

ALGORITHMIC ASPECTS OF MAJORITY DOMINATION*

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Abstract. This paper studies algorithmic aspects of majority domination, which is a variation of domination in graph theory. A majority dominating function of a graph $G = (V, E)$ is a function g from V to $\{-1, 1\}$ such that $\sum_{u \in N[v]} g(u) \geq 1$ for at least half of the vertices $v \in V$. The majority domination problem is to find a majority dominating function g of a given graph $G = (V, E)$ such that $\sum_{v \in V} g(v)$ is minimized. The concept of majority domination was introduced by Hedetniemi and studied by Broere *et al.*, who gave exact values for the majority domination numbers of complete graphs, complete bipartite graphs, paths, and unions of two complete graphs. They also proved that the majority domination problem is NP-complete for general graphs; and asked if the problem NP-complete for trees. The main result of this paper is to give polynomial-time algorithms for the majority domination problem in trees, cographs, and k -trees with fixed k .

1. INTRODUCTION

In this paper we study algorithmic aspects of majority domination, which is a variation of domination in graph theory.

In a graph $G = (V, E)$, the *neighborhood* of a vertex v is $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* $\deg_G(v)$ of a vertex v is the size of $N_G(v)$. We denote $\Delta(G)$ as the *maximum degree* of a vertex of G . For any real-valued function g on V and $S \subseteq V$, let $g(S) = \sum_{v \in S} g(v)$. For two disjoint graphs $G = (V, E)$ and $H = (W, F)$, the *union* of G and H is the graph $G \cup H = (V \cup W, E \cup F)$ and the *join* of G and H is $G \times H = (V \cup W, E \cup F \cup \{xy : x \in V \text{ and } y \in W\})$.

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A *majority dominating function* of $G = (V, E)$ is a function $g : V \rightarrow \{-1, 1\}$ such that $g(N_G[v]) \geq 1$ for at least half of the vertices $v \in V$. The *majority domination problem* involves determining the *majority domination number* $\gamma_{maj}(G)$ of a graph G , where $\gamma_{maj}(G) = \min\{g(V) : g \text{ is a majority dominating function of } G = (V, E)\}$.

The concept of majority domination was introduced by Hedetniemi [5] and studied by Broere *et al.* [2], who gave exact values for the majority domination numbers of complete graphs, complete bipartite graphs, paths, and unions of two complete graphs. They also proved that the majority domination problem is \mathcal{NP} -complete for general graphs. They raised several interesting questions; e.g., finding a good upper bound on $\gamma_{maj}(G \cup H)$ for connected graphs G and H , is the majority domination problem \mathcal{NP} -complete for trees?

In this paper, we study the majority domination problem from an algorithmic point of view. In particular, we give polynomial-time algorithms for the majority domination problem in trees, cographs, and k -trees with fixed k . For technical reasons, we consider several generalizations of the majority domination problem in the following sections.

2. MAJORITY DOMINATION IN TREES

This section gives a polynomial-time algorithm for the majority domination problem in trees. For technical reasons, we consider the following generalization of the majority domination problem. Suppose $G = (V, E)$ is a graph rooted at x , $s \in \{-1, 1\}$, t is an integer with $|t| \leq \deg_G(x)$, and i is an integer with $0 \leq i \leq |V|$. An (x, s, t, i) -*signed function* of G is a function $g : V \rightarrow \{-1, 1\}$ such that $g(x) = s$ and $m(G, x, s, t, g) \geq i$, where

$$m(G, x, s, t, g) = |\{v : g(N_G[v]) + t \geq 1 \text{ when } v = x \text{ and } g(N_G[v]) \geq 1 \text{ when } v \neq x\}|.$$

The (x, s, t, i) -*domination number* of $G = (V, E)$ is

$$\gamma(G, x, s, t, i) = \min\{g(V) : g \text{ is an } (x, s, t, i) \text{ - signed function of } G\}.$$

Note that $\min \emptyset$ is considered to be $+\infty$.

Lemma 1. *If G is a graph rooted at x and with n vertices, then $\gamma_{maj}(G) = \min\{\gamma(G, x, -1, 0, \lceil n/2 \rceil), \gamma(G, x, 1, 0, \lceil n/2 \rceil)\}$.*

Proof. The lemma follows from the fact that g is a majority dominating function of G if and only if g is an $(x, s, 0, \lceil n/2 \rceil)$ -signed function of G for some $s \in \{-1, 1\}$. ■

Lemma 2. *Suppose G and H are two disjoint graphs rooted at x and y , respectively. Let F be a graph rooted at x and obtained from the union of*

FIG. 1. Graphs G, H , and F .

G and H by joining a new edge xy (see Figure 1). If $s \in \{-1, 1\}$, t is an integer with $|t| \leq \deg_F(x)$, and i is an integer with $0 \leq i \leq |V(F)|$, then $\gamma(F, x, s, t, i) = \min\{\gamma(G, x, s, t + s', i') + \gamma(H, y, s', s, i - i') : s' \in \{-1, 1\} \text{ and } 0 \leq i' \leq i\}$.

Proof. Suppose f is an (x, s, t, i) -signed function of F such that $f(V(F)) = \gamma(F, x, s, t, i)$. Let g (respectively, h) be the function f restricted over the set $V(G)$ (respectively, $V(H)$) and $s^* = h(y) = f(y)$. Note that $f(N_F[x]) + t = g(N_G[x]) + (t + s^*)$; $f(N_F[v]) = g(N_G[v])$ for $v \in V(G) - \{x\}$; $f(N_F[y]) = h(N_H[y]) + s$; $f(N_F[v]) = h(N_H[v])$ for $v \in V(H) - \{y\}$. Therefore, $m(F, x, s, t, f) = m(G, x, s, t + s^*, g) + m(H, y, s^*, s, h)$. Since f is an (x, s, t, i) -signed function of F , g is an $(x, s, t + s^*, i^*)$ -signed function of G and h is a $(y, s^*, s, i - i^*)$ -signed function of H for some $0 \leq i^* \leq i$. Thus, we have $\gamma(F, x, s, t, i) = f(V(F)) = g(V(G)) + h(V(H)) \geq \min\{\gamma(G, x, s, t + s', i') + \gamma(H, y, s', s, i - i') : s' \in \{-1, 1\} \text{ and } 0 \leq i' \leq i\}$.

On the other hand, for any $s' \in \{-1, 1\}$ and i' any integer with $0 \leq i' \leq i$. Suppose g is an $(x, s, t + s', i')$ -signed function of G such that $g(V(G)) = \gamma(G, x, s, t + s', i')$ and h is a $(y, s', s, i - i')$ -signed function of H such that $h(V(H)) = \gamma(H, y, s', s, i - i')$. Define a function f of F by $f(v) = g(v)$ if $v \in V(G)$ and $f(v) = h(v)$ if $v \in V(H)$. Since $f(x) = g(x) = s$ and $m(F, x, s, t, f) = m(G, x, s, t + f(y), g) + m(H, y, f(y), f(x), h) = m(G, x, s, t + s', g) + m(H, y, s', s, h) \geq i' + (i - i') = i$, we see that f is an (x, s, t, i) -signed function of F and hence $\gamma(F, x, s, t, i) \leq f(V(F)) = g(V(G)) + h(V(H)) = \gamma(G, x, s, t + s', i') + \gamma(H, y, s', s, i - i')$. It follows that $\gamma(F, x, s, t, i) \leq \min\{\gamma(G, x, s, t + s', i') + \gamma(H, y, s', s, i - i') : s' \in \{-1, 1\} \text{ and } 0 \leq i' \leq i\}$. ■

Theorem 3. *There is an $O(n^2)$ -time algorithm for computing the majority domination number of a tree T with n vertices.*

Proof. Note that a tree can be obtained from isolated vertices by a sequence

of graph operations as described in Lemma 2. Using Lemma 2, at any iteration, $\gamma(F, x, s, t, i)$ for $s \in \{-1, 1\}$ and $|t| \leq \deg_F(x)$ and $0 \leq i \leq |V(F)|$ can be obtained in $O(n \deg(x))$ time. Once $\gamma(T, x, s, t, i)$'s are obtained, we can compute $\gamma_{maj}(T)$ in a constant time with Lemma 1. Thus, the total running time to compute $\gamma_{maj}(T)$ is $O(n \sum_x \deg(x)) = O(n^2)$. ■

3. MAJORITY DOMINATION IN COGRAPHS

This section gives a polynomial-time algorithm for the majority domination in cographs. Recall that *cographs* are defined by the following rules: K_1 is a cograph; if G and H are cographs, then so are $G \cup H$ and $G \times H$ (see [3, 4]).

For an integer $p \geq 0$, a *signed p -function* of $G = (V, E)$ is a function $g : V \rightarrow \{-1, 1\}$ such that there are exactly p vertices $v \in V$ with $g(v) = 1$. For any integers t and $p \geq 0$, define $\gamma(G, t, p) = \max\{m(G, t, g) : g \text{ is a signed } p\text{-function of } G\}$, where $m(G, t, g) = |\{v \in V : g(N_G[v]) + t \geq 1\}|$. Note that $\max \emptyset$ is considered as $-\infty$.

Lemma 4. *If G and H are two disjoint graphs, t is an integer, and p is a nonnegative integer, then $\gamma(G \cup H, t, p) = \max_{0 \leq p' \leq p} \{\gamma(G, t, p') + \gamma(H, t, p - p')\}$.*

Proof. Suppose f is a signed p -function of $G \cup H$ such that $m(G \cup H, t, f) = \gamma(G \cup H, t, p)$. Let g (respectively, h) be the function f restricted on the set $V(G)$ (respectively, $V(H)$). Then, g (respectively, h) is a signed p^* -function (respectively, $(p - p^*)$ -function) of G (respectively, H) for some $0 \leq p^* \leq p$. Therefore, $\gamma(G \cup H, p, t) = m(G \cup H, t, f) = m(G, t, g) + m(H, t, h) \leq \gamma(G, t, p^*) + \gamma(H, t, p - p^*) \leq \max_{0 \leq p' \leq p} \{\gamma(G, t, p') + \gamma(H, t, p - p')\}$.

On the other hand, for any integer p' with $0 \leq p' \leq p$, suppose g is a signed p' -function of G with $m(G, t, g) = \gamma(G, t, p')$, and h is a signed $(p - p')$ -function of H with $m(H, t, h) = \gamma(H, t, p - p')$. Define a function f of $G \cup H$ as follows: $f(x) = g(x)$ if $x \in V(G)$ and $f(x) = h(x)$ if $x \in V(H)$. Note that f is a signed p -function of $G \cup H$. Therefore, $\gamma(G \cup H, t, p) \geq m(G \cup H, t, f) = m(G, t, g) + m(H, t, h) = \gamma(G, t, p') + \gamma(H, t, p - p')$ for any integer p' with $0 \leq p' \leq p$. Thus, $\gamma(G \cup H, t, p) \geq \max_{0 \leq p' \leq p} \{\gamma(G, t, p') + \gamma(H, t, p - p')\}$. ■

Lemma 5. *If G and H are two disjoint graphs with ν_G and ν_H vertices respectively, t is an integer, and p a nonnegative integer, then $\gamma(G \times H, t, p) = \max_{0 \leq p' \leq p} \{\gamma(G, t + 2p - 2p' - \nu_H, p') + \gamma(H, t + 2p' - \nu_G, p - p')\}$.*

Proof. Suppose f, g , and h are three functions from $V(G \times H), V(G)$ and $V(H)$ to $\{-1, 1\}$, respectively, having $f(x) = g(x)$ for all $x \in V(G)$ and

$f(x) = h(x)$ for all $x \in V(H)$. Note that f is a signed p -function of $G \times H$ if and only if g (respectively, h) is a signed p' -function (respectively, $(p - p')$ -function) of G (respectively, H) for some $0 \leq p' \leq p$. Additionally, in this case we have $f(N_{G \times H}[v]) + t = g(N_G[v]) + t + 2p - 2p' - \nu_H$ for $v \in V(G)$ and $f(N_{G \times H}[v]) + t = h(N_H[v]) + t + 2p' - \nu_G$ for $v \in V(H)$. These observations imply that the lemma can be proved with an argument similar to that used in the proof of Lemma 4. ■

Lemma 6. *If $G = (V, E)$ is a graph with n vertices, then $\gamma_{maj}(G) = -n + 2 \min \mathcal{F}$, where $\mathcal{F} = \{p : 0 \leq p \leq n \text{ and } \gamma(G, 0, p) \geq n/2\}$.*

Proof. Suppose g is a majority dominating function of G with $\gamma_{maj}(G) = g(V)$. Let p^* be the number of vertices $v \in V$ with $g(v) = 1$. Then, g is a signed p^* -function of G and $\gamma(G, 0, p^*) \geq m(G, 0, g) = |\{v \in V : g(N_G[v]) \geq 1\}| \geq n/2$. So, $p^* \in \mathcal{F}$ and hence $\gamma_{maj}(G) = g(V) = p^* - (n - p^*) = -n + 2p^* \geq -n + 2 \min_{p \in \mathcal{F}} p$.

On the other hand, let $p' \in \mathcal{F}$ with $p' = \min_{p \in \mathcal{F}} p$. Since $\gamma(G, 0, p') \geq n/2$, there exists a signed p' -function g of G such that $|\{v \in V : g(N_G[v]) \geq 1\}| = m(G, 0, g) = \gamma(G, 0, p') \geq n/2$. Therefore, g is a majority dominating function of G and hence $\gamma_{maj}(G) \leq g(V) = p' - (n - p') = -n + 2p' = -n + 2 \min_{p \in \mathcal{F}} p$. Thus, $\gamma_{maj}(G) = -n + 2 \min_{p \in \mathcal{F}} p$. ■

Theorem 7. *There is an $O(n^3)$ -time algorithm for the majority domination problem in cographs.*

Proof. The theorem follows from Lemmas 4 to 6 and the definition of cographs. Note that in Lemmas 4 and 5, we only need to consider integers t for $|t| \leq |V(G)| + |V(H)|$. ■

4. MAJORITY DOMINATION IN k -TREES

This section gives a polynomial-time algorithm for the majority domination problem in k -trees for any fixed k . Recall that k -trees are defined recursively as follows: the complete graph of k vertices is a k -tree; the graph obtained from a k -tree by adding a new vertex joints to all vertices of a k -clique is a k -tree, by a k -clique we mean a pairwise adjacent vertex set of size k . A vertex v is *simplicial* if $N(v)$ is a clique. An ordering (v_1, v_2, \dots, v_n) of V is a *perfect elimination scheme* if each v_i is simplicial in the induced subgraph $G_i = G[\{v_i, v_{i+1}, \dots, v_n\}]$. Note that a graph is a k -tree if and only if G has a perfect elimination scheme such that $|N_{G_i}(v_i)| = k$ for $1 \leq i \leq n - k$. A

FIG. 2. A 2-tree G .

2-tree is shown in Fig. 2, where the vertex numbering indicates the ordering of a perfect elimination scheme.

To compute the majority domination number of a k -tree, we generalize the notation used in Section 2. Suppose $G = (V, E)$ is a graph with specified set $X \subseteq V$, $s = (s_x)_{x \in X}$, $t = (t_x)_{x \in X}$, $q = (q_x)_{x \in X}$ are vectors with integer components $s_x \in \{-1, 1\}$, $|t_x| \leq \deg_G(x)$ and $|q_x| \leq \deg_G(x)$ for each $x \in X$, and i is an integer with $0 \leq i \leq |V|$. An (X, s, t, q, i) -signed function of G is a function $g : V \rightarrow \{-1, 1\}$ such that $g(x) = s_x, g(N_G(x)) = q_x$ for each $x \in X$ and $m(G, X, s, t, q, g) \geq i$, where $m(G, X, s, t, q, g) = |\{v : g(N_G[v]) + t_v \geq 1 \text{ when } v \in X \text{ and } g(N_G[v]) \geq 1 \text{ when } v \notin X\}|$. The (X, s, t, q, i) -domination number of $G = (V, E)$ is $\gamma(G, X, s, t, q, i) = \min\{g(V) : g \text{ is an } (X, s, t, q, i)\text{-signed function of } G\}$. Note that $\min \emptyset$ is considered to be $+\infty$.

Lemma 8. *If G is a graph of n vertices with a specified k -clique X , then $\gamma_{maj}(G) = \min\{\gamma(G, X, s, \vec{0}, q, \lceil n/2 \rceil) : s = (s_x)_{x \in X} \in \{-1, 1\}^k, \vec{0} = (0)_{x \in X}, q = (q_x)_{x \in X} \text{ with } |q_x| \leq \Delta(G)\}$.*

Lemma 9. *Suppose G_1, G_2, \dots, G_{k+1} are $k+1$ disjoint graphs with specified k -cliques X_1, X_2, \dots, X_{k+1} respectively, where $X_i = \{x_1^i, x_2^i, \dots, x_{k+1}^i\} - \{x_i^i\}$. Let F be the graph with specified k -clique $X = \{x_1, x_2, \dots, x_k\}$, which is obtained from the union of G_1, G_2, \dots, G_{k+1} by identifying $\{x_j^1, x_j^2, \dots, x_j^{k+1}\} - \{x_j^j\}$ as a new vertex x_j for $1 \leq j \leq k+1$ (see Fig. 3 for $k = 2$). If $s = (s_x)_{x \in X}, t = (t_x)_{x \in X}, q = (q_x)_{x \in X}$ are vectors with integer components $s_x \in \{-1, 1\}, |t_x| \leq \deg_F(x)$, and $|q_x| \leq \deg_F(x)$ for each $x \in X$, and I an integer with $0 \leq I \leq |V(F)|$, then $\gamma(F, X, s, t, q, I) = \min\{\sum_{i=1}^{k+1} \gamma(G_i, X_i, (s_x)_{x \in X_i}, (t_x)_{x \in X_i}, (q_x)_{x \in X_i}, I_i) - (k-1) \sum_{j=1}^{k+1} s_{x_j} :$*

FIG. 3. Graphs G_1, G_2, G_3 , and F .

$$\begin{aligned}
 & s_{x_{k+1}} \in \{-1, 1\}, t_{x_{k+1}} = 0, |q_{x_j^i}| \leq \Delta(G_i) \text{ for all } i \neq j \text{ and } q_{x_i^i} = 0 \text{ for all } i, \\
 & s_{x_j^i} = s_{x_j} \text{ and } t_{x_j^i} = t_{x_j} + \sum_{l \neq i} q_{x_j^l} - (k-2) \sum_{l \neq j} s_{x_l} \text{ for all } i \neq j, \\
 & \sum_{i=1}^{k+1} q_{x_j^i} - (k-2) \sum_{i \neq j} s_{x_i} = q_{x_j} \text{ for } 1 \leq j \leq k+1, \text{ and} \\
 & \sum_{i=1}^{k+1} I_i - (k-1) \sum_{j=1}^{k+1} \mu(q_{x_j}, s_{x_j}, t_{x_j}) = I,
 \end{aligned}$$

where $\mu(z_1, z_2, z_3) = 1$ if $z_1 + z_2 + z_3 \geq 1$ and $\mu(z_1, z_2, z_3) = 0$ otherwise.

Although the proofs of Lemmas 8 and 9 are complicated, they are similar to those for Lemmas 1 and 2, therefore we omit the details.

Theorem 10. *There is a polynomial-time algorithm for computing the majority domination number of a k -tree for any fixed k .*

Proof. The theorem follows from Lemmas 8 and 9 and the fact that k -trees can be obtained from complete graphs of k vertices by a sequence of operations described in Lemma 9. ■

We conclude this paper by noting that since it is possible to embed a partial k -tree for any fixed k into a k -tree in polynomial time [1], there is also a polynomial-time algorithm for computing the majority domination number of a partial k -tree.

REFERENCES

1. S. Arnborg, D. G. Corneil, and A. Proskurowski, Complexity of finding embeddings in a k -tree, *SIAM J. Alg. Disc. Meth.* **8(2)**, (1987) 277-284.
2. I. Broere, J. H. Hattingh, M. A. Henning, and A. A. McRae, Majority domination in graphs, *Discr. Math.* **138**, (1995) 125-135.

3. D. G. Corneil, H. Lerchs, and L. Stewart Burlingham, Complement reducible graphs, *Discr. Appl. Math.* **3**, (1981) 163-174.
4. D. G. Corneil, Y. Perl, and L. K. Stewart, A linear recognition algorithm for cographs, *SIAM J. Comput.* **14**, (1985) 926-934.
5. S. T. Hedetniemi, private communication to Broere *et al.*

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