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A CLASS OF SMALL DEVIATION THEOREMS FOR THE SEQUENCES OF NONNEGATIVE INTEGER-VALUED RANDOM VARIABLES

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Abstract. In virtue of the notion of likelihood ratio the limit properties of the sequences of dependent nonnegative integer-valued random variables are studied, and a class of small deviation theorems are obtained. In the proof an approach of applying the tool of generating function together with the method of splitting intervals to the study of the strong laws is proposed.

1. Introduction

Let $\{X_n, n > 1\}$ be a sequence of nonnegative integer-valued variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with the joint distribution

$$(1)\quad f_n(x_1, \cdots, x_n) = P(X_1 = x_1, \cdots, X_n = x_n) > 0,\ x_k \in S, \quad 1 \leq k \leq n,$$

where $S = \{0, 1, 2, \cdots\}$. Let

$$(2)\qquad (p_k(0), p_k(1), \cdots), \quad k = 1, 2, \cdots,$$

be a sequence of probability distribution on S, and let

$$(3)\qquad q_n(x_1, \cdots, x_n) = \prod_{k=1}^{n} p_k(x_k), \quad x_k \in S, \quad 1 \leq k \leq n,$$

be the product distribution generated by (2). The random variable

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$$(4) r_n(\omega) = q_n(X_1, \cdots, X_n)/f_n(X_1, \cdots, X_n) = \left[\prod_{k=1}^{n} p_k(X_k)\right] / f_n(X_1, \cdots, X_n)$$

is called the likelihood ratio, which is of fundamental importance in the theory of testing statistical hypotheses (cf. Karlin and Taylor, 1975, p. 245; Laha and Rohatgi, 1979, p. 388; Billingsley, 1986, p. 483).

In the above definition $f_n(x_1, \cdots, x_n)$ is the joint distribution of $\{X_k, 1 \leq k \leq n\}$, and $q_n(x_1, \cdots, x_n)$ is called the reference distribution.

In this paper we use the likelihood ratio $r_n(\omega)$ as a measure of the deviation between $\{X_k, 1 \leq k \leq n\}$ and the sequence of independent random variables with the product distribution (3). A subset of sample space is determined by restricting $r_n(\omega)$, and on this subset a class of strong limit theorems represented by inequalities, which we call the small deviation theorems, are established. The usual strong limit theorems represented by equalities are special cases of the corresponding small deviation theorem.

2. Main Results

Let

$$m_k = \sum_{i=1}^{\infty} i\, p_k(i) < \infty, \tag{5}$$

$$P_k(s) = \sum_{i=0}^{\infty} p_k(i) s^i \tag{6}$$

be, respectively, the mathematical expectation and the generating function of a random variable with distribution (2).

Theorem 1. *Let $\{X_n, n \geq 1\}$, $r_n(\omega)$, m_k, $P_k(s)$ be given as above, and $c \geq 0$ a constant. Let*

$$D(c) = \{\omega : \liminf_n (1/n) \ln\ r_n(\omega) \geq -c\}. \tag{7}$$

Then

$$\liminf_n (1/n) \sum_{k=1}^{n} (X_k - m_k) \geq \alpha(c) \quad \text{a. e. } \omega \in D(c), \tag{8}$$

where

$$\alpha(c) = \sup\{\varphi(s),\ 0 < s < l\}, \tag{9}$$

$$(10)\quad \varphi(s)=\liminf_n (1/n)\sum_{k=1}^{n}[\ln\ P_k(s)/\ln\ s-m_k]+c/\ln\ s,\ 0<s<1.$$

Remark 1. It will be shown below that

$$(11)\qquad \limsup_n (1/n)\ln\ r_n(\omega)\le 0 \text{ a.e.}$$

Hence the inequality $\liminf_n (1/n)\ln\ r_n(\omega)\ge -c$ in (7) may be looked upon as a restriction to the deviation between $\{X_n,\ n\ge 1\}$ and the sequence of independent random variables with distribution (3). The smaller is c, the smaller is the deviation. The theorems we obtained assert that under the restriction in (7) the ratio $(1/n)\sum_{k=1}^{n}(X_k-m_k)$ is restricted correspondingly, and formulas (8) and (30) give respectively an estimation of the lower and upper bounds of its inferior and supperior limits corresponding to c. (49) and (61) assert that under appropriate conditions the bounds given by (46) and (58) tend to zero as $c\to 0$. This means that the behaviour described above is similar in some sense to that described in the theorems on the stability of solutions of differential equations.

Remark 2. By the definition of a.e. convergence on a measurable set (8) holds trivially if $P(D(c))=0$. This case goes beyond the scope of small deviation.

Proof. In what follows we shall use the method of splitting interval proposed by the author (see Liu Wen, 1990) together with the tool of generating functions. It is different from the traditional probabilistic method. The crucial part is the application of Lebesgue's theorem on differentiability of monotone function to the study of a.e. convergence.

Throughout this paper we deal with the underlying probability space $(\Omega,\mathcal{F},P)$, where $\Omega=[0,1)$, $\mathcal{F}$ is the class of Borel measurable sets in the interval $[0,1)$, and P is the Lebesgue measure. For the sake of completion, we first give, in the above probability space, a realization of non-negative integer-valued random variable sequence with distribution (1).

Split the interval [0, 1) into countably many right-semiopen intervals $I_{x_1}(x_1=0,1,2,\cdots)$ according to the ratio $f_1(0):f_1(1):\cdots$, i.e.,

$$I_o=[0,f_1(0)),\quad I_1=[f_1(0),\ f_1(0)+f_1(1)),\ \cdots.$$

These intervals will be called the I-interval of the first order. Proceeding inductively, suppose the nth order I-intervals $\{I_{x_1\cdots x_n},\ x_i\in S,\ 1\le i\le n\}$ have been defined. Then split the right-semiopen interval $I_{x_1\cdots x_n}$ into countably many right-semiopen intervals

$$I_{x_1\cdots x_n 0},\quad I_{x_1\cdots x_n 1},\ \cdots$$

according to the ratio $f_{n+1}(x_1,\cdots,x_n,0):f_{n+1}(x_1,\cdots,x_n,1)$, the I-intervals of $(n+1)$th order are created. It is easy to see that for $n\geq 1$,

$$P(I_{x_1\cdots x_n})=f_n(x_1,\cdots,x_n). \tag{12}$$

Define, for $n\geq 1$, a random variable $X_n:[0,1)\to S$ as follows:

$$X_n(\omega)=x_n,\quad \text{if } \omega\in I_{x_1\cdots x_n}. \tag{13}$$

By (12) and (13),

$$\{\omega : X_1=x_1,\cdots,\ X_n=x_n\}=I_{x_1\cdots x_n}; \tag{14}$$

$$P(X_1=x_1,\cdots,\ X_n=x_n)=P(I_{x_1\cdots x_n})=f_n(x_1,\cdots,x_n), \tag{15}$$

hence $\{X_i,\ 1\leq i\leq n\}$ has distribution (1).

For the need of the proof, we first construct an auxiliary function. Let the radius of convergence of (6) be R_k, $R=\inf\{R_k,\ k=1,2,\cdots\}$. Since $P_k(s)$ is a probability generating function, $R\geq 1$. Let $s\in(0,R)\cup\{1\}$, and put

$$P_k(s,i)=[1/P_k(s)]p_k(i)s^i,\quad i=0,1,2,\cdots. \tag{16}$$

In a similar way we can create the J-intervals by splitting the interval [0, 1) successively as follows: Split [0, 1) into countably many right-semiopen intervals

$$J_o=[0,P_1(s,0)),\quad J_1=[P_1(s,0),\quad P_1(s,0)+P_1(s,1)),\cdots.$$

These intervals will be called the J-intervals of the first order. Suppose the nth order J-interval $J_{x_1\cdots x_n}$ has be defined. Then split it into countably many right-semiopen intervals $J_{x_1\cdots x_n x_{n+1}}(X_{n+1}=0,1,2,\cdots)$ according to the ratio $P_{n+1}(s,0):P_{n+1}(s,1):\cdots$, the J-intervals of $(n+1)$th order are created. It is easy to see that

$$P(J_{x_1\cdots x_n})=\prod_{k=1}^{n}P_k(s,x_k)=\prod_{k=1}^{n}[1/P_k(s)]p_k(x_k)s^{x_k}. \tag{17}$$

Let $I^-_{x_1\cdots x_n}$ and $I^+_{x_1\cdots x_n}$ be, respectively, the left and right end-points of $I_{x_1\cdots x_n}$; define $J^-_{x_1\cdots x_n}$ and $J^+_{x_1\cdots x_n}$ similarly. Let Q be the set of end-points of I-intervals of all orders. Define a function $g_s:[0,1]\to[0,1]$ as follows:

$$g_s\,(I^-_{x_1\cdots x_n}) = J^-_{x_1\cdots x_n}, \quad g_s\,(I^+_{x_1\cdots x_n}) = J^+_{x_1\cdots x_n}; \tag{18}$$

$$g_s(x) = \sup\{g_s(t), \;\; t \in Q \cap [0,x)\}, \;\; x \in [0,1] - Q. \tag{19}$$

Let

$$t_n(s,\omega) = \frac{g_s(I^+_{x_1\cdots x_n}) - g_s(I^-_{x_1\cdots x_n})}{I^+_{x_1\cdots x_n} - I^-_{x_1\cdots x_n}}, \;\; \omega \in I_{x_1\cdots x_n}. \tag{20}$$

We have by (4), (13), (15) and (17),

$$t_n(s,\omega) = P(J_{X_1\cdots X_n})/P(I_{X_1\cdots X_n}) = r_n(\omega)\, s^{\sum_{k=1}^{n} X_k} \prod_{k=1}^{n} [1/P_k(s)]. \tag{21}$$

Let $A(s)$ be the set of points of differentiability of g_s. Then $P(A(s)) = 1$ by the theorem on the existence of derivative of monotone function (cf. Billingsley, 1986, p. 424). In virtue of a property of derivative (cf. Billingsley, 1986, p. 423) we have

$$\lim_n \; t_n(s,\omega) = \text{a finite number}, \;\; \omega \in A(s). \tag{22}$$

This implies that

$$\limsup_n\,(1/n) \ln\; t_n(s,\omega) \le 0, \;\; \omega A(s). \tag{23}$$

We have by (21) and (23),

$$\limsup_n\,(1/n)\,[\ln\; r_n(\omega) + \sum_{k=1}^{n} X_k \ln\; s] \le \limsup_n (1/n) \cdot \sum_{k=1}^{n} \ln\; P_k(s), \; \omega \in A(s). \tag{24}$$

Letting $s = 1$ in (24) we obtain

$$\limsup_n\,(1/n) \ln\; r_n(\omega) \le 0, \;\; \omega \in A(1). \tag{25}$$

Since $P(A(1)) = 1$, (11) follows from (25). We have by (24) and (7),

$$\limsup_n\,(1/n) \sum_{k=1}^{n} X_k \;\ln\; s \le \limsup_n\,(1/n) \sum_{k=1}^{n} \ln\; P_k(s) + c, \quad \omega \in A(s) \cap D(c). \tag{26}$$

Assume $0 < s < 1$. Dividing the two sides of (26) by $\ln\, s$, we obtain

$$\begin{aligned}(27)\qquad &\liminf_n (1/n)\sum_{k=1}^{n}(X_k - m_k) \geq \liminf_n (1/n)\sum_{k=1}^{n}\\ &[\ln\ P_k(s)/\ln\ s - m_k] + c/\ln\ s = \varphi(s),\\ &\qquad \omega \in A(s)\cap D(c).\end{aligned}$$

From (9), there exist $s_i \in (0,1),\ i = 1, 2, \cdots$, such that

$$\lim_i \varphi(s_i) = \alpha(c). \tag{28}$$

Let $A = \bigcap_{i=1}^{\infty} A(s_i)$. We have by (27) and (28),

$$\liminf_n (1/n)\sum_{k=1}^{n}(X_k - m_k) \geq \alpha\ (c),\quad \omega \in A\cap D(c). \tag{29}$$

Since $P(A) = 1$, (8) follows from (29). The theorem is proved.

Remark. Since the probability-theoretic properties of any family $\{X_t,\ t \in T\}$ of random variables can be expressed in terms of the distributions of its finite subfamilies (cf. Loeve, 1977, p. 174), we may use a special realization of $\{X_n,\ n \geq 1\}$ to prove our theorems without loss of generality.

Theorem 2. *Let R be defined as above, and suppose that $R > 1$. Then under the hypotheses of Theorem 1 we have*

$$\limsup_n (1/n)\sum_{k=1}^{n}(X_k - m_k) \leq \beta(c) \quad \text{a.e.}\ \ \omega \in D(c), \tag{30}$$

where

$$\beta(c) = \inf\{\psi(s),\ 1 < s < R\}, \tag{31}$$

$$\Psi(s) = \limsup_n (1/n)\sum_{k=1}^{n}\{[\ln\ P_k(s)]/\ln\ s - m_k\} + c/\ln\ s,\ 1 < s < R. \tag{32}$$

Proof. Assume $1 < s < R$. Dividing the two sides of (26) by $\ln s$, we obtain

$$\begin{aligned}(33)\qquad &\limsup_n (1/n)\sum_{k=1}^{n}[X_k - m_k \leq \limsup_n (1/n)\sum_{k=1}^{n}\{[\ln\ P_k(s)]/\ln\ s - m_k\}\\ &+c/\ln\ s = \psi(s),\quad \omega \in A(s)\cap D(c).\end{aligned}$$

From (31), there exist $s_i \in (1, R),\ i = 1, 2, \cdots$, such that

$$\lim_i \psi(s_i) = \beta(c). \tag{34}$$

Let $A = \bigcap_{i=1}^{\infty} A(s_i)$. By (33) and (34) we obtain

$$\limsup_n (1/n) \sum_{k=1}^{n} (X_k - m_k) \le \beta(c), \quad \omega A \cap D(c). \tag{35}$$

Since $P(A) = 1$, (30) follows from (35). The theorem is proved.

Corollary 1. *Let $\{p_k(i)\}(i = 0, 1, 2, \cdots)$ in the definition of $r_n(\omega)$ be the Poisson distribution with parameter $\lambda_k > 0$, and let*

$$\lambda = \limsup_n (1/n) \sum_{k=1}^{n} \lambda_k < \infty. \tag{36}$$

Then

$$\limsup_n (1/n) \sum_{k=1}^{n} (X_k - \lambda_k) \le 2\sqrt{\lambda c} + c \quad \text{a.e. } \omega D(c); \tag{37}$$

and for $0 \le c < \lambda$

$$\liminf_n (1/n) \sum_{k=1}^{n} (X_k - \lambda_k) \ge -2\sqrt{\lambda c} \quad \text{a.e. } \omega \in D(c). \tag{38}$$

Proof. In this case $P_k(s) = e^{\lambda_k(s-1)},\ m_k = \lambda_k$, let

$$\begin{aligned} g_n(s) &= (1/n) \sum_{k=1}^{n} [\ln\ P_k(s)/\ln\ s - m_k] + c/\ln s \\ &= [(s-1)/\ln s - 1](1/n) \sum_{k=1}^{n} \lambda_k + c/\ln s, \quad s \in (0,1) \cup (1, R). \end{aligned} \tag{39}$$

We have by (39), (32) and the inequality $1 - 1/s \le \ln\ s \le s - 1 (s > 0)$.

$$\begin{aligned} \psi(s) &= \limsup_n\ g_n(s) = [(s-1)/\ln\ s - 1]\lambda + c/\ln\ s \\ &\le \left[\frac{s-1}{1-1/s} - 1\right]\lambda + \frac{c}{1-1/s} \\ &= \lambda(s-1) + c/(s-1) + c = g(s), \quad s \in (1, R). \end{aligned} \tag{40}$$

It is easy to see that $g(s)$ attains, at $s = 1 + \sqrt{c/\lambda}$, its smallest value $v(c) = g(1 + \sqrt{c/\lambda}) = 2\sqrt{\lambda c} + c$ on the interval $(1, R)$ if $c > 0$ and $\lambda > 0$, and (40) implies that $v(c) \geq \beta(c)$. Hence (37) follows from (30). Similarly, we have

$$\begin{aligned}\varphi(s) &= \liminf_n \ g_n(s) = [(s-1)/\ln\ s - 1]\lambda + c/\ln\ s \\ &\geq \left[\frac{s-1}{1-1/s} - 1\right]\lambda + \frac{c}{s-1} \\ &= \lambda(s-1) + c/(s-1) = h(s), \quad s \in (0,1).\end{aligned} \tag{41}$$

It is easy to see that $h(s)$ attains, at $s = 1 - \sqrt{c/\lambda}$, its largest value $u(c) = -2\sqrt{\lambda c}$ on the interval $(0, 1)$ if $0 < c < \lambda$, and (41) implies that $u(c) \leq \alpha(c)$. Hence (38) follows from (8). Obviously, $\alpha(0) = \beta(0) = 0$, and $\alpha(c) = \beta(c) = 0$ if $\lambda = 0$. Hence in the case $c = 0$ or $\lambda = 0$, (37) and (38) are also true.

Theorem 3. *Let*

$$q_k(i) = \sum_{j=i+1}^{\infty} p_k(j), \ i \in S; \tag{42}$$

$$Q_k(s) = \sum_{i=0}^{\infty} q_k(i) s^i \tag{43}$$

be, respectively, the tail probability and the tail probability generating functions of the distribution (2). *If there exists a sequence of positive numbers* $\{q(i),\ i \geq 0\}$ *such that*

$$q_k(i) \leq q(i), \ i \geq 0, \ k \geq 1; \tag{44}$$

$$\sum_{i=0}^{\infty} q(i) = m < \infty, \tag{45}$$

then under the assumption of Theorem 1 we have

$$\liminf_n\, (1/n) \sum_{k=1}^{n} (X_k - m_k) \geq \alpha_*(c) \quad \text{a.e. } \omega \in D(c), \tag{46}$$

where

$$\alpha_*(c) = \sup\{\varphi_*(s),\ 0 < s < 1\}, \tag{47}$$

$$\varphi_*(s) = \liminf_n\, (1/n) \sum_{k=1}^{n} [s\ Q_k(s) - Q_k(1)] + c/\ln\ s, \ 0 < s < 1. \tag{48}$$

Moreover, $\alpha * (c) \le 0$, *and*

$$\lim_{c \to 0^+} \alpha_*(c) = \alpha_*(0) = 0. \tag{49}$$

Proof. Applying the inequality $1 - 1/x < \ln\ x < x - 1 (0 < x < 1)$ and the property of generating function

$$Q_k(s) = [1 - P_k(s)]/(1 - s), \quad |s| < 1; \quad Q_k(1) = m_k \tag{50}$$

(cf. Hunter, 1983, p. 39), we obtain from (27),

$$\begin{aligned} &\liminf_n (1/n) \sum_{k=1}^{n} (X_k - m_k) \\ &\ge \liminf_n\ (1/n) \sum_{k=1}^{n} \left[\frac{P_k(s) - 1}{1 - 1/s} - m_k \right] + c/\ln\ s \\ &= \liminf_n\ (1/n) \sum_{k=1}^{n} [s\ Q_k(s) - Q_k(1)] + c/\ln\ s \\ &= \varphi_*(s),\ 0 < s < 1,\ \omega \in A(s) \cap D(c). \end{aligned} \tag{51}$$

From (47), there exist $s_i \in (0, 1),\ i = 1, 2, \cdots$, such that

$$\lim_i \varphi_*(s_i) = \alpha_*(c). \tag{52}$$

Let $A = \bigcap_{i=1}^{\infty} A(s_i)$. We have by (51) and (52),

$$\liminf_n\ (1/n) \sum_{k=1}^{n} (X_k - m_k) \ge \alpha_*(c), \quad \omega A \cap D(c). \tag{53}$$

Since $P(A) = 1$, (46) follows from (53).

For $0 < s < 1,\ s\ Q_k(s) - Q_k(1) < 0$, and $\varphi(s) < 0$. Hence $\alpha(c) \le 0$. Let

$$Q(s) = \sum_{i=0}^{\infty} q(i) s^i \tag{54}$$

be the generating function of $\{q(i),\ i \ge 0\}$. We have by (42) and (43),

$$(55)\qquad \begin{aligned}&(1/n)\sum_{k=1}^{n}[s\ Q_k(s)-Q_k(1)]\\ &\quad=[(s-1)/n]\sum_{k=1}^{n}Q_k(s)+(1/n)\sum_{k=1}^{n}\sum_{i=0}^{\infty}q_k(i)(s^i-1)\\ &\quad\ge(s-1)Q(s)+Q(s)-Q(1)\\ &\quad=s\ Q(s)-Q(1),\quad 0\le s\le 1.\end{aligned}$$

By (47), (48) and (55),

$$(56)\ \alpha_*(c)\ge\varphi_*(1-\sqrt{c})\ge(1-\sqrt{c})Q(1-\sqrt{c})-Q(1)+c/\ln(1-\sqrt{c}),\ 0<c<1;$$

$$(57)\qquad \alpha_*(0)\ge\varphi_*(1-1/n)\ge(1-1/n)Q(1-1/n)-Q(1),\ n\ge 2.$$

Since $\alpha_*(c)\le 0$, it is obvious that (56) and (57) imply (49). The theorem is proved.

Theorem 4. *Let R be the radius of convergence of* (54)*, and suppose that $R>1$. Then under the hypotheses of Theorem 3 we have*

$$(58)\qquad \limsup_n\,(1/n)\sum_{k=1}^{n}(X_k-m_k)\le\beta_*(c),\quad \omega\in D(c),$$

where

$$(59)\qquad \beta_*(c)=\inf\{\psi_*(s),\quad 1<s<R\},$$

$$(60)\quad \psi_*(s)=\limsup_n\,(1/n)\sum_{k=1}^{n}[s\ Q_k(s)-Q_k(1)]+c/\ln\ s,\ 1<s<R.$$

Moreover, $\beta_*(c)\ge 0$, and

$$(61)\qquad \lim_{c\to 0^+}\beta_*(c)=\beta\ (0)=0.$$

Proof. It is easy to see that the hypotheses of the theorem imply that the radiuses of (6) and (43) are not less than R, and

$$(62)\ \ Q_k(s)=[1-P_k(s)]/(1-s),\ 1<s<R.\ \text{(cf. Hunter, 1983, p. 39)}$$

Applying the inequality $0<1-1/x<\ln\ x<x-1(x>l)$ and the property (62) of the generating function, we obtain from (33),

$$\limsup_n (1/n)\sum_{k=1}^{n} (X_k - m_k) \le \limsup_n (1/n)\sum_{k=1}^{n} [s\,Q_k(s) - Q_k(1)] + c/\ln s \tag{63}$$
$$= \psi_*(s),\ 1 < s < R,\quad \omega \in A(s) \cap D(c).$$

From (59), there exist $s_i \in (1, R),\ i = 1, 2, \cdots$, suth that

$$\lim_i\ \psi_*(s_i) = \beta_*(c). \tag{64}$$

Let $A = \bigcap_{i=1}^{\infty} A(s_i)$. By (63) and (64) we obtain

$$\limsup_n (1/n) \sum_{k=1}^{n} (X_k - m_k) \le \beta_*(c),\quad \omega \in A \cap D(c). \tag{65}$$

Since $P(A) = 1$, (58) follows from (65). Imitating the proof of (49), (61) can be established. The theorem is proved.

Corollary 1. *Under the hypotheses of Theorem* 3 *we have*

$$\lim_n (1/n) \sum_{k=1}^{n} (X_k - m_k) = 0 \ \text{ a.e.},\quad \omega \in D(0). \tag{66}$$

Proof. Letting $c = 0$, (66) follows from (46) and (58) immediately.

Corollary 2. *Let* $\{X_n,\ n \ge 1\}$ *be independent and have the distribution* (2). *Then under the hypotheses of Theorem* 3 *we have*

$$\lim_n (1/n) \sum_{k=1}^{n} (X_k - m_k) = 0 \text{ a.e.} \tag{67}$$

Proof. In this case $r_n(\omega) \equiv 1$, and $D(0) = [0, 1)$. Thus (67) follows immediately from (66).

3. An Open Problem

We have seen that in the deriving of (30) and (58) the condition $R > 1$ is essential. On the other hand, it is well-known that for the strong law of large numbers of a sequence of independent and identically distributed random variables the first moment is the only one assumed in the hypothesis. This motivates the following problem: under the conditions of Theorem 4 except the assumption $R > 1$, does there exsist a nondecreasing function $\beta : [0, \infty) \to [0, \infty)$ satisfying the following condition?

$$\lim_{c \to 0^+} \beta(c) = \beta(0) = 0 \tag{68}$$

such that for each $c \geq 0$,

$$\limsup_n \, (1/n) \sum_{k=1}^{n} (X_k - m_k) \leq \beta(c) \quad \text{a.e.,} \quad \omega \in D(0). \tag{69}$$

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