3.11. *Let M be a closed subspace of X. Suppose that X/M has the $\mathcal{W}$-BAP. If for every reflexive Banach space Y and $T \in \mathcal{A}(X, Y)$ there exists $\lambda_T > 0$ and a net (S_α) in $\mathcal{F}(Y^*, X^*; \lambda_T)$ so that $(S_\alpha y^*)m \longrightarrow y^*(Tm)$ for every $m \in M$ and $y^* \in Y^*$, then for every reflexive Banach space Y and $T \in \mathcal{A}(X, Y)$ there exists $\lambda_T > 0$ so that $T \in \overline{\mathcal{F}(X, Y; \lambda_T)}^{\tau}$.*

4. Proofs of Main Results

Since the proofs of Theorem 1.1(a), (b), and (c) are the same, we only prove Theorem 1.1(c).

Proof of Theorem 1.1(c). Let Y be a Banach space. Since X has the $\mathcal{A}$-λ-BAP, by Proposition 3.4(d) $\mathcal{A}(Y^*, X^*; 1) \subset \overline{\mathcal{F}^*(X, Y; \lambda)}^{weak^*}$. Now let $T \in \mathcal{A}(M, Y; 1)$. Then $UT^* \in \mathcal{A}(Y^*, X^*; \|U\|) \subset \overline{\mathcal{F}^*(X, Y; \lambda\|U\|)}^{weak^*}$. Thus there exists a net $(T_\alpha^*) \subset \mathcal{F}^*(X, Y; \lambda\|U\|)$ so that $T_\alpha^* \overset{weak^*}{\longrightarrow} UT^*$. That is,

$$\sum_n y_n^*(T_\alpha x_n) \longrightarrow \sum_n (UT^* y_n^*) x_n$$

for every $(x_n) \subset X$ and $(y_n^*) \subset Y^*$ satisfying $\sum_n \|x_n\|\|y_n^*\| < \infty$.

Consider the net $(T_\alpha I_M) \subset \mathcal{F}(M, Y; \lambda\|U\|)$, where $I_M : M \longrightarrow X$ is the inclusion map, and assume that sequences $(m_n) \subset M$ and $(y_n^*) \subset Y^*$ satisfy $\sum_n \|m_n\|\|y_n^*\| < \infty$. Then we have

$$\begin{aligned} &\sum_n y_n^*(T_\alpha I_M m_n) = \sum_n y_n^*(T_\alpha m_n) \longrightarrow \sum_n (UT^* y_n^*)(m_n) \\ &= \sum_n (T^* y_n^*)(m_n) = \sum_n y_n^*(T m_n). \end{aligned}$$

Therefore $T \in \overline{\mathcal{F}(M, Y; \lambda\|U\|)}^{swo} = \overline{\mathcal{F}(M, Y; \lambda\|U\|)}^{\tau}$ from Lemma 3.2. Hence M has the $\mathcal{A}$-$\lambda\|U\|$-BAP. ■

Since the proofs of Theorem 1.2(a), (b), and (c) are the same, we only prove Theorem 1.2(c).

Proof of Theorem 1.2(c). Let Y be a Banach space and $T \in \mathcal{A}(X, Y; 1)$. Since $TI_M \in \mathcal{A}(M, Y; 1)$ and M has the $\mathcal{A}$-μ-BAP, there exists a net $(S_\alpha) \subset \mathcal{F}(M, Y; \mu)$ so that for every $m \in M$ and $y^* \in Y^*$

$$y^*(S_\alpha m) \longrightarrow y^*(TI_M m) = y^*(Tm).$$

We consider the net $(US_\alpha^*) \subset \mathcal{F}(Y^*, X^*; \mu\|U\|)$. Then for every $m \in M$ and $y^* \in Y^*$

$$(US_\alpha^* y^*)m = (S_\alpha^* y^*)m \longrightarrow y^*(Tm).$$

Hence, by Proposition 3.8 X has the $\mathcal{A}$-$\mu\|U\| + (\mu\|U\| + 1)\lambda$-BAP. ■

Proof of Theorem 1.3. Let W be the Willis space [14] which is separable and reflexive. Then $\mathcal{K}(W, W) \not\subset \overline{\mathcal{F}(W, W)}^{\tau}$ (cf. [2, Example 2.3]). By a result of Lindenstrauss [7] (cf. [1, Proposition 1.3]) there exists a Banach space Z with a basis so that Z^{**} has a basis and $Z^{**}/Q_Z(Z)$ is isomorphic to W. Observe that $Q_Z(Z)^\perp$ is complemented in Z^{***} (cf. [11, Exercise 3.23]). Then Z has the BAP and so $Q_Z(Z)$ has the BAP. Also Z^{**} has the 1-BAP (cf. [1, Theorem 3.6]) because it is a separable dual space. But $\mathcal{K}(Z^{**}/Q_Z(Z), Z^{**}/Q_Z(Z)) \not\subset$

$\overline{\mathcal{F}(Z^{**}/Q_Z(Z), Z^{**}/Q_Z(Z))}^{\tau}$. If we let $Y = Z^{**}$ and $M = Q_Z(Z)$, then we complete the proof. ■

In Theorems 1.1 and 1.2, if we let $\mathcal{A} = \mathcal{B}$, then we have [4, Theorem 2.4] and from Propositions 2.1, 2.3, Corollaries 3.7, 3.9, 3.11, and the proofs of Theorems 1.1 and 1.2, we have

Theorem 4.1. *Suppose that M is a closed subspace of X and $M^{\perp}$ is complemented in X^*.*

(a) *If X has the weak λ-BAP, then M has the weak $\lambda\|U\|$-BAP.*
(b) *If X has the $\mathcal{W}$-BAP, then M has the $\mathcal{W}$-BAP.*
(c) *If M has the $\mathcal{W}$-BAP and X/M has the $\mathcal{W}$-AP, then X has the $\mathcal{W}$-AP (AP).*
(d) *If M has the weak μ-BAP and X/M has the weak λ-BAP, then X has the weak $\mu\|U\| + (\mu\|U\| + 1)\lambda$-BAP.*
(e) *If M has the $\mathcal{W}$-BAP and X/M has the $\mathcal{W}$-BAP, then X has the $\mathcal{W}$-BAP.*

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