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LOCAL EXISTENCE OF SOME NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. The paper will investigate the local existence, uniqueness of solution to integrodifferential equations with infinite delay. We assume that the linear part is not necessarily densely defined and satisfies a Hille-Yosida condition. Moreover, the continuity of solutions with respect to initial conditions is also studied.

1. Introduction

In this paper we study the well posedness of the following integrodifferential equation with infinite delay in a Banach space $(X, ||\cdot||)$

$$\text{(RVID)} \qquad \begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t k(t,\theta,u(\theta))d\theta + F(t,u_t), & t \in [0,T], \\ u_0 = \varphi \in \mathcal{P} \end{cases}$$

where $A : D(A) \to X$ denotes a Hille-Yosida operator, the phase space $\mathcal{P}$ is a linear space of functions mapping $(-\infty, 0]$ into X satisfying some axioms which will be described later, F is an X-valued function defined on $[0,T] \times \mathcal{P}$ and the function $u_t(\cdot) \in \mathcal{P}$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in (-\infty, 0].$$

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A is a Hille-Yosida operator with type (M, ω) means that there exist $\omega \in \mathbb{R}$ and $M \geq 1$ such that $(\omega, +\infty) \subset \rho(A)$ ($\rho(A)$ is the usual resolvent set of A) and satisfies

$$\|(\lambda - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for all} \quad \lambda > \omega \quad \text{and} \quad n \in \mathbb{N}.$$

The $D(A)$ may be nondense in X. The domain $D(A)$ endowed with the graph norm $||\cdot||_{D(A)}$ becomes a Banach space. The map $x \mapsto k(t, s, x)$ is assumed to be defined from $(D(A), ||\cdot||_{D(A)})$ to X in this work.

In [12], the authors consider the following equation

$$\begin{cases} u'(t) = Au(t) + \int_0^t k(t, \theta, u(\theta))d\theta + f(t), \quad t \in [0, T], \\ u(0) = z \in D(A). \end{cases} \tag{1.1}$$

They have established local existence and uniqueness for the equation (1.1) under some suitable assumptions. Moreover, some properties of solutions are also studied. As in [12], the equation (1.1) is the abstract version of the initial boundary value problem for a nonlinear Volterra integrodifferential equation of hyperbolic type

$$\begin{aligned} &u_t(t, x) = Au(t, x) + \int_0^t G(t, s, u(s, x), \ldots, D_x^\beta u(s, x))ds + f(t, x), \\ &(t, x) \in [0, \infty) \times \Omega, \\ &Bu(t, x) = 0, (t, x) \in [0, \infty) \times \partial\Omega, \\ &u(0, x) = z(x), x \in \Omega, \end{aligned}$$

where A and B denote suitable linear partial differential operators on a subset Ω of $\mathbb{R}^n$ and on its boundary $\partial\Omega$, respectively, and the kernel G does not depend on the derivatives $D_x^\beta u$ of order greater than the order of A. For the study of equation (1.1), we also refer to [6, 7, 13] and [17].

Recently, the following class of differential equations with delay is studied by many authors ([1, 2, 8, 9, 10, 14])

$$\begin{cases} u'(t) = Au(t) + F(t, u_t), \quad t \in [0, T], \\ u_0 = \varphi \in \mathcal{P}. \end{cases} \tag{1.2}$$

There have been a great deal of works contributed to the study of partial differential equations with delay by using different methods under different conditions. The most classical work is due to Travis and Webb [16]. The investigation of functional differential equations with infinite delay in an abstract admissible phase space was initiated by Hale and Kato [5], Kappel and Schappacher [9], and Schumacher [15]. The method of using admissible phase spaces enables one to treat a large class of functional differential equations with infinite delay in the same time and obtain general results. For a detailed discussion on this topic, we refer the reader to the book by Hino et al. [8].

So, it is easy to see that equation (RVID) is the mixed type of equation (1.1) and equation (1.2). We shall investigate it basing on the results on equation (1.2) and generalize the method used in [12] to solve it. Under different assumptions, equation (RVID) was also studied in [11]. In [11], we assume that F and k satisfy a local Lipschitz and global Lipschitz condition, respectively. In this paper, we consider the symmetric conditions: k satisfies the global Lipschitz condition and k satisfies the local Lipschitz condition. Using the similar computation under different space, we also show the local existence to equation (RVID).

In section 2, we recall some preliminary results about the equation (1.1) and equation (1.2). Some basic notations and assumptions are also given in this section. In section 3, we prove the local existence and uniqueness of solutions to equation (RVID) which are the main results of this paper. Moreover, some properties of solutions are also studied. In section 4, we give an example to show that our results are valuable.

2. Linear Cauchy Problems with Infinite Delay

First, A always denotes a Hille-Yosida operator with type (M, ω) in this paper. Suppose that T is a positive number, $\Delta(0,T)$ denotes the set $\{(t,s); 0 \leq s \leq t \leq T\} \subseteq \mathbb{R}^2$ and D is the Banach space $D(A)$ equipped with the graph norm $||\cdot||_{D(A)}$. Let $k : \Delta(0,T) \times D \to X$. We make the following assumptions.

(H1) k is continuous. The derivative $\partial_t k(t,s,x)$ exists and is continuous from $\Delta(0,T) \times D$ into X. k and $\partial_t k$ satisfy the local Lipschitz condition: for each $r > 0$ there exists $b(r)$ such that

$$||k(t,s,x) - k(t,s,y)|| \leq b(r)||x - y||_{D(A)}$$

and

$$||\partial_t k(t,s,x) - \partial_t k(t,s,y)|| \leq b(r)||x - y||_{D(A)}$$

for each $(t,s) \in \Delta(0,T)$ and $||x||_{D(A)}, ||y||_{D(A)} \leq r$.

Throughout this paper, we assume that the phase space $(\mathcal{P}, ||\cdot||_{\mathcal{P}})$ is a Banach space consisting of functions from $\mathbb{R}^-$ into X satisfying the following axioms introduced at first by Hale and Kato in [5].

(A1) There exist a positive constant H and functions $K(\cdot)$, $M(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a \geq 0$, if $x : (-\infty, \sigma + a] \to X$, $x_\sigma \in \mathcal{P}$ and $x(\cdot)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold.

(i) $x_t \in \mathcal{P}$,

(ii) $||x(t)|| \leq H||x_t||_{\mathcal{P}}$,

(iii) $||x_t||_{\mathcal{P}} \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} ||x(s)|| + M(t - \sigma)||x_\sigma||_{\mathcal{P}}$.

(A2) For each function $x(\cdot)$ in (A1), $t \mapsto x_t$ is a $\mathcal{P}$-valued continuous function for $t \in [\sigma, \sigma + a]$.

In our situations, the following more restrictive conditions are needed.

(B) If $\{\phi_n\}$ is a Cauchy sequence in $\mathcal{P}$ and $\{\phi_n\}$ converges compactly to ϕ on $(-\infty, 0]$, then $\phi \in \mathcal{P}$ and $||\phi_n - \phi||_{\mathcal{P}} \to 0$.

(C) For a sequence $\{\phi_n\}$ in $\mathcal{P}$, if $||\phi_n||_{\mathcal{P}} \to 0$ as $n \to \infty$, then $||\phi_n(\theta)|| \to 0$, as $n \to \infty$, for each $\theta \in (-\infty, 0]$.

Under the conditions (B) and (C), the following properties hold.

Theorem 2.1. [[8]]. *Let $\mathcal{P}$ satisfy axiom (B) and let $f : [0, a] \to \mathcal{P},\ a > 0$, be a continuous function such that $f(t)(\theta)$ is continuous for $(t, \theta) \in [0, a] \times (-\infty, 0]$. Then*

$$\left[\int_0^a f(t)dt\right](\theta) = \int_0^a f(t)(\theta)dt$$

for $\theta \in (-\infty, 0]$.

Theorem 2.2. [3]). *Let $\mathcal{P}$ satisfy axiom (C) and let $f : [0, a] \to \mathcal{P},\ a > 0$, be a continuous function. Then for all $\theta \in (-\infty, 0]$, the function $f(\cdot)(\theta)$ is continuous and*

$$\left[\int_0^a f(t)dt\right](\theta) = \int_0^a f(t)(\theta)dt$$

for $\theta \in (-\infty, 0]$.

The following consequence is useful in our proof. Before stating a useful consequence, we note that u'_t means $(u')_t$. The proof can be found in [11].

Theorem 2.3. *Suppose that phase space $\mathcal{P}$ that satisfies axioms (B) or (C). If $a > 0$ and $u : (-\infty, a] \to X$ is continuously differentiable on $[0, T]$ with $u_0 \in \mathcal{P}$, then $t \mapsto u_t$ is continuously differentiable on $[0, a]$ with $(u_t)' = u'_t$, where $(u_t)'$ denotes the Fréchet derivative of u_t.*

The following assumptions are considered.

(H2) $F \in C^1([0,T] \times \mathcal{P}, X)$ and for $t \in [0,T]$, there is a constant L such that

$$||F(t,\psi_1) - F(t,\psi_2)|| \leq L||\psi_1 - \psi_2||_{\mathcal{P}}$$

and

$$||D_1F(t,\psi_1) - D_1F(t,\psi_2)|| + ||D_2F(t,\psi_1) - D_2F(t,\psi_2)|| \leq L||\psi_1 - \psi_2||_{\mathcal{P}}$$

for $\psi_1,\ \psi_2 \in \mathcal{P}$ and D_i denotes the derivative with respect to the ith variable.

Remark 2.4. Let $\mathcal{P}$ satisfy axiom (B) or (C). Suppose that $u : (-\infty, T] \to X$ is continuously differentiable on $[0,T]$ with $u_0, u_0' \in \mathcal{P}$. If F satisfies (H2), then it is easy to see that

$$\frac{d}{ds}F(s,u_s) = D_1F(s,u_s) + D_2F(s,u_s)(u_s')$$

for $s \in [0,T]$.

For solving the equation (RVID), we recall the basic estimates for abstract Cauchy problems.

Theorem 2.5. ([4]). *Suppose that $f \in W^{1,1}([0,T],X)$ and $x \in D(A)$ such that $Ax + f(0) \in \overline{D(A)}$. Then there exists a unique $u \in C^1([0,T],X) \cap C([0,T],D)$ that satisfies*

$$\text{(ACP)} \qquad \begin{cases} u' = Au(t) + f(t), \quad t \in [0,T], \\ u(0) = x. \end{cases}$$

Moreover, one has the following estimates for each $t \in [0,T]$.

(a) $||u(t)|| \leq Me^{\omega t}(||u(0)|| + \int_0^t e^{-\omega s}||f(s)||ds)$.

(b) $||Au(t)|| \leq ||f(t)|| + Me^{\omega t}(||Au(0) + f(0)|| + \int_0^t ||e^{-\omega s}f'(s)||ds)$.

The existence of solutions to equation (1.2) can be found in [1].

Theorem 2.6. ([1]). *Suppose that F satisfies the hypothesis (H2), $\mathcal{P}$ satisfies the assumption (B) or (C), $\varphi \in \mathcal{P}$ is continuously differentiable with $\varphi' \in \mathcal{P}$, $\varphi(0) \in D(A)$ and $\varphi'(0) = A\varphi(0) + F(0,\varphi) \in \overline{D(A)}$. Then there exists a unique $u \in C^1([0,T],X) \cap C([0,T],D)$ which satisfies the equation* (1.2).

In the whole paper, we assume that k satisfies the hypothesis (H1), $\mathcal{P}$ always satisfies the axiom (B) or (C), and F satisfies the hypothesis (H2). According to Theorem 2.6, we assume that φ always denotes the initial condition of equation (RVID) and satisfies the assumptions in Theorem 2.6.

3. Local Existence and Uniqueness of Solutions to Equation (RVID)

In this section, the local existence and uniqueness to equation (RVID) will be proved.

Definition 3.1. We say that a function $u : (-\infty, T] \to X$ is a classical solution of equation (RVID) on $[0, T]$ if u satisfies the following conditions.

(i) $u|_{[0,T]} \in C^1([0, T], X) \cap C([0, T], D)$.

(ii) u satisfies $(RVID)$ on $[0, T]$.

(iii) $u(t) = \varphi(t)$ for $-\infty < t \leq 0$.

Let $\tau \in [0, T]$ and define the set $Z(\tau)$ by

$$Z(\tau) := \{u : (-\infty, \tau] \to X; u_0 \in \mathcal{P} \text{ and } u|_{[0,\tau]} \in C([0, \tau], D)\}.$$

Obviously, $Z(\tau)$ is a vector space. Moreover, it is easy to see that $(Z(\tau), ||\cdot||_{Z(\tau)})$ becomes a normed space if we define

$$||u||_{Z(\tau)} := ||u_0||_{\mathcal{P}} + \sup_{0\leq s\leq\tau} ||u(s)||_{D(A)}$$

for $u \in Z(\tau)$. We show that $(Z(\tau), ||\cdot||_{Z(\tau)})$ is a Banach space. It is sufficient to show the completeness of $Z(\tau)$. To do this, let $\{u^n\}$ be a Cauchy sequence of $Z(\tau)$. From the definition of $||\cdot||_{Z(\tau)}$, it follows that $\{(u^n)_0\}$ and $\{u^n|_{[0,\tau]}\}$ are Cauchy sequences of $\mathcal{P}$ and $C([0, \tau], D)$, respectively. So, $\{(u^n)_0\}$ and $\{u^n|_{[0,\tau]}\}$ are convergent in $\mathcal{P}$ and $C([0, \tau], D)$, respectively. Suppose that $f_1 = \lim_{n\to\infty} (u^n)_0$ and $f_2 = \lim_{n\to\infty} u^n|_{[0,\tau]}$, by hypothesis (A1)(ii), one can derive that

$$f_1(0) = \lim_{n\to\infty} (u^n)_0(0) = \lim_{n\to\infty} u^n|_{[0,\tau]}(0) = f_2(0).$$

So, $\{u^n\}$ converges to v in $(Z(\tau), ||\cdot||_{Z(\tau)})$ where v is defined by $v_0 = f_1$ and $v|_{[0,\tau]} = f_2$. Consequently, $(Z(\tau), ||\cdot||_{Z(\tau)})$ is a Banach space.

Next, we define

$$Z_\varphi(\tau) := \{u \in Z(\tau); u_0 = \varphi \text{ and } \sup_{0\leq s\leq\tau} ||u(s) - \varphi(0)||_{D(A)} \leq 1\}.$$

Then it is easy to see that $Z_\varphi(\tau)$ is a nonempty closed subset of $Z(\tau)$. For each $u \in Z_\varphi(\tau)$, we consider the following abstract Cauchy problem:

$$(\mathrm{ACP}_u) \qquad \begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t k(t,\theta,u(\theta))d\theta + F(t,x_t), \quad t \in [0,\tau], \\ x_0 = \varphi. \end{cases}$$

By Theorem 2.6, we know that equation (ACP$_u$) has a unique solution, say Su. So, we define P from $Z_\phi(\tau)$ into $Z(\tau)$ by $Pu := Su$ for $u \in Z_\varphi(\tau)$. If we can find a τ such that the range of P is contained in $Z_\phi(\tau)$ and P has a fixed point, then the proof for the local existence of solutions to equation (RVID) is completed. For finding this τ and the fixed point, the following lemmas are needed. For convenience, we define $K_\tau := \sup\limits_{0\le s\le\tau} K(s)$, $M_\tau := \sup\limits_{0\le s\le\tau} M(s)$ and the function $J : C([0,\tau],D) \to C([0,\tau],X)$ by $(Ju)(t) = \int_0^t k(t,\theta,u(\theta))d\theta$ for each $u \in C([0,\tau],D)$. Let Y be a Banach space. For each $u \in C([0,\tau],Y)$ we use $||u||_{C([0,\tau],Y)}$ to denote the sup-norm.

Lemma 3.2. *For each $\tau \in [0,T]$ and u_1, u_2 in $Z_\phi(\tau)$, there is a $\alpha_1(\tau)$ such that*

$$||Pu_1 - Pu_2||_{C([0,\tau],X)} \le \alpha_1(\tau)||u_1 - u_2||_{C([0,\tau],D)}$$

where $\alpha_1(\tau)$ depends on M, ω, L and b, and increases in τ with $\lim\limits_{\tau\to0^+} \alpha_1(\tau) = 0$.

Proof. From the definition of P, $Pu_1 - Pu_2$ is the unique solution to the following equation:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t k(t,\theta,u_1(\theta))d\theta - \int_0^t k(t,\theta,u_2(\theta))d\theta \\ \qquad\qquad +F(t,(Pu_1)_t) - F(t,(Pu_2)_t), \quad t \in [0,\tau], \\ x_0 = 0. \end{cases}$$

By Theorem 2.5, we know that

$$\begin{aligned} &||(Pu_1)(t) - (Pu_2)(t)|| \\ &\le Me^{|\omega|\tau}[\int_0^t \int_0^s ||k(s,\theta,u_1(\theta))d\theta - k(s,\theta,u_2(\theta))||d\theta ds \\ &\quad + \int_0^t ||F(s,(Pu_1)_s) - F(s,(Pu_2)_s)||ds] \\ &\le Me^{|\omega|\tau}\big[\tau^2 b(1)||u_1 - u_2||_{C([0,\tau],D)} + LK_\tau \int_0^t ||(Pu_1) - (Pu_2)||_{C([0,s],X)}ds\big]. \end{aligned}$$

for $t \in [0,\tau]$. Then the existence of $\alpha_1(\tau)$ comes from Gronwall's inequality. ∎

Lemma 3.3. *Suppose that $\tau \in [0, T]$, $t \in [0, \tau]$ and $u \in Z_\phi(\tau)$, then the following properties hold.*

(a) *There is a constant $\beta_1(\tau)$ independent of u such that*

$$||Pu||_{C([0,\tau],X)} \leq \beta_1(\tau).$$

(b) *There is a constant $\beta_2(\tau)$ independent of u such that*

$$\sup_{0\leq t\leq \tau} ||D_2F(t, (Pu)_t)|| \leq \beta_2(\tau).$$

(c) *There is a constant $\beta_3(\tau)$ independent of u_1 and u_2 such that*

$$||(Pu_1)_t' - (Pu_2)_t'||_{\mathcal{P}} \leq K_\tau ||A(Pu_1) - A(Pu_2)||_{C([0,\tau],X)} + \beta_3(\tau).$$

(d) *There is a constant $\beta_4(\tau)$ independent of u such that*

$$\sup_{0\leq t\leq \tau} ||(Pu)_t'||_{\mathcal{P}} \leq \beta_4(\tau).$$

β_i can be chosen so that it depends on ω, M, L, b and τ. Moreover, β_i is increasing in τ for $i = 1, 2, 3, 4$.

Proof.

(a) Let $u_1 \in Z_\phi(\tau)$ be a fixed element. From the definition of $Z_\phi(\tau)$, we know that $||u||_{C([0,\tau],D)} \leq 1 + ||\varphi(0)||_{D(A)}$ for each $u \in Z_\varphi(\tau)$. So, by Lemma 3.2, it is easy to see that

$$||Pu||_{C([0,\tau],X)} \leq \alpha_1(\tau)2(1 + ||\varphi(0)||_{D(A)}) + ||Pu_1||_{C([0,\tau],D)} := \beta_1(\tau).$$

(b) From the assumption of F, we know that

$$\begin{aligned}
&\sup_{0\leq t\leq \tau} ||D_2F(t, (Pu)_t)|| \\
&\leq L \sup_{0\leq t\leq \tau} ||(Pu)_t||_{\mathcal{P}} + \sup_{0\leq t\leq \tau} ||D_2F(t, 0)|| \\
&\leq LM_\tau ||\varphi||_{\mathcal{P}} + LK_\tau ||Pu||_{C([0,\tau],X)} + \sup_{0\leq t\leq \tau} ||D_2F(t, 0)|| \\
&\leq LM_\tau ||\varphi||_{\mathcal{P}} + LK_\tau \beta_1(\tau) + \sup_{0\leq t\leq \tau} ||D_2F(t, 0)|| := \beta_2(\tau).
\end{aligned}$$

(c) Since $Pu_1 - Pu_2$ is a solution of

$$\begin{cases} x'(t) = Ax(t) + (Ju_1)(t) - (Ju_2)(t) + F(t, (u_1)_t) - F(t, (u_2)_t), \quad t \in [0, \tau], \\ x(0) = 0 \end{cases}$$

and Theorem 2.3, we have

$$\begin{aligned}
& ||(Pu_1)'_t - (Pu_2)'_t||_{\mathcal{P}} \\
\leq & K_\tau \sup_{0 \leq s \leq t} ||A(Pu_1)(s) - A(Pu_2)(s) + (Ju_1)(s) - (Ju_2)(s) \\
& + F(s, (Pu_1)_s) - F(s, (Pu_2)_s)|| \\
\leq & K_\tau \{ ||A(Pu_1) - A(Pu_2)||_{C([0,t],X)} + ||Ju_1 - Ju_2||_{C([0,t],X)} \\
& + \sup_{0 \leq s \leq t} ||F(s, (Pu_1)_s) - F(s, (Pu_2)_s)|| \} \\
\leq & K_\tau \{ ||A(Pu_1) - A(Pu_2)||_{C([0,t],X)} + 2b(1)\tau(||\varphi(0)||_{D(A)} + 1) \\
& + 2LK_\tau \beta_1(\tau) \} := K_\tau ||A(Pu_1) - A(Pu_2)||_{C([0,t],X)} + \beta_3(\tau).
\end{aligned} \tag{3.1}$$

(d) According to (c), it is sufficient to show that $||A(Pu)||_{C([0,\tau],X)}$ is uniformly bounded. Let $u_1 \in Z_\phi(\tau)$ be a fixed element. By Theorem 2.5, Remark 2.4 and definition of P, we see that

$$\begin{aligned}
& ||A(Pu)(t) - A(Pu_1)(t)|| \\
\leq & ||F(t, (Pu)_t) - F(t, (Pu_1)_t)|| + ||(Ju)(t) - (Ju_1)(t)|| \\
& + Me^{|\omega|t} [\int_0^t ||(Ju)'(s) - (Ju_1)'(s)|| ds \\
& + \int_0^t || \frac{d}{ds} (F(s, (Pu)_s) - F(s, (Pu_1)_s)|| ds] \\
\leq & LK_\tau ||Pu - Pu_1||_{C([0,\tau],X)} + ||(Ju)(t) - (Ju_1)(t)|| \\
& + Me^{|\omega|t} [\int_0^t ||(Ju)'(s) - (Ju_1)'(s)|| ds \\
& + \int_0^t ||D_1(F(s, (Pu_1)_s) - D_1 F(s, (Pu)_s)|| ds \\
& + \int_0^t ||D_2 F(s, (Pu_1)_s) - D_2 F(s, (Pu)_s)|| \cdot ||(Pu_1)'_s||_{\mathcal{P}} ds \\
& + \int_0^t ||D_2 F(s, (Pu)_s)|| \cdot ||(Pu_1)'_s - (Pu)'_s||_{\mathcal{P}} ds] \\
:= & A_1 + A_2 + A_3 + A_4 + A_5 + A_6
\end{aligned} \tag{3.2}$$

for $u \in Z_\phi(\tau)$. For A_1, (a) implies that

$$A_1 = LK_\tau ||Pu - Pu_1||_{C([0,\tau],X)} \le LK_\tau 2\beta_1(\tau).$$

For A_2, hypothesis (H1) implies that

$$A_2 \le b(1)\tau ||u - u_1||_{C([0,\tau],D)} \le 2b(1)\tau(||\varphi(0)||_{D(A)} + 1).$$

For A_3, hypothesis (H1) also implies that

$$\begin{aligned} A_3 \le{}& Me^{|\omega|t}[\int_0^t ||k(t,\theta,u(\theta)) - k(t,\theta,u_1(\theta))||d\theta \\ &+ \int_0^t \int_0^s ||k_s(s,\theta,u(\theta)) - k_s(s,\theta,u_1(\theta))||d\theta ds] \\ &\le Me^{|\omega|\tau}[b(1)\tau||u-u_1||_{C([0,\tau],D)} + b(1)\tau^2||u-u_1||_{C([0,\tau],D)}] \\ &\le 2Me^{|\omega|\tau}b(1)[\tau+\tau^2](||\varphi(0)||_{D(A)}+1). \end{aligned}$$

For (A_4), it is easy to see that

$$A_4 \le 2\tau Me^{|\omega|\tau} LK_\tau \beta_1(\tau).$$

From (c) and axiom (A2), we know that

$$\begin{aligned} ||(Pu)'_s||_{\mathcal{P}} \le{}& K_\tau ||A(Pu_1) - A(Pu)||_{C([0,s],X)} \\ &+\beta_3(\tau) + \sup_{0\le r\le\tau} ||(Pu_1)'_r||_{\mathcal{P}}. \end{aligned} \tag{3.3}$$

So, (b) and (3.3) imply that

$$\begin{aligned} A_5 \le{}& Me^{|\omega|\tau} 2\beta_2(\tau) \int_0^t \big\{K_\tau ||A(Pu_1) - A(Pu)||_{C([0,s],X)} + \beta_3(\tau) \\ &+ \sup_{0\le r\le\tau} ||(Pu_1)'_r||_{\mathcal{P}}\big\} ds \\ \le{}& Me^{|\omega|\tau} 2\beta_2(\tau) \big\{ \int_0^t K_\tau ||A(Pu_1) - A(Pu)||_{C([0,s],X)} ds + \tau\beta_3(\tau) \\ &+ \tau \sup_{0\le r\le\tau} ||(Pu_1)'_r||_{\mathcal{P}} \big\}. \end{aligned}$$

For (A_6), (b) and (c) imply that

$$A_6 \le \beta_2(\tau) Me^{|\omega|\tau} \big\{ \int_0^t K_\tau ||A(Pu) - A(Pu_1)||_{C([0,s],X)} ds + \tau\beta_3(\tau) \big\}.$$

Consequently, from the estimates for A_i for $i = 1, 2 \cdots 6$, it follows that there exist $C_1(\tau)$ and $C_2(\tau)$ which are increasing in τ such that

$$||(APu_1)(t) - (APu_2)(t)|| \le C_1(\tau) + C_2(\tau) \int_0^t ||APu_1 - APu_2||_{C([0,s],X)} ds.$$

By Gronwall's inequality, there exists $\tilde{\beta}_4(\tau)$ such that

$$||A(Pu) - A(Pu_1)||_{C([0,\tau],X)} \le \tilde{\beta}_4(\tau).$$

So, we can choose $\beta_4(\tau) := \tilde{\beta}_4(\tau) + ||A(Pu_1)||_{C([0,\tau],X)}$.

Finally, the properties of β_i are easy to see from their choices. ■

Lemma 3.4. *For each $\tau \in [0, T]$ and u_1, $u_2 \in Z_\phi(\tau)$, there is an $\alpha_2(\tau)$ such that*

$$||Pu_1 - Pu_2||_{C([0,\tau],D)} \le \alpha_2(\tau)||u_1 - u_2||_{C([0,\tau],D)}$$

where $\lim_{\tau \to 0} \alpha_2(\tau) = 0$ and it depends on L, M, ω and b.

Proof. Let $t \in [0, \tau]$. According to Lemma 3.2, it is sufficient to find a $C(\tau)$ such that $||A(Pu_2) - A(Pu_1)||_{C([0,\tau],X)} \le C(\tau)||u_1 - u_2||_{C([0,\tau],D)}$ with $\lim_{\tau \to 0} C(\tau) = 0$. First, according to Lemma 3.2, we see that

$$\begin{aligned} & ||(Pu_1)'_t - (Pu_2)'_t|| \\ & \le ||A(Pu_1)(t) - A(Pu_2)(t)|| + ||(Ju_1)(t) - (Ju_2)(t)|| \\ & \quad + ||F(t, (Pu_1)_t - F(t, (Pu_2)_t|| \\ & \le ||A(Pu_1)(t) - A(Pu_2)(t)|| + [b(1)\tau + LK_\tau \alpha_1(\tau)]||u_1 - u_2||_{C([0,\tau],D)}. \end{aligned} \tag{3.4}$$

Using (3.2), (3.4) and a similar argument to Lemma 3.3, we have

$$\begin{aligned} & ||A(Pu_2)(t) - A(Pu_1)(t)|| \\ & \le ||F(t, (Pu_1)_t) - F(t, (Pu_2)_t)|| + ||(Ju_1)(t) - (Ju_2)(t)|| \\ & \quad + Me^{|\omega|t}\Big[\int_0^t ||(Ju_1)'(s) - (Ju_2)'(s)||ds \\ & \quad + ||\frac{d}{ds}(F(s, (Pu_2)_s) - F(s, (Pu_1)_s)||ds\Big] \\ & \le LK_\tau ||Pu_2 - Pu_1||_{C([0,\tau],X)} + ||(Ju_1)(t) - (Ju_2)(t)|| \\ & \quad + Me^{|\omega|t}\Big[\int_0^t ||(Ju_1)'(s) - (Ju_2)'(s)||ds \\ & \quad + \int_0^t ||D_1(F(s, (Pu_1)_s) - D_1F(s, (Pu_2)_s)||ds \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&+\int_0^t ||D_2F(s,(Pu_1)_s)-D_2F(s,(Pu_2)_s)||\cdot||(Pu_1)'_s||_{\mathcal{P}}ds\\
&+\int_0^t ||D_2F(s,(Pu_2)_s)||\cdot||(Pu_1)'_s-(Pu_2)'_s||_{\mathcal{P}}ds]\\
\le&\ \big\{LK_\tau\alpha_1(\tau)+b(1)\tau+Me^{|\omega|\tau}b(1)[\tau+\tau^2]\\
&+\tau Me^{|\omega|\tau}LK_\tau\alpha_1(\tau)+\tau LK_\tau Me^{|\omega|\tau}\beta_4(\tau)\alpha_1(\tau)\\
&+\tau Me^{|\omega|\tau}\beta_2(\tau)[b(1)\tau+LK_\tau\alpha_1(\tau)]\big\}||u_1-u_2||_{C([0,\tau],D)}\\
&+Me^{|\omega|\tau}K_\tau\beta_2(\tau)\int_0^t ||A(Pu_2)-A(Pu_1)||_{C([0,s],X)}ds,
\end{aligned}
$$

where β_i is the number in Lemma 3.3. Then, the existence of $C(\tau)$ can be found by Gronwall's inequality and definitions of $\beta_i(\tau)$ and $\alpha_1(\tau)$. ■

Theorem 3.5. *There is a* $\tau\in[0,T]$ *such that equation (RVID) has a solution on* $[0,\tau]$.

Proof. According to Lemma 3.4, there is a $\tau_1\in[0,T]$ such that $||Pu_1-Pu_2||_{Z(\tau_1)}\le\frac{1}{2}||u_1-u_2||_{Z(\tau_1)}$ for each $u_1,u_2\in Z_\varphi(\tau_1)$. If we can find a $\tau\in[0,\tau_1]$ such that $P(Z_\varphi(\tau))\subset Z_\varphi(\tau)$, then the existence of solution on $[0,\tau]$ is deduced by Contraction Mapping Principle.

Define $h_1(t)=\begin{cases}\varphi(0), & t\in[0,\tau_1],\\ \varphi(t), & t\in(-\infty,0].\end{cases}$ Obviously, $h_1\in Z_\varphi(\tau_1)$. From the definiton of P, there is a $\tau\le\tau_1$ such that $\sup\limits_{0\le t\le\tau}||(Ph_1)(t)-\varphi(0)||_{D(A)}<\frac{1}{2}$. Hence,

$$
\begin{aligned}
&\sup_{0\le t\le\tau}||(Pu)(t)-\varphi(0)||_{D(A)}\\
\le&\sup_{0\le t\le\tau}||(Ph_1)(t)-\varphi(0)||_{D(A)}+||Ph_1-Pu||_{C([0,\tau],D)}\\
<&\ \frac{1}{2}+\frac{1}{2}||h_1-u||_{Z(\tau_1)}\\
<&\ \frac{1}{2}+\frac{1}{2}=1
\end{aligned}
$$

for $u\in Z_\varphi(\tau)$. So, $Pu\in Z_\varphi(\tau)$ for $u\in Z_\varphi(\tau)$ and the proof is completed. ■

Theorem 3.6. *The solution of equation (RVID) is unique on any interval* $[0,\tau]\subset[0,T]$ *if it exists.*

Proof. Assume that u and v are two solutions in $[0,\tau]$. Let $\hat{t} := \max\{t \in [0,\tau]; u \equiv v \text{ on } [0,t]\}$. Suppose that $\hat{t} < \tau$ and set $y_0 := v_{\hat{t}} = u_{\hat{t}}$. Then

$$h(t) := \int_0^{\hat{t}} k(t,\theta,u(\theta))d\theta = \int_0^{\hat{t}} k(t,\theta,v(\theta))d\theta$$

for each $t \in [0,\hat{t}]$. We consider the following equation

$$\text{(3.6)} \qquad \begin{cases} w'(t) = Aw(t) + \displaystyle\int_{\hat{t}}^{t} k(t,\theta,w(\theta))d\theta + h(t) + F(t,w_t), \quad t \in [\hat{t},\tau], \\ w_{\hat{t}} = y_0. \end{cases}$$

Noting that $y_0 \in \mathcal{P}$ and is continuously differentiable by axiom (A2) and Theorem 2.3. By a similar argument in Theorem 3.5, we know that there is an $\varepsilon \in [0,\tau-\hat{t}]$ such that equation (3.6) has a unique solution on $[\hat{t},\hat{t}+\varepsilon]$. But this contradicts the definiton of $\hat{t}$. It follows that $u \equiv v$ and the solution is unique. ∎

We say that $u : [0,\tau^*) \to D(A)$ is a maximal solution of equation (RVID) if $v \in C^1([0,\tau'],X) \cap C([0,\tau'],D)$ is another solution of equation (RVID), then $\tau' \leq \tau^*$.

From the previous discussion, the maximal solution can be obtained via the usual extension procedure. Next, we study the property of the maximal solution to equation (RVID).

Lemma 3.7. *Suppose that $u : [0,\tau) \to D(A)$ is a maximal solution of equation (RVID) with $\tau < T$ and $\limsup_{t\to\tau} ||u(t)||_{D(A)}$ is finite. Then the following conditions hold.*

(i) *There are constants c_1, $c_2 \in \mathbb{R}^+$ such that*

$$c_1 = \sup\{||k(t,s,u(s))||; 0 \leq s \leq t < \tau\}$$

and

$$c_2 = \sup\{||\partial_t k(t,s,u(s))||; 0 \leq s \leq t < \tau\}.$$

(ii) *There are constants c_3, c_4, $c_5 \in \mathbb{R}^+$ such that*

$$c_3 = \sup\{||F(t,u_t)||; 0 \leq t < \tau\},$$

$$c_4 = \sup\{||D_1 F(t,u_t)||; 0 \leq t < \tau\}$$

and

$$c_5 = \sup\{||D_2 F(t,u_t)||; 0 \leq t < \tau\}.$$

(iii) *There are constants* $c_6,\ c_7 \in \mathbb{R}^+$ *such that*

$$c_6 = \sup\{||u'(t))||; 0 \leq t < \tau\}$$

and

$$c_7 = \sup\{||u_t'||_{\mathcal{P}}; 0 \leq t < \tau\}.$$

Proof. Since $\limsup_{t\to\tau} ||u(t)||_{D(A)}$ is a finite number, there is a constant $C > 0$ such that $\sup_{0\leq t<\tau} ||u(t)||_{D(A)} \leq C$. Let $0 \leq s \leq t < \tau$. From assumption (H1), it follows that

$$\begin{aligned} ||k(t,s,u(s))|| &\leq ||k(t,s,u(s)) - k(t,s,0)|| + ||k(t,s,0)|| \\ &\leq b(C)||u(s)||_{D(A)} + \sup_{0\leq s\leq t\leq\tau} ||k(t,s,0)|| \\ &\leq C\cdot b(C) + \sup_{0\leq s\leq t\leq\tau} ||k(t,s,0)||. \end{aligned} \tag{3.7}$$

According to k is continuous on $\Delta(0,T)\times D$, it follows that $\sup_{0\leq s\leq t\leq\tau} ||k(t,s,0)||$ is finite. So, from (3.7), the existence of c_1 is deduced.

The rest of proofs for (i) and (ii) are similar.

Finally, we show the existence of c_6 and c_7. Since u is a maximal solution of equation (RVID), it follows that

$$||u'(t)|| \leq ||Au(t)|| + \int_0^t ||k(t,s,u(s))||ds + ||F(t,u_t)||$$

for $t\in[0,\tau)$. Then the existence of c_6 is deduced from the boundedness of $||Au(\cdot)||$, (i) and (ii). The existence of c_7 immediately comes from c_6 and hypothesis (A1). ■

Theorem 3.8. *Suppose that* $u : [0,\tau) \to D(A)$ *is a maximal solution of equation (RVID). Then one of the following properties holds.*

(a) $\tau = T$ *and* u *can be extended to a solution of equation (RVID) on* $[0,T]$.

(b) $\limsup_{t\to\tau} ||u(t)||_{D(A)} = \infty$.

Proof. Suppose that (b) is not true. It implies that the maximal solution u of equation (RVID) is contained in a bounded set of D and $\tau \neq T$. Let $\gamma > 0$ such that $0 \leq t < t+\gamma \leq \tau$. Define

$$v(t) := u(t+\gamma) - u(t)$$

and

$$G(t) := F(t+\gamma, u_{t+\gamma}) - F(t, u_t) + \int_0^{t+\gamma} k(t+\gamma, \theta, u(\theta))d\theta - \int_0^t k(t, \theta, u(\theta))d\theta$$

for $0 \leq t < t+\gamma \leq \tau$. Then v satisfies

$$\begin{cases} v'(t) = Av(t) + G(t), \quad t \in [0, \tau - \gamma), \\ v_0 = u_\gamma - \varphi. \end{cases}$$

So, Theorem 2.5 implies that there is a $\zeta > 0$ independent of t and γ such that

$$\begin{aligned} &||v(t)||_{D(A)} = ||u(t+\gamma) - u(t)||_{D(A)} \\ &\leq \zeta \big\{ ||u(\gamma) - u(0)|| + ||Av(0) + G(0)|| + ||G(t)|| \\ &+ \int_0^t ||G(s)||ds + \int_0^t ||\frac{d}{ds}G(s)||ds \big\} \\ &:= \zeta(A_1 + A_2 + A_3 + A_4 + A_5) \end{aligned}$$

for $t \in [0, \tau - \gamma)$. We will show that for any $t \in [0, \tau - \gamma]$ and $\varepsilon > 0$, there is a $\delta > 0$ such that $||u(t+\gamma) - u(t)||_{D(A)} \leq \varepsilon$ for $0 \leq \gamma \leq \delta$. If it is true, then u is uniformly continuous on $[0, \tau)$. So, u can be extended to $[0, \tau]$ and the maximallity implies that $\tau = T$. It follows that (a) is true. Hence, the main work of the rest proof is to estimate A_i for $= 1, 2, 3, 4, 5$. It is easy to see that $A_2 = ||u'(\gamma) - u'(0)||$. So, the continuous differentiability of u implies there is a $\delta_1 \leq \tau$ such

$$\zeta(A_1 + A_2) \leq \frac{\varepsilon}{4}$$

for $\gamma \leq \delta_1$. Next, from Lemma 3.7, we know that there are c_1, $c_2 \in \mathbb{R}^+$ such that

$$\begin{aligned} A_3 = ||G(t)|| &\leq ||F(t+\gamma, u_{t+\gamma}) - F(t, u_t)|| \\ &+ \int_0^t ||k(t+\gamma, \theta, u(\theta)) - k(t, \theta, u(\theta))||d\theta + \int_t^{t+\gamma} ||k(t+\gamma, \theta, u(\theta))||d\theta \\ &\leq \int_t^{t+\gamma} ||\frac{d}{ds}F(s, u_s)||ds + c_2\tau\gamma + c_1\gamma. \end{aligned}$$

By the definitons of F and u, there is a δ_2 such that

$$\zeta A_3 \leq \frac{\varepsilon}{4}$$

for $\gamma \leq \delta_2$. Moreover, from the estimate of A_3, it follows that there is a δ_3 such that

$$\zeta A_4 = \zeta \int_0^t ||G(s)||ds \leq \frac{\varepsilon}{4}$$

for $\gamma \leq \delta_3$. It is easy to see that

$$\frac{d}{ds}G(s) = \frac{d}{ds}F(s+\gamma, u_{s+\gamma}) - \frac{d}{ds}F(s, u_s) + k(s+\gamma, s+\gamma, u(s+\gamma)) \\ - k(s,s,u(s)) + \int_0^{s+\gamma} k_s(s+\gamma, \theta, u(\theta))d\theta - \int_0^s k_s(s,\theta,u(\theta))d\theta$$

for $s \in [0, \tau - \gamma)$. By lemma 3.7(ii) and (iii), it follows that the set

$$\{||\frac{d}{ds}F(s+\gamma, u_{s+\gamma}) - \frac{d}{ds}F(s, u_s)||; s \in [0, \tau-\gamma)\}$$

is bounded. So,

$$\begin{aligned} &\int_0^t ||\frac{d}{ds}G(s)||ds \leq \int_0^t ||\frac{d}{ds}F(s+\gamma, u_{s+\gamma}) - \frac{d}{ds}F(s,u_s)||ds \\ &+ \int_0^t ||k(s+\gamma, s+\gamma, u(s+\gamma)) - k(s,s,u(s))||ds \\ &+ \int_0^t \int_{-\gamma}^0 ||k_s(s+\gamma, \theta+\gamma, u(\theta+\gamma))||d\theta ds \\ &+ \int_0^t \int_0^s ||k_s(s+\gamma, \theta+\gamma, u(\theta+\gamma)) - k_s(s,\theta,u(\theta))||d\theta ds. \end{aligned} \tag{3.8}$$

By (3.8), Lemma 3.7(i) and boundedness of $||\frac{d}{d\cdot}F(\cdot+\gamma, u_{\cdot+\gamma}) - \frac{d}{d\cdot}F(\cdot, u_\cdot)||$, we know that there is a δ_4 such that

$$\zeta A_5 \leq \frac{\varepsilon}{4}$$

for $\gamma \leq \delta_4$. Finally, $\delta := \min(\delta_1, \delta_2, \delta_3, \delta_4)$ is the desired number. ∎

We conclude this section by showing the well posedness of solutions to the equation (RVID). The following lemma is needed.

Lemma 3.9. *Suppose that u is the unique solution to equation (RVID) with initial condition φ on $[0,t]$ for some $t > 0$. In addition that $\psi \in \mathcal{P}$ is continuously differentiable with $\psi' \in \mathcal{P}$, $\psi(0) \in D(A)$, $\psi'(0) = A\psi(0) + F(0,\psi) \in \overline{D(A)}$ and the equation $(RVID)$ with initial data ψ also has a unique solution v on $[0,t]$. If $||v||_{C([0,s],D)} \leq 1 + ||u||_{C([0,s],D)}$ for $s \in [0,t]$, then there is an increasing function $C : [0,t] \to \mathbb{R}^+$ such that*

$$||u-v||_{C([0,s],D)} \leq C(s)(||\varphi - \psi||_{\mathcal{P}} + ||\varphi' - \psi'||_{\mathcal{P}} + ||\varphi(0) - \psi(0)||_{D(A)}).$$

Proof. Let $w := u - v$. Then w is the unique solution of the following equation

$$\begin{cases} w'(s) = Aw(s) + (Ju)(s) - (Jv)(s) + F(s, u_s) - F(s, v_s), \quad s \in [0, t], \\ w_0 = \varphi - \psi. \end{cases}$$

Following the argument in the proof of Lemma 3.2 and Lemma 3.4, we can get the desired conclusion. We give a sketch of the proof here.

Let $s \in [0, t]$ and set the constants $\alpha := 1 + ||u||_{C([0,t],D)}$ and $\lambda := Me^{|\omega|t}$.

From hypothesis (H1), (H2) and axiom (A1), it follows that the following inequalities hold.

$$||u'_s||_{\mathcal{P}} \leq K_t||u'||_{C([0,s],D)} + M_t||\varphi'||_{\mathcal{P}}, \tag{3.9}$$

$$\begin{aligned} &||D_2F(s, v_s)|| \leq L||v_s||_{\mathcal{P}} + \sup_{0\leq s\leq t} ||D_2F(s, 0)|| \\ &\leq L(K_t\alpha + M_t||\psi||_{\mathcal{P}}) + \sup_{0\leq s\leq t} ||D_2F(s, 0)||, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &||w'_s||_{\mathcal{P}} \leq K_t||w'||_{C([0,s],D)} + M_t||\varphi' - \psi'||_{\mathcal{P}} \\ &\leq K_t\big[||Aw||_{C([0,s],X)} + \textstyle\int_0^s b(\alpha)||w||_{C([0,r],D)}dr \\ &+LK_t||w||_{C([0,s],D)} + LM_t||\varphi - \psi||_{\mathcal{P}}\big] + M_t||\varphi' - \psi'||_{\mathcal{P}}, \end{aligned} \tag{3.11}$$

$$||D_2F(r, u_s) - D_2F(r, v_s)|| \leq LK_t||w||_{C([0,s],D)} + LM_t||\varphi - \psi||_{\mathcal{P}}. \tag{3.12}$$

Moreover, we have the fact

$$\begin{aligned} &|| \int_0^s \frac{d}{dr}[F(r, u_r) - F(r, v_r)]dr|| \\ &\leq \int_0^s ||D_1F(r, u_r) - D_1F(r, v_r)||dr \\ &+ \int_0^s ||D_2F(r, u_r) - D_2F(r, v_r)|| \cdot ||u'_r||_{\mathcal{P}}dr \\ &+ \int_0^s ||D_2F(r, v_r)|| \cdot ||u'_r - v'_r||_{\mathcal{P}}dr. \end{aligned} \tag{3.13}$$

On the other hand, by Theorem 2.5, we have

$$\begin{aligned} &||w(s)|| \leq \lambda\big[||\varphi(0) - \psi(0)||_{D(A)} + \int_0^s \int_0^r b(\alpha)||w||_{C([0,\theta],D)}d\theta dr \\ &+ \int_0^s (LK_t||w||_{C([0,r],D)} + LM_t||\varphi - \psi||_{\mathcal{P}})dr\big] \end{aligned} \tag{3.14}$$

and

$$
\begin{aligned}
&||Aw(s)|| \\
&\leq ||(Ju)(s) - (Jv)(s) + F(s, u_s) - F(s, v_s)|| \\
&\quad +\lambda(||A\varphi(0) - A\psi(0)|| + L||\varphi - \psi||_{\mathcal{P}}) \\
&\quad +\lambda \int_0^s \{||\frac{d}{dr}[(Ju)(r) - (Jv)(r)]|| + ||\frac{d}{dr}[F(r, u_r) - F(r, v_r)]||\} dr.
\end{aligned} \tag{3.15}
$$

Adding (3.14) to (3.15), we substitute (3.9)-(3.13) into (3.15), then a similar argument to (3.5) and Gronwall's inequality deduces the existence of $C(\cdot)$. ∎

Theorem 3.10. *Suppose that $u(\cdot, \varphi)$ is the unique solution to equation (RVID) on a maximal interval of existence $[0, \tau(\varphi))$ and $t \in [0, \tau(\varphi))$, then there exist positive constants C and δ such that for $\psi \in \mathcal{P}$, ψ is continuously differentiable with $\psi' \in \mathcal{P}$, $\psi(0) \in D(A)$, $\psi'(0) = A\psi(0) + F(0, \psi) \in \overline{D(A)}$ and*

$$||\varphi - \psi||_{\mathcal{P}} + ||\varphi' - \psi'||_{\mathcal{P}} + ||\varphi(0) - \psi(0)||_{D(A)} \leq \delta,$$

we have

$$||u(s, \varphi) - u(s, \psi)||_{D(A)} \leq C(||\varphi - \psi||_{\mathcal{P}} + ||\varphi' - \psi'||_{\mathcal{P}} + ||\varphi(0) - \psi(0)||_{D(A)})$$

for all $s \in [0, t]$.

Proof. Let $t \in [0, \tau(\varphi))$. We put $\alpha := 1 + ||u(\cdot, \varphi)||_{C([0,t],D)}$. Let $C(t)$ be the number defined in Lemma 3.9. Choose $\delta \in (0, 1)$ such that $C(t)\delta < 1$ and let

$$
\begin{aligned}
B_\delta := \{&\psi \in \mathcal{P}; \psi' \in \mathcal{P},\ \psi(0) \in D(A),\ \psi'(0) = A\psi(0) + F(0, \psi) \in \overline{D(A)}, \\
&||\varphi - \psi||_{\mathcal{P}} + ||\varphi' - \psi'||_{\mathcal{P}} + ||\varphi(0) - \psi(0)||_{D(A)} \leq \delta\}.
\end{aligned}
$$

We will show that

$$||u(s, \varphi) - u(s, \psi)|| \leq C(t)(||\varphi - \psi||_{\mathcal{P}} + ||\varphi' - \psi'||_{\mathcal{P}} + ||\varphi(0) - \psi(0)||_{D(A)})$$

for $s \in [0, t]$ but first, we prove the following claim.

Claim. Let $\psi \in B_\delta$ and let $T_0 = \sup\{s > 0; ||u(\tau, \psi)||_{D(A)} \leq \alpha$ for $\tau \in [0, s]\}$. Then $t < T_0$.

Suppose the claim is false, i.e. $T_0 \leq t$. By Lemma 3.9, we know that there exists $C(T_0)$ such that $C(T_0) \leq C(t)$ and

$$||u(\tau, \varphi) - u(\tau, \psi)||_{D(A)} \leq C(T_0)(||\varphi - \psi||_{\mathcal{P}} + ||\varphi' - \psi'||_{\mathcal{P}} + ||\varphi(0) - \psi(0)||_{D(A)})$$

for $\tau \in [0, T_0]$. So, we obtain that

$$||u(s,\psi)||_{D(A)} \leq C(t)\delta + \alpha - 1 < \alpha$$

for $s \in [0, T_0]$. By the continuity of $u(\cdot,\psi)$, it implies that T_0 can not be the largest number $s > 0$ such that $||u(\zeta,\psi)||_{D(A)} \leq \alpha$ for each $\zeta \in [0, s]$. So, it follows that $T_0 > t$. We complete the proof of the claim.

Finally, we know that $||u(s,\psi)||_{D(A)} \leq \alpha$ for $s \in [0,t]$ and $\psi \in B_\delta$ from the claim. Using Lemma 3.9 again, we deduce the conclusion by setting $C := C(t)$. ■

4. An Example

In this section, we apply our abstract results to the following partial differential equation

$$(4.1)\qquad \begin{aligned} \omega_t(t,x) + \omega_x(t,x) &= f(\omega(t-\eta,x)) + c\int_{-\infty}^{0} G_1(s)\omega(t+s,x)ds \\ &\quad + \int_0^t G_2(t-s,\omega_x(s,x))ds,\ (t,x)\in[0,T]\times[0,1], \\ \omega(t,0) &= \omega(t,1), \quad t\in[0,T], \\ \omega(t,x) &= \omega_0(t,x), \quad (t,x)\in(-\infty,0]\times[0,1], \end{aligned}$$

where η and c are positive constants, $G_1 : (-\infty, 0] \to \mathbb{R}$, $G_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$.

Let $X = C([0,1],\mathbb{R})$ equipped with supnorm $||\cdot||$ and define the linear operator A from $D(A)$ into X by $Ay = -y'$ for $y \in D(A) = \{y \in C^1([0,1],\mathbb{R}); y(0) = y(1)\}$. Noting that A is not densely defined and $\overline{D(A)} = \{y \in C([0,1],\mathbb{R}); y(0) = y(1)\}$. In [12], it has been shown that A is a Hille-Yosida operator on X. Let $\gamma > 0$. We choose phase space

$$\mathcal{P} := \{\phi \in C((-\infty,0],X); \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } X\}$$

with $||\psi||_{\mathcal{P}} := \sup_{\theta\leq 0} ||e^{\gamma\theta}\psi(\theta)||$ for $\psi \in \mathcal{P}$. It had been shown that $\mathcal{P}$ satisfies the axiom (A1) and (A2). Furthermore, it had been also shown that $\mathcal{P}$ satisfies the axioms (C) in [8].

The equation (4.1) can be transformed into

$$(4.2)\qquad \begin{aligned} u'(t) &= Au(t) + \int_0^t k(t,s,u(s))ds + F(u_t),\ t\in[0,T] \\ u_0 &= \varphi, \end{aligned}$$

by setting

$$\begin{aligned}
u(t)(\xi) &= \omega(t,\xi),\ (t,\xi)\in[0,T]\times[0,1],\\
\varphi(\theta)(\xi) &= \omega_0(\theta,\xi),\ (\theta,\xi)\in(-\infty,0]\times[0,1],\\
k(t,s,\psi)(\xi) &= G_2(t-s,\psi'(\xi)),\psi\in D(A),\\
F(\phi)(\xi) &= f(\phi(-\eta)(\xi))+\int_{-\infty}^{0} G_1(s)(\phi(s))(\xi)ds,\ \phi\in\mathcal{P},\xi\in[0,1].
\end{aligned}$$

In addition, we suppose that

(i) f is continuously differentiable and f' is Lipschitz continuous.

(ii) $G_1(\cdot)e^{-\gamma\cdot}$ is integrable on $(-\infty,0]$.

(iii) $G_2\in C^1([0,T]\times\mathbb{R},\mathbb{R})$ and for each $r>0$ there is a constant $b(r)>0$ such that

$$|G_2(t,x)-G_2(t,y)|\le b(r)|x-y|$$

and

$$|\partial_t G_2(t,x)-\partial_t G_2(t,y)|\le b(r)|x-y|$$

for each $t\in[0,T]$ and $|x|,|y|\le r$.

(iv) $\omega_0\in C^1((-\infty,0]\times[0,1],\mathbb{R})\cap\mathcal{P}$ with $\lim\limits_{\theta\to-\infty}(e^{\gamma\theta}\sup\limits_{0\le\xi\le1}|\frac{\partial}{\partial\theta}\omega_0(\theta,\xi)|)<\infty$, $\omega_0(0,0)=\omega_0(0,1)$, $\frac{\partial}{\partial\theta}\omega_0(0,0)=\frac{\partial}{\partial\theta}\omega_0(0,1)$ and $\frac{\partial}{\partial\theta}\omega_0(0,\xi)=-\frac{\partial}{\partial\xi}\omega_0(0,\xi)+f(\omega_0(-\eta,\xi))+\int_{-\infty}^{0}G_1(s)\omega_0(s,\xi)ds$ for $\xi\in[0,1]$.

First, we show that k satisfies the hypothesis (H1). Obviously, from the definition of k and G_2, it is easy to see that $(t,s,\psi)\mapsto k(t,s,\psi)$ and $(t,s,\psi)\mapsto\partial_t k(t,s,\psi)$ are continuous functions from $\Delta(0,T)\times D(A)$ into X. Moreover, by assumption (iii), it follows that

$$\begin{aligned}
||k(t,s,\psi_1)-k(t,s,\psi_2)|| &= \sup_{0\le\zeta\le1}|G_2(t-s,\psi_1'(\zeta))-G_2(t-s,\psi_2'(\zeta))|\\
&\le \sup_{0\le\zeta\le1} b(r)|\psi_1'(\zeta)-\psi_2'(\zeta)|\\
&= b(r)||\psi_1'-\psi_2'||\le b(r)||\psi_1-\psi_2||_{D(A)}
\end{aligned}$$

for $(t,s)\in\Delta(0,T)$ and $\psi_1,\psi_2\in D(A)$ with $||\psi_1||_{D(A)},||\psi_1||_{D(A)}\le r$. Similarly, we also can show that

$$||\partial_t k(t,s,\psi_1)-\partial_t k(t,s,\psi_2)||\le b(r)||\psi_1-\psi_2||\le b(r)||\psi_1-\psi_2||_{D(A)}$$

for $(t,s) \in \Delta(0,T)$ and $\psi_1, \psi_2 \in D(A)$ with $||\psi_1||_{D(A)}, ||\psi_1||_{D(A)} \leq r$. Consequently, k satisfies the hypothesis (H1).

Next, for $\phi_1, \phi_2 \in \mathcal{P}$, in [2] the authors show that

$$\sup_{0\leq\zeta\leq 1} \int G_1(s)|(\phi_1)(s)(\zeta) - \phi_2(s)(\zeta)|ds$$
$$\leq \int_{-\infty}^{0} e^{-\gamma s} G_1(s) ds \sup_{-\infty<s\leq 0} [\sup_{0\leq\zeta\leq 1} e^{-\gamma s}|(\phi_1)(s)(\zeta) - \phi_2(s)(\zeta)|].$$

Hence, from assumption (i) and (ii), it follows that F satisfies the hypothesis (H2) with

$$F'(\phi)(\psi)(\xi) = f'(\phi(-\eta)(\xi))\psi(-\eta)(\xi) + c\int_{-\infty}^{0} G_1(s)(\psi(s))(\xi)ds$$

for $\phi,\ \psi \in\ \mathcal{P}$. Finally, from Theorem 3.5 and Theorem 3.6, it follows that there exists a $\tau \in [0,T]$ such that equation (4.2) admits a unique solution on $[0,\tau]$.

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