TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 1, pp. 195-211, February 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

VOLTERRA TYPE OPERATORS ON Q_K SPACES

Songxiao Li and Hasi Wulan

Abstract. The boundness of Volterra type operators on Q_K space is investigated in this paper. A new generalized Carleson measure and Logarithmic Q_K spaces has been introduced and studied. In addition, we give a new characterization of Q_K space and $Q_{K,0}$ space.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk of complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. Denote by $H(\mathbb{D})$ the class of functions analytic in $\mathbb{D}$. For $a \in \mathbb{D}$, $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$ is the Green function in $\mathbb{D}$, where $\varphi_a(z) = (a - z)/(1 - \overline{a}z)$ is the Mӧbius map of $\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space $\mathcal{B}$ if $B(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty$. The expression $B(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}} = |f(0)| + B(f)$. The norm makes $\mathcal{B}$ into a conformally invariant Banach space.

For any nonnegative, nondecreasing and Lebesgue measurable function $K : (0,\infty) \to [0,\infty)$, we say that f belongs to the space Q_K if

$$\|f\|^2_{Q_K} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a))dA(z) < \infty, \tag{1}$$

where dA is an area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. It is easy to see that Q_K is Mӧbius invariant, that is,

$$\|f \circ \varphi_a\|_{Q_K} = \|f\|_{Q_K},$$

whenever $f \in Q_K$ and $a \in \mathbb{D}$.

Received May 29, 2007, accepted April 11, 2008.
Communicated by Der-Chen Chang.
2000 *Mathematics Subject Classification*: Primary 47B38, Secondary 30H05.
Key words and phrases: Volterra type operator, Q_K space, Carleson measure.

The space $Q_{K,0}$ consists of analytic functions f on $\mathbb{D}$ for which

$$\lim_{|a|\to 1}\int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))dA(z) = 0. \tag{2}$$

It is easy to see that $Q_{K,0}$ is a closed subspace in Q_K. We know that the Green function $g(a,z)$ in (1) and (2) can be replaced by the expression $1-|\varphi_a(z)|^2$ (see [8]).

For $0<p<\infty$, $K(t)=t^p$ gives the space Q_p. $K(t)=1$ gives the Dirichlet space $\mathcal{D}$. For more results on Q_p spaces and Q_K spaces, see [5, 6, 8, 9, 19-22].

If the function K is only defined on $(0,1]$, then we extend it to $(0,\infty)$ by setting $K(t)=K(1)$ for $t>1$. We define an auxiliary function (see [9] or [21]) as follow:

$$\varphi_K(s) = \sup_{0<t\le 1}\frac{K(st)}{K(t)},\ 0<s<\infty. \tag{3}$$

We assume that K is continuous and nondecreasing on $(0,1]$. This ensures that the function φ_K is continuous and nondecreasing on $(0,\infty)$. Moreover we need the following constraints on K: $\varphi_K(2)<\infty$ and

$$\int_0^1 \varphi_K(s)\frac{ds}{s} < \infty. \tag{4}$$

Suppose that $f,g\in H(\mathbb{D})$. A class of integral operators introduced by Pommerenke in [16] is defined by

$$J_g f(z) = \int_0^z f(\xi)g'(\xi)d\xi, \qquad z\in\mathbb{D}. \tag{5}$$

We call J_g Volterra type operator(see, e.g. [17]), which can be viewed as a generalization of the Cesàro operator(see, e.g. [7]).

Similarly, another integral operator is defined by(see, e.g. [24])

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)d\xi. \tag{6}$$

The importance of the operator J_g and I_g comes from the fact that

$$J_g f + I_g f = M_g f - f(0)g(0),$$

where the multiplication operator M_g is defined by $(M_g f)(z) = g(z)f(z)$.

In [16] Pommerenke showed that J_g is a bounded operator on the Hardy space H^2 if and only if $g\in$ BMOA. Aleman and Siskakis considered J_g on the Hardy space, $1\le p<\infty$, and weighted Bergman space in [2, 3]. Recently, the boundedness and compactness of J_g and I_g between some spaces of analytic functions, as well as their n-dimensional extensions, were investigated in [1, 7, 10-13, 17, 18, 23, 24] (see also the related references therein).

The paper is organized as follows. In the first section, we introduce the concept of the Q_K space and the Volterra type operators J_g, I_g. The second section is devoted to study the boundedness of Volterra type operators J_g and I_g on the Q_K space. In the third section, we introduce a new Carleson type measure, i.e. $p-$logarithmic $K-$Carleson measure and characterized it. In the fourth section, we introduce two new spaces, logarithmic Q_K space and logarithmic $Q_{K,0}$ space, denoted by $Q_K^{\log}$ and $Q_{K,0}^{\log}$ respectively. Some characterizations of $Q_K^{\log}$ and $Q_{K,0}^{\log}$ are given. In addition, a new characterization of Q_K space is given in the last section. These results can be viewed as a development of our early study on Q_K spaces, see [8, 9, 14, 19, 20, 21].

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. $a \preceq b$ means that there is a positive constant C such that $a \leq Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. The Operators J_g and I_g on Q_K Spaces

In this section, we give the characterization of boundedness of the operators J_g and I_g on the Q_K space. We state the first result of this section as follows.

Theorem 2.1. *Let K satisfy (4).*

(a) *If*

$$\sup_{I\subset\partial\mathbb{D}} \int_{S(I)} \Big(\log\frac{1}{1-|z|^2}\Big)^2 |g'(z)|^2 K(\frac{1-|z|}{|I|}) dA(z) < \infty, \tag{7}$$

then J_g is bounded on Q_K.

(b) *If J_g is bounded on Q_K, then*

$$\sup_{I\subset\partial\mathbb{D}} \Big(\log\frac{2}{|I|}\Big)^2 \int_{S(I)} |g'(z)|^2 K(\frac{1-|z|}{|I|}) dA(z) < \infty. \tag{8}$$

To prove the above theorem, we need the following two results which can be found in [9].

Lemma 2.1. *Let K satisfy (4) and $f \in H(\mathbb{D})$. Then the following are equivalent.*

(i) $f \in Q_K$.

(ii) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f\circ\varphi_a)'(z)|^2 K(1-|z|^2) dA(z) < \infty.$

(*iii*) $|f'(z)|^2dA(z)$ *is a* K*-Carleson measure on* $\mathbb{D}$*, that is*

$$\sup_{I\subset\partial\mathbb{D}}\int_{S(I)}|f'(z)|^2K(\frac{1-|z|}{|I|})dA(z)<\infty.$$

Lemma 2.2. *Let* K *satisfy* (4). *Then* $\log(1-z)$ *belongs to* Q_K.

Proof of Theorem 2.1. Suppose (7) holds and let $f\in Q_K$. By Lemma 2.1, we will show $F=J_g(f)\in Q_K$ by proving that $|F'(z)|^2dA(z)$ is a K-Carleson measure. Let $I\in\partial\mathbb{D}$ be an arc. Using the fact that $f\in\mathcal{B}$ and $\|f\|_{\mathcal{B}}\le\|f\|_{Q_K}$, we have

$$\begin{aligned}
&\int_{S(I)}|F'(z)|^2K(\frac{1-|z|}{|I|})dA(z)\\
&=\int_{S(I)}|f(z)|^2|g'(z)|^2K(\frac{1-|z|}{|I|})dA(z)\\
&\le\|f\|^2_{\mathcal{B}}\int_{S(I)}\Big(\log\frac{1}{1-|z|^2}\Big)^2|g'(z)|^2K(\frac{1-|z|}{|I|})dA(z)\\
&\le\|f\|^2_{Q_K}\int_{S(I)}\Big(\log\frac{1}{1-|z|^2}\Big)^2|g'(z)|^2K(\frac{1-|z|}{|I|})dA(z)<\infty.
\end{aligned}\tag{9}$$

From Lemma 2.1, we get the desired result.

Conversely, suppose that $J_g:Q_K\to Q_K$ is bounded. For $a\in\mathbb{D}$, set $f_a(z)=\log\frac{1}{1-\bar{a}z}$. Since K satisfies (4), by Lemma 2.2, we see that $f_a\in Q_K$. For an arc $I\subset\partial\mathbb{D}$, let $a=(1-|I|)e^{i\theta}$ with the midpoint $e^{i\theta}$ of I. Then there is a constant C such that

$$\frac{1}{C}\log\frac{2}{|I|}\le|f_a(z)|\le C\log\frac{2}{|I|}\tag{10}$$

for all $z\in S(I)$. Therefore we get

$$\begin{aligned}
&\Big(\log\frac{2}{|I|}\Big)^2\int_{S(I)}|g'(z)|^2K(\frac{1-|z|}{|I|})dA(z)\\
&\preceq\int_{S(I)}|f_a(z)|^2|g'(z)|^2K(\frac{1-|z|}{|I|})dA(z)\\
&=\int_{S(I)}|(J_gf_a)'(z)|^2K(\frac{1-|z|}{|I|})dA(z)\\
&\preceq\|J_g(f_a)\|^2_{Q_K}\preceq\|J_g\|^2.
\end{aligned}\tag{11}$$

It follows that (8) holds. This finishes the proof.

Theorem 2.2. *Let K satisfy (4). The operator I_g is bounded on Q_K if and only if $g \in H^\infty$. Moreover*

$$\|I_g\| \asymp \|g\|_\infty. \tag{12}$$

Proof. By the definition of I_g, we have that $(I_g f)' = f'(z)g(z)$ and $I_g f(0) = 0$. Assume that $g \in H^\infty$. For an $f \in Q_K$,

$$\begin{aligned} \|I_g f\|_{Q_K}^2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(I_g f)'(z)|^2 K(1-|\varphi_a(z)|^2)dA(z) \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g(z)|^2 |f'(z)|^2 K(1-|\varphi_a(z)|^2)dA(z) \\ &\preceq \|g\|_\infty^2 \|f\|_{Q_K}^2. \end{aligned} \tag{13}$$

It follows that I_g is bounded on Q_K and $\|I_g\| \preceq \|g\|_\infty$.

Conversely, assume that the operator $I_g : Q_K \to Q_K$ is bounded. For any $a \in \mathbb{D}$ such that $|a| > 1/2$, taking $f_a = \log \frac{1}{1-\bar{a}z}$, then $f_a \in Q_K$. Hence

$$\begin{aligned} |g(z)||f_a'(z)|(1-|z|^2) &= |(I_g f_a)'(z)|(1-|z|^2) \le \|I_g f_a\|_{\mathcal{B}} \\ &\preceq \|I_g f_a\|_{Q_K} \\ &\preceq \|I_g\| \|f_a\|_{Q_K}. \end{aligned} \tag{14}$$

Letting $z = a$, we get

$$|\bar{a}||g(a)| \preceq \|I_g\| \|f_a\|_{Q_K} \preceq \|I_g\| \| \log \frac{1}{1-z} \|_{Q_K}. \tag{15}$$

Taking supremum in the last inequality over the set $1/2 \le |a| < 1$ and noticing that by the maximum modulus principle there is a positive constant C independent of $g \in H(\mathbb{D})$ such that

$$\sup_{a\in\mathbb{D}} |g(a)| \le C \sup_{1/2\le|a|<1} |\bar{a}||g(a)|. \tag{16}$$

From (15) and (16), for any $a \in \mathbb{D}$, we have

$$|g(a)| \preceq \|I_g\|. \tag{17}$$

From (13) and (17) we obtain (12). It completes the proof of this theorem.

3. Logarithmic $K-$Carleson Measure and Characterization

Let μ denote a positive Borel measure on $\mathbb{D}$. For a subarc $I \in \partial\mathbb{D}$, let

$$S(I) = \{r\zeta \in \mathbb{D} : 1 - |I| < r < 1, \zeta \in I\}.$$

If $|I| \geq 1$, then we set $S(I) = \mathbb{D}$. For $0 < p < \infty$, we say that μ is a p-Carleson measure on $\mathbb{D}$ if

$$\sup_{I \subset \partial\mathbb{D}} \mu(S(I))/|I|^p < \infty.$$

Here and henceforth $\sup_{I\subset\partial\mathbb{D}}$ indicates the supremum taken over all subarcs I of $\partial\mathbb{D}$. Note that $p = 1$ gives the classical Carleson measure.

From (8), if we let $d\mu = |g'(z)|^2 dA(z)$, then we obtain a natural expression

$$\Big(\log\frac{2}{|I|}\Big)^2 \int_{S(I)} K\big(\frac{1-|z|}{|I|}\big) d\mu(z).$$

Motivated by the above formula, we define a new measure and give some characterizations of it.

Definition 3.1. For $0 \leq p < \infty$, a positive Borel measure μ on $\mathbb{D}$ is called a $p-$logarithmic $K-$Carleson measure if

$$\sup_{I\subset\partial\mathbb{D}} \Big(\log\frac{2}{|I|}\Big)^p \int_{S(I)} K\big(\frac{1-|z|}{|I|}\big) d\mu(z) < \infty. \tag{18}$$

A positive Borel measure μ on $\mathbb{D}$ is called a vanishing $p-$logarithmic K-Carleson measure if

$$\lim_{|I|\to 0} \Big(\log\frac{2}{|I|}\Big)^p \int_{S(I)} K\big(\frac{1-|z|}{|I|}\big) d\mu(z) = 0. \tag{19}$$

Remark 1. Note that μ is called K-Carleson measure if $p = 0$, see [9] for more results about K-Carleson measures. The related $p-$logarithmic $s-$Carleson measure was studied in [15, 25].

Theorem 3.1. *Let μ be a positive Borel measure on $\mathbb{D}$ and $0 \leq p < \infty$. Let K satisfy (4). Then μ is a $p-$logarithmic $K-$Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\log\frac{2}{|1-\overline{a}z|})^p K(1-|\varphi_a(z)|^2) d\mu(z) < \infty. \tag{20}$$

Proof. Sufficiency. Assume that (20) holds. For a subarc $I \in \partial\mathbb{D}$, suppose that $e^{i\theta}$ is the midpoint of I. Then by taking $a = e^{i\theta}(1-|I|)$, we have

$$\frac{1}{|I|} \preceq \frac{1-|a|^2}{|1-\bar{a}z|^2} \preceq \frac{1}{|1-\bar{a}z|}, \qquad z \in S(I).$$

Consequently,

$$\begin{aligned}(\log\frac{2}{|I|})^p \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z) &= \int_{S(I)} (\log\frac{2}{|I|})^p K\left(\frac{1-|z|}{|I|}\right) d\mu(z) \\ &\preceq \int_{\mathbb{D}} (\log\frac{2}{|1-\overline{a}z|})^p K(1-|\varphi_a(z)|^2) d\mu(z).\end{aligned}$$

Thus μ is a $p-$logarithmic $K-$Carleson measure.

Necessity. We suppose that μ is a $p-$logarithmic $K-$Carleson measure. Now, for $|a| > 3/4$, let I be the subarc centered at $a/|a|$ of length $\frac{(1-|a|)}{2\pi}$. Consider

$$S_n = \{z \in \mathbb{D} : |z - \frac{a}{|a|}| \leq 2^n(1-|a|)\}, \quad n = 1, 2, \cdots.$$

We have that

$$\frac{1-|a|^2}{|1-\bar{a}z|^2} \preceq \frac{1}{2^{2n}|I|}, \quad z \in S_n \setminus S_{n-1}, \quad n = 2, \cdots.$$

Thus

$$\begin{aligned}&\int_{\mathbb{D}} \left(\log\frac{2}{|1-\overline{a}z|}\right)^p K(1-|\varphi_a(z)|^2) d\mu(z) \\ &= \int_{\mathbb{D}} \left(\log\frac{2}{|1-\overline{a}z|}\right)^p K\left(\frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2}\right) d\mu(z) \\ &\preceq \int_{S_1} \left(\log\frac{2}{|1-\overline{a}z|}\right)^p K\left(\frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2}\right) d\mu(z) \\ &\quad + \sum_{n=2}^{\infty} \int_{S_n \setminus S_{n-1}} \left(\log\frac{2}{|1-\overline{a}z|}\right)^p K\left(\frac{(1-|z|^2)(1-|a|^2)}{|1-\bar{a}z|^2}\right) d\mu(z) \\ &\preceq C + \sum_{n=2}^{\infty} \int_{S_n \setminus S_{n-1}} \left(\log\frac{2}{|1-\overline{a}z|}\right)^p K\left(\frac{1-|z|}{2^{2n}|I|}\right) d\mu(z) \\ &\preceq \sum_{n=2}^{\infty} \left(\log\frac{2}{2^n|I|}\right)^p \sup_{z\in S_n} \frac{K(\frac{1-|z|}{2^{2n}|I|})}{K(\frac{1-|z|}{2^n|I|})} \int_{S_n} K\left(\frac{1-|z|}{2^n|I|}\right) d\mu(z).\end{aligned}$$

Putting $\frac{1-|z|}{2^n|I|} = t$, we have

$$\sup_{z\in S_n} \frac{K(\frac{1-|z|}{2^{2n}|I|})}{K(\frac{1-|z|}{2^n|I|})} \leq \sup_{0\leq t\leq 1} \frac{K(2^{-n}t)}{K(t)} = \varphi_K(2^{-n}).$$

Since μ is a $p-$logarithmic $K-$Carleson measure,

$$\left(\log\frac{2}{2^n|I|}\right)^p\int_{S_n}K\left(\frac{1-|z|}{2^n|I|}\right)d\mu(z)\preceq 1,$$

for all $n=1,2,\cdots$. Thus

$$\sum_{n=2}^{\infty}\left(\log\frac{2}{2^n|I|}\right)^p\sup_{z\in S_n}\frac{K(\frac{1-|z|}{2^{2n}|I|})}{K(\frac{1-|z|}{2^n|I|})}\int_{S_n}K\left(\frac{1-|z|}{2^n|I|}\right)d\mu(z)$$
$$\preceq\sum_{n=2}^{\infty}\varphi_K(2^{-n})\preceq\int_0^1\frac{\varphi_K(s)}{s}ds.$$

Therefore

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(\log\frac{2}{|1-\overline{a}z|})^pK(1-|\varphi_a(z)|^2)d\mu(z)\preceq\int_0^1\frac{\varphi_K(s)}{s}ds<\infty.$$

The proof is completed.

Carefully check the proof of the above theorem, we have the following result. We omit the details.

Theorem 3.2. *Let $0\le p<\infty$ and μ be a positive Borel measure on $\mathbb{D}$. Let K satisfy (4). Then μ is a vanishing $p-$logarithmic $K-$Carleson measure if and only if*

$$\lim_{|a|\to 1}\int_{\mathbb{D}}\Big(\log\frac{2}{|1-\overline{a}z|}\Big)^pK(1-|\varphi_a(z)|^2)d\mu(z)=0. \tag{21}$$

4. The Logarithmic Q_K Spaces

From the above section, it is natural to consider the following spaces $Q_K^{\log}$ and $Q_{K,0}^{\log}$ defined as follows.

For any nonnegative, nondecreasing and Lebesgue measurable function $K:(0,\infty)\to[0,\infty)$, we say that f belongs to the logarithmic Q_K space, denoted by $Q_K^{\log}$, if

$$\|f\|^2_{Q_K^{\log}}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\Big(\log\frac{2}{|1-\overline{a}z|}\Big)^2|f'(z)|^2K(g(z,a))dA(z)<\infty,$$

and f belongs to the space $Q_{K,0}^{\log}$ if

$$\lim_{|a|\to 1}\int_{\mathbb{D}}\Big(\log\frac{2}{|1-\overline{a}z|}\Big)^2|f'(z)|^2K(g(z,a))dA(z)=0.$$

To study the spaces $Q_K^{\log}$ and $Q_{K,0}^{\log}$, we consider the logarithmic Bloch space $\mathcal{B}^{\log}$ and the little logarithmic Bloch space $\mathcal{B}_0^{\log}$. We say $f \in \mathcal{B}^{\log}$ if

$$\|f\|_{\mathcal{B}^{\log}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)\log\frac{2}{1-|z|^2} < \infty.$$

f belongs to the little logarithmic Bloch space $\mathcal{B}_0^{\log}$ if

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2)\log\frac{2}{1-|z|^2} = 0.$$

In [4], Attete proved that if $f \in L_a^1$ then the Hankel operator $H_{\bar f}$ is bounded on L_a^1 if and only if $f \in \mathcal{B}^{\log}$.

The first result concering the relationship between $Q_K^{\log}$ and $\mathcal{B}^{\log}$, is follows.

Theorem 4.1. $Q_K^{\log} \subset \mathcal{B}^{\log}$; $Q_{K,0}^{\log} \subset \mathcal{B}_0^{\log}$.

Proof. For $0 < r < 1$, let $\mathbb{D}(a,r) = \{a \in \mathbb{D} : |\varphi_a(z)| < r\}$ be the pseudohyperbolic disk with center $a \in \mathbb{D}$ and radius r. By [27] we see that

$$\frac{1}{|1-\overline{a}z|^2} \asymp \frac{1}{(1-|z|^2)^2} \asymp \frac{1}{(1-|a|^2)^2} \asymp \frac{1}{|\mathbb{D}(a,r)|},\ z \in \mathbb{D}(a,r).$$

Choose an $r_0 \in (0,1)$ such that $g(z,a) \geq \log\frac{1}{r_0}$ for $z \in \mathbb{D}(a,r)$. By the subharmonicity, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}}\Big(\log\frac{2}{|1-\overline{a}z|}\Big)^2 |f'(z)|^2 K(g(z,a))dA(z)\\
&\succeq K(\log\frac{1}{r_0})\int_{\mathbb{D}(a,r_0)}\Big(\log\frac{2}{|1-\overline{a}z|}\Big)^2 |f'(z)|^2 dA(z)\\
&\succeq K(\log\frac{1}{r_0})\Big(\log\frac{2}{1-|a|^2}\Big)^2\int_{\mathbb{D}(a,r_0)}|f'(z)|^2 dA(z)\\
&\succeq K(\log\frac{1}{r_0})\Big(\log\frac{2}{1-|a|^2}\Big)^2(1-|a|^2)^2|f'(a)|^2,
\end{aligned}$$

which means that $Q_K^{\log} \subset \mathcal{B}^{\log}$. The proof of the inclusion $Q_{K,0}^{\log} \subset \mathcal{B}_0^{\log}$ is similar to the former.

Theorem 4.2. *If*

$$\int_0^1 K(\log(1/r))(1-r^2)^{-2} r\,dr < \infty, \tag{22}$$

then (i) $Q_K^{\log} = \mathcal{B}^{\log}$; *(ii)* $Q_{K,0}^{\log} = \mathcal{B}_0^{\log}$.

Proof.

(i) From Theorem 4.1, we know that $Q_K^{\log} \subset \mathcal{B}^{\log}$. Now we assume that $f \in \mathcal{B}^{\log}$ and observe that

$$\begin{aligned}
&\int_{\mathbb{D}} \Big(\log \frac{2}{|1-\overline{a}z|} \Big)^2 |f'(z)|^2 K(g(z,a))dA(z) \\
&\preceq \int_{\mathbb{D}} \Big(\log \frac{2}{1-|z|} \Big)^2 |f'(z)|^2 K(g(z,a))dA(z) \\
&\preceq \|f\|^2_{\mathcal{B}^{\log}} \int_{\mathbb{D}} (1-|z|^2)^{-2} K(g(z,a))dA(z) \\
&\preceq \|f\|^2_{\mathcal{B}^{\log}} \int_0^1 K(\log(1/r))(1-r^2)^{-2} r dr < \infty.
\end{aligned}$$

Hence $f \in Q_K^{\log}$.

(ii) From Theorem 4.1, it suffices to prove that $\mathcal{B}_0^{\log} \subset Q_{K,0}^{\log}$. Suppose that $f \in \mathcal{B}_0^{\log}$. From the assumption, for given $\varepsilon > 0$ there exists an r, $0 < r < 1$, such that

$$\int_r^1 K(\log(1/r))(1-r^2)^{-2} r dr < \varepsilon.$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{D}\setminus\mathbb{D}(a,r)} \Big(\log \frac{2}{|1-\overline{a}z|} \Big)^2 |f'(z)|^2 K(g(z,a))dA(z) \\
&\preceq \int_{\mathbb{D}\setminus\mathbb{D}(a,r)} \Big(\log \frac{2}{1-|z|} \Big)^2 |f'(z)|^2 K(g(z,a))dA(z) \\
&\preceq \|f\|^2_{\mathcal{B}^{\log}} \int_{\mathbb{D}\setminus\mathbb{D}(a,r)} (1-|z|^2)^{-2} K(g(z,a))dA(z) \\
&\preceq \|f\|^2_{\mathcal{B}^{\log}} \int_r^1 K(\log(1/r))(1-r^2)^{-2} r dr \\
&\preceq \|f\|^2_{\mathcal{B}^{\log}} \varepsilon.
\end{aligned} \tag{23}$$

Since $f \in \mathcal{B}_0^{\log}$, we see that for given $\varepsilon > 0$, there existing $\delta > 0$, such that for $\delta < |z| < 1$

$$\log \frac{2}{1-|a|}(1-|a|^2)|f'(a)| < \varepsilon.$$

For $z \in \mathbb{D}(a,r)$, we can choose ρ, $0 < \rho < 1$, such that $\rho < |a| < 1$ implies $\delta < |z| < 1$. Then for $\rho < |a| < 1$

$$
\begin{aligned}
&\int_{\mathbb{D}(a,r)} \Big(\log \frac{2}{|1-\overline{a}z|}\Big)^2 |f'(z)|^2 K(g(z,a))dA(z)\\
&\preceq \int_{\mathbb{D}(a,r)} \Big(\log \frac{2}{1-|z|^2}\Big)^2 |f'(z)|^2 K(g(z,a))dA(z)\\
&\preceq \varepsilon^2 \int_{\mathbb{D}(a,r)} (1-|z|^2)^{-2} K(g(z,a))dA(z)\\
&\preceq \varepsilon^2 \int_0^r K(\log(1/r))(1-r^2)^{-2} r dr.
\end{aligned} \tag{24}
$$

Combining (23) and (24), we get

$$
\lim_{|a|\to 1} \int_{\mathbb{D}} \Big(\log \frac{2}{|1-\overline{a}z|}\Big)^2 |f'(z)|^2 K(g(z,a))dA(z) = 0,
$$

which shows that $f \in Q_{K,0}^{\log}$. We complete the proof.

Theorem 4.3. *Let K satisfy (4) and $f \in H(\mathbb{D})$. Then the following statements are equivalent.*
(a) $f \in Q_K^{\log}$.
(b) $|f'(z)|^2 dA(z)$ is a 2-logarithmic K-Carleson measure.

Proof. $(a) \Rightarrow (b)$. Suppose that $f \in Q_K^{\log}$, by $1-|\varphi_a(z)|^2 \le g(z,a)$, we obtain

$$
\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \Big(\log \frac{2}{|1-\overline{a}z|}\Big)^2 |f'(z)|^2 K(1-|\varphi_a(z)|^2)dA(z) < \infty. \tag{25}
$$

From Theorem 3.1, we see that (b) holds.

Assume that (b) holds, i.e (25) holds. From the proof of Theorem 4.1 we know that (25) implies $f \in \mathcal{B}^{\log}$. Therefore

$$
\begin{aligned}
&\int_{|g(z,a)|>1} \Big(\log \frac{2}{|1-\overline{a}z|}\Big)^2 |f'(z)|^2 K(g(z,a))dA(z)\\
&\preceq \int_{|g(z,a)|>1} \Big(\log \frac{2}{1-|z|}\Big)^2 |f'(z)|^2 K(g(z,a))dA(z)\\
&\preceq \|f\|_{\mathcal{B}^{\log}}^2 \int_{|g(z,a)|>1} (1-|z|^2)^{-2} K(g(z,a))dA(z)\\
&\preceq \|f\|_{\mathcal{B}^{\log}}^2 \int_{|w|<1/e} (1-|w|^2)^{-2} K(\log \frac{1}{|w|})dA(w).
\end{aligned} \tag{26}
$$

On the other hand,

$$\begin{aligned}&\int_{|g(z,a)|\le 1}\Big(\log\frac{2}{|1-\overline{a}z|}\Big)^2|f'(z)|^2K(g(z,a))dA(z)\\ &\preceq \int_{|\varphi_a(z)|\ge\frac{1}{e}}\Big(\log\frac{2}{|1-\overline{a}z|}\Big)^2|f'(z)|^2K(1-|\varphi_a(z)|^2)dA(z)\\ &\preceq \int_{\mathbb{D}}\Big(\log\frac{2}{|1-\overline{a}z|}\Big)^2|f'(z)|^2K(1-|\varphi_a(z)|^2)dA(z)\end{aligned}$$

which, together with (26), shows that $f\in Q_K^{\log}$.

Similarly, we have the following theorem.

Theorem 4.4. *Let K satisfy (4) and $f\in H(\mathbb{D})$. Then the following statements are equivalent.*

(a) $f\in Q_{K,0}^{\log}$;

(b) $|f'(z)|^2dA(z)$ *is a vanishing 2-logarithmic K-Carleson measure.*

5. A New Characterization of Q_K Space

In [21], the high order derivative characterizations of Q_K and $Q_{K,0}$ spaces were given by the second author and Zhu which can be stated as follows.

Theorem 5.1. *Suppose the function K satisfies (4) or that there exists some $p<2$ such that*

$$\int_1^\infty \frac{\varphi_K(s)}{s^p}ds<\infty.$$

Then for any positive integer n, an $f\in H(\mathbb{D})$ belongs to Q_K if and only if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|\varphi_a(z)|^2)dA(z)<\infty. \tag{27}$$

In this section, we give another characterizations of Q_K and $Q_{K,0}$ as follows.

Theorem 5.2. *Suppose the function K satisfies (4) or that there exists some $p<2$ such that*

$$\int_1^\infty \frac{\varphi_K(s)}{s^p}ds<\infty.$$

Then for any positive integer n, an $f \in H(\mathbb{D})$ belongs to Q_K if and only if

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f\circ\varphi_a)^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|z|^2)dA(z)<\infty. \tag{28}$$

To prove the Theorem 5.2, we need the following lemma (see [26]).

Lemma 5.1. *Suppose f is analytic in $\mathbb{D}$, $a \in \mathbb{D}$, and n is a positive integer. Then*

$$(f\circ\varphi_a)^{(n)}(z)=\sum_{k=1}^{n}c_k f^{(k)}(\varphi_a(z))\frac{(1-|a|^2)^k}{(1-\overline{a}z)^{n+k}},$$

and

$$f^{(n)}(\varphi_a(z))\frac{(1-|a|^2)^n}{(1-\overline{a}z)^{2n}}=\sum_{k=1}^{n}\frac{d_k}{(1-\overline{a}z)^{n-k}}(f\circ\varphi_a)^{(k)}(z),$$

where c_k and d_k are polynomials of $\overline{a}$.

Proof of Theorem 5.2. By a change of variables, we get

$$\begin{aligned}
&\int_{\mathbb{D}}|f^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|\varphi_a(z)|^2)dA(z)\\
&=\int_{\mathbb{D}}|f^{(n)}(\varphi_a(z))|^2(1-|\varphi_a(z)|^2)^{2n-2}K(1-|z|^2)\frac{(1-|a|^2)^2}{|1-\overline{a}z|^4}dA(z)\\
&=\int_{\mathbb{D}}|f^{(n)}(\varphi_a(z))|^2\frac{(1-|a|^2)^{2n}}{|1-\overline{a}z|^{4n}}(1-|z|^2)^{2n-2}K(1-|z|^2)dA(z)\\
&=\int_{\mathbb{D}}|\sum_{k=1}^{n}\frac{d_k}{(1-\overline{a}z)^{n-k}}(f\circ\varphi_a)^{(k)}(z)|^2(1-|z|^2)^{2n-2}K(1-|z|^2)dA(z)\\
&\le\sum_{k=1}^{n}n^2|d_k|^2\int_{\mathbb{D}}|(f\circ\varphi_a)^{(k)}(z)|^2\frac{(1-|z|^2)^{2n-2}}{|1-\overline{a}z|^{2(n-k)}}K(1-|z|^2)dA(z)\\
&\le\sum_{k=1}^{n}n^2|d_k|^2\int_{\mathbb{D}}|(f\circ\varphi_a)^{(k)}(z)|^2\frac{(1-|z|^2)^{2(n-k)}}{|1-\overline{a}z|^{2(n-k)}}\\
&\qquad(1-|z|^2)^{2k-2}K(1-|z|^2)dA(z)\\
&\le\sum_{k=1}^{n}n^2|d_k|^2\int_{\mathbb{D}}|(f\circ\varphi_a)^{(k)}(z)|^2(1-|z|^2)^{2k-2}K(1-|z|^2)dA(z).
\end{aligned} \tag{29}$$

Since for any positive integer $m \ge 2$,

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f\circ\varphi_a)^{(m)}(z)|^2(1-|z|^2)^{2m-2}K(1-|z|^2)dA(z)<\infty$$

implies that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f\circ\varphi_a)^{(m-1)}(z)|^2(1-|z|^2)^{2(m-1)-2}K(1-|z|^2)dA(z)<\infty.$$

Therefore (28) together with (29) imply

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|\varphi_a(z)|^2)dA(z)<\infty.$$

From (27), we see that $f\in Q_K$.

Conversely, assume that $f\in Q_K$. By (27), for any positive integer k,

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f^{(k)}(z)|^2(1-|z|^2)^{2k-2}K(1-|\varphi_a(z)|^2)dA(z)<\infty.$$

Hence for any positive integer n, by Lemma 5.1, we have

$$\begin{aligned}
&\int_{\mathbb{D}}|(f\circ\varphi_a)^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|z|^2)dA(z)\\
\le&\sum_{k=1}^{n}n^2|c_k|^2\int_{\mathbb{D}}|f^{(k)}(\varphi_a(z))|^2\frac{(1-|a|^2)^{2k}(1-|z|^2)^{2n-2}}{|1-\overline{a}z|^{2(n+k)}}K(1-|z|^2)dA(z)\\
\le&\sum_{k=1}^{n}n^2|c_k|^2\int_{\mathbb{D}}|f^{(k)}(z)|^2\frac{(1-|a|^2)^{2k}(1-|\varphi_a(z)|^2)^{2n-2}}{|1-\overline{a}\varphi_a(z)|^{2(n+k)}}K(1-|\varphi_a(z)|^2)dA(\varphi_a(z))\\
\le&\sum_{k=1}^{n}n^2|c_k|^2\int_{\mathbb{D}}|f^{(k)}(z)|^2\frac{(1-|z|^2)^{2(n-k)}}{|1-\overline{a}z|^{2(n-k)}}(1-|z|^2)^{2k-2}K(1-|\varphi_a(z)|^2)dA(z)\\
\le&\sum_{k=1}^{n}n^2|c_k|^2\int_{\mathbb{D}}|f^{(k)}(z)|^2(1-|z|^2)^{2k-2}K(1-|\varphi_a(z)|^2)dA(z)<\infty.
\end{aligned}$$

The proof is completed.

Remark 2. Since our estimates are pointwise estimates with respect to $a\in\mathbb{D}$, we have the corresponding little oh version characterizations of $Q_{K,0}$ spaces as follows.

Theorem 5.3. *Suppose the function K satisfies (4) or that there exists some $p<2$ such that*

$$\int_1^\infty\frac{\varphi_K(s)}{s^p}ds<\infty.$$

Then for any positive integer n, an $f\in H(\mathbb{D})$ belongs to $Q_{K,0}$ if and only if

$$\lim_{|a|\to1}\int_{\mathbb{D}}|(f\circ\varphi_a)^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|z|^2)dA(z)=0.$$

As a corollary, we obtain the following new characterizations of Q_p and $Q_{p,0}$ space.

Corollary 5.1. *For any positive integer* n *and* $0 < p < \infty$, *an* $f \in H(\mathbb{D})$ *belongs to* Q_p *if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f \circ \varphi_a)^{(n)}(z)|^2 (1 - |z|^2)^{2n-2+p} dA(z) < \infty.$$

An $f \in H(\mathbb{D})$ *belongs to* $Q_{p,0}$ *if and only if*

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |(f \circ \varphi_a)^{(n)}(z)|^2 (1 - |z|^2)^{2n-2+p} dA(z) = 0.$$

ACKNOWLEDGMENT

The authors of this paper are supported by the NNSF of China (No. 10671115), the Research Fund for the doctoral program of Higher Education (No. 20060560002) and NSF of Guangdong Province (No. 06105648).

REFERENCES

1. A. Aleman and J. Cima, An integral operator on H^p and Hardy's inequality, *J. Anal. Math.*, **85** (2001), 157-176.
2. A. Aleman and A. Siskakis, An integral operator on H^p, *Complex Variables*, **28** (1995), 140-158.
3. A. Aleman and A. Siskakis, Integral operators on Bergman spaces, *Indiana Univ. Math. J.*, **46** (1997), 337-356.
4. K. Attele, Toeplitz and Hankel operators on Bergman space, *Hokkaido Math. J.*, **211** (1992), 279-293.
5. R. Aulaskari, D. Stegenga and J. Xiao, Some subclasses of BMOA and their characterization in terms of Carleson measure, *Rocky Mountain J. Math.*, **26** (1996), 485-506.
6. R. Aulaskari, J. Xiao and R. Zhao, Some subspaces and subsets of BMOA and UBC, *Analysis*, **15** (1995), 101-121.
7. D. Chang, S. Li and S. Stević, Om some integral operators on the unit polydisk and the unit ball, *Taiwanese J. Math.*, **11(5)** (2007), 1251-1286.

8. M. Essén and H. Wulan, On analytic and meromorphic functions and spaces of Q_K-type, *Illinois J. Math.*, **46** (2002), 1233-1258.

9. M. Essén, H. Wulan and J. Xiao, Several function-theoretic characterizations of Möbius invariant Q_K spaces, *J. Funct. Anal.*, **230** (2006), 78-115.

10. Z. Hu, Extended Cesàro operators on mixed norm spaces, *Proc. Amer. Math. Soc.*, **131(7)** (2003), 2171-2179.

11. Z. Hu, Extended Cesàro operators on the Bloch space in the unit ball of $\mathbb{C}^n$, *Acta Math. Sci. Ser. B Engl. Ed.*, **23(4)** (2003), 561-566.

12. S. Li, Riemann-Stieltjes operators from $F(p, q, s)$ to Bloch space on the unit ball, *J. Ineq. Appl.*, Volume 2006, Article ID 27874, Pages 1-14.

13. S. Li and S. Stević, Riemann-Stieltjes operators on Hardy spaces in the unit ball of $\mathbb{C}^n$, *Bull. Belg. Math. Soc. Simon Stevin*, **14** (2007), 621-628.

14. S. Li and H. Wulan, Composition operators on Q_K spaces, *J. Math. Anal. Appl.*, **327** (2007), 948-958.

15. B. MacCluer and R. Zhao, Vanishing logarithmic Carleson measure, *Illinois J. Math.*, **46(2)** (2002), 507-518.

16. C. Pommerenke, Schlichte funktionen und analytische funktionen von beschränkter mittlerer oszillation, *Comment. Math. Helv.*, **52** (1977), 591-602.

17. A. Siskakis and R. Zhao, A Volterra type operator on spaces of analytic functions, *Contemp. Math.*, **232** (1999), 299-311.

18. S. Stević, On an integral operator on the unit ball in $\mathbb{C}^n$, *J. Inequal. Appl.*, **1** (2005), 81-88.

19. H. Wulan and P. Wu, Characterizations of Q_T spaces, *J. Math. Anal. Appl.*, **254** (2001), 484-497.

20. H. Wulan and K. Zhu, Derivative-free characterizations of Q_K spaces, *J. Austra Math.*, **82** (2007), 283-295.

21. H. Wulan and K. Zhu, Q_K spaces via higher order derivatives, *Rocky Mountain J. Math.*, **38(1)** (2008), 329-350.

22. J. Xiao, *Holomorphic Q classes*, Berlin-Heidelberg, New York: Springer-Verlag, 2001.

23. J. Xiao, Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball, *J. London. Math. Soc.*, **70(2)** (2004), 199-214.

24. R. Yoneda, Pointwise multipliers from $BMOA^\alpha$ to $BMOA^\beta$, *Complex Variables*, **49(14)** (2004), 1045-1061.

25. R. Zhao, On logarithmic Carleson measure, *Acta Math.* $($*Szego*$)$, **69** (2003), 605-618.

26. K. Zhu, A class of Möbius invariant function spaces, *Illinois J. Math.*, **51(3)** (2007), 977-1002.

27. K. Zhu, *Operator Theory in Function Space*, Marcel Dekker, New York, 1990.

Songxiao Li
Department of Mathematics,
JiaYing University,
Meizhou 514015, GuangDong,
P. R. China
E-mail: jyulsx@163.com
lsx@mail.zjxu.edu.cn

Hasi Wulan
Department of Mathematics,
Shantou University,
Shantou 515063, GuangDong,
P. R. China
E-mail: wulan@stu.edu.cn