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ON THE EXISTENCE OF INVARIANT SUBSPACES AND REFLEXIVITY OF N-TUPLES OF OPERATORS

Marek Ptak

Abstract. Recent results concerning the existence of a common non-trivial invariant subspace and reflexivity for families of commuting linear bounded Hilbert space operators will be presented; starting with the families of linear transformations on finite dimensional space, through families of isometries, jointly quasinormal operators and spherical isometries, finishing with N-tuples of contractions with dominating spectra.

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1. Introduction

In what follows we will deal with N-tuples of commuting linear bounded Hilbert space operators. Two problems will be considered:

(1) Existence of a Common Non-trivial Invariant Subspace – whether there is a non-trivial (not equal to the whole space or the zero space) closed subspace invariant for all operators from the N-tuple;

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(2) Reflexivity Problem–whether the lattice of all common invariant subspaces for the N-tuple is so rich, that it determines the algebra generated by the N-tuple in the sense that any operators leaving invariant all subspaces, which are invariant for the N-tuple, have to belong to the smallest closed (in the weak operator topology) algebra containing the N-tuple and the identity.

It is straightforward that if Reflexivity Problem has a positive answer then so does the Existence of a Common Non-trivial Invariant Subspace Problem.

The second motivation for studying the Reflexivity Problem comes from von Neumann algebras. The commutant of any von Neumann algebra is generated by all projections in the commutant. Since von Neumann algebra is self–adjoint, it is generated by all its reducing projections. Thus, considering the N-tuple of operators and the smallest (non–selfadjoint) weak operator topology closed algebra generated by them, it is natural to consider the set of all invariant subspaces (invariant projections) instead of set of reducing projections. For a von Neumann algebra, we are considering the double–commutant, the set of all operators which commute with all operators from the commutant, in other words, which commute with all projections that reduce all operators from the given von Neumann algebra. Moreover, the well–known double–commutant theorem shows that the double–commutant of a von Neumann algebra is equal to the algebra itself.

Thus, in the non–selfadjoint case, we can consider all operators which leave invariant all subspaces which are invariant for a given N-tuple. Now, one can ask whether such an operator belongs to the algebra generated by the given N-tuple. In some sense we are asking whether this algebra fulfills the non–selfadjoint version of the double commutant theorem.

The paper is generally based on the results from [2], [7]-[8], [13], [46]-[47], [56]-[58]; however it also contains some new material.

1.1. The basic definition: invariant subspace, reflexive algebra, reflexive space, reflexive operator, reflexive family.

Throughout this paper we will mostly deal with bounded operators on a finite-dimensional or separable infinite-dimensional complex Hilbert space $\mathcal{H}$. Let $\mathcal{S}$ be a family of operators acting on a common Hilbert space $\mathcal{H}$. Then we denote by $\mathcal{W}(\mathcal{S})$ (respectively, $\mathcal{A}(\mathcal{S})$) and $\mathcal{S}'$, the WOT (= weak operator topology)-closed (respectively, the weak star closed) subalgebra of $L(\mathcal{H})$ generated by $\mathcal{S}$ and the identity I and the commutant of $\mathcal{S}$. The subspace $\mathcal{L} \subset \mathcal{H}$ is called *invariant* (respectively, *hyperinvariant*) for the family $\mathcal{S}$ if $T\mathcal{L} \subset \mathcal{L}$ for all operators $T \in \mathcal{S}$ (respectively, for all operators $T \in \mathcal{S}'$). *Lat*

$\mathcal{S}$ will be the lattice of all (closed) invariant subspaces for $\mathcal{S}$, and *Alg Lat* $\mathcal{S}$ is as usual the algebra of all $T \in L(\mathcal{H})$ such that *Lat* $\mathcal{S} \subset$ *Lat* T.

Let $\mathcal{W}$ be a WOT-closed algebra of operators containing the identity I. $\mathcal{W}$ is said to be *reflexive* if it is determined by its lattice of invariant subspaces in the sense that $\mathcal{W} =$*Alg Lat* $\mathcal{W}$. An individual operator T is called *reflexive* if the operator algebra $\mathcal{W}(T)$ it generates is reflexive. A family of operators $\mathcal{S}$ is said to be *reflexive* if the algebra $\mathcal{W}(\mathcal{S})$ is reflexive.

Recall that an algebra $\mathcal{A}$ of operators has property $\mathbb{A}_1(1)$ if for a given weak-star continuous linear functional ϕ on $L(\mathcal{H})$ and $\varepsilon > 0$, there are $a, b \in \mathcal{H}$ such that $\|a\| \cdot \|b\| \leq (1+\varepsilon)\|\phi\|$ and $\phi(A) = (A\,a, b)$ for all $A \in \mathcal{A}$.

We say that a commutative set $\mathcal{S} \subset L(\mathcal{H})$ is *doubly commuting* (respectively, *almost doubly commuting*) if $ST^* - T^*S$ is a zero (respectively, compact) operator for all $S, T \in \mathcal{S}$ with $S \neq T$. In particular, sets consisting of a single operator are doubly commuting.

Let us recall an extension of the reflexivity concept originally due to A. I. Loginov and V. I. Sulman [51]. The *reflexive closure* of an operator space $\mathcal{S} \subset L(\mathcal{H})$ is defined by $Ref\mathcal{S} = \{T \in L(\mathcal{H}) : Tx \in \mathcal{S}x \text{ for all } x \in \mathcal{H}\}$. The space $\mathcal{S}$ is called *reflexive* if $\mathcal{S} = Ref\mathcal{S}$. When $\mathcal{S}$ is an algebra with identity, then $Ref\mathcal{S} = Alg\ Lat\mathcal{S}$, and the current notion of reflexivity reduces to the classical concept.

We will denote by $\mathcal{C}_1(\mathcal{H})$ the ideal of trace-class operators on a Hilbert space $\mathcal{H}$. Recall that $L(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$ and the duality is given by the form $< T, S > := tr(TS)$ for $T \in L(\mathcal{H})$, $S \in \mathcal{C}_1(\mathcal{H})$.

1.2. Basic Theorems and Examples.

In this section we will present some basic theorems and examples about reflexivity. The first result was proved in [66].

Theorem 1.2.1. *Any algebra of normal operators is reflexive.*

The following example shows that the problem is non-trivial even in the finite-dimensional case.

Example 1.2.2. Let us consider the following algebras:

$$\mathcal{W}_1 = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{C} \right\}$$

and

$$\mathcal{W}_2 = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \oplus [a] : a, b \in \mathbb{C} \right\}.$$

One can easily see that $\mathcal{W}_1$ is not reflexive, but $\mathcal{W}_2$ is reflexive.

In [66], it was also shown

Theorem 1.2.3. *The shift operator is reflexive* ($(Sf)(z) = zf(z)$ *for* $f \in H^2$).

The result was extended in [34].

Theorem 1.2.4. *Every isometry is reflexive.*

The reflexive nilpotents in finite-dimensional spaces are completely characterized [35].

Theorem 1.2.5. *A nilpotent in the finite-dimensional space is reflexive if and only if the two largest blocks in its Jordan decomposition differ no more than one in size.*

The following example shows that the question about reflexivity for N-tuples is non-trivial.

Example 1.2.6. There is a pair $\{T_1, T_2\} \subset L(\mathcal{H})$ of commuting operators such that T_i is reflexive for $i = 1, 2$, but $\mathcal{W}(T_1, T_2)$ is not reflexive.

Let us consider $\mathcal{H} = \mathbb{C}^2 \oplus \mathbb{C}^2$ and define $T_1 = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $T_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \oplus \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. It is straightforward that T_1 and T_2 commute. Moreover, T_1 is reflexive by Theorem 1.2.4 and T_2 is reflexive as a normal operator (see Theorem 1.2.1). It is easy to see that

$$\mathcal{W}(T_1, T_2) = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \oplus \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} : a, b, c, d \in \mathbb{C} \right\}.$$

On the other hand, one can check that

$$\begin{aligned} Lat(T_1T_2, I - T_2) = \{L_1 \oplus L_2 : L_1 \in \{\{0\}\} \oplus \{0\}, \mathbb{C} \oplus \{0\}, \mathbb{C} \oplus \mathbb{C}\} \\ \textit{and } L_2 \textit{ is any subspace of } \mathbb{C}^2\} \end{aligned}$$

Similarly,

$$\begin{aligned} Lat(T_1(I - T_2), T_2) = \{L_1 \oplus L_2 : L_1 \textit{ is any subspace of } \mathbb{C}^2 \\ \textit{and } L_2 \in \{\{0\} \oplus \{0\}, \mathbb{C} \oplus \{0\}, \mathbb{C} \oplus \mathbb{C}\}\}. \end{aligned}$$

Hence

$$\begin{aligned} &Lat(T_1, T_2) \subset Lat(T_1T_2, I - T_2) \cap Lat(T_1(I - T_2), T_2) \\ &\quad = \{\{0\} \oplus \{0\} \oplus \{0\} \oplus \{0\}, \{0\} \oplus \{0\} \oplus \mathbb{C} \oplus \{0\}, \{0\} \oplus \{0\} \oplus \mathbb{C} \oplus \mathbb{C}, \\ &\qquad \mathbb{C} \oplus \{0\} \oplus \{0\} \oplus \{0\}, \mathbb{C} \oplus \{0\} \oplus \mathbb{C} \oplus \{0\}, \mathbb{C} \oplus \{0\} \oplus \mathbb{C} \oplus \mathbb{C}, \\ &\qquad \mathbb{C} \oplus \mathbb{C} \oplus \{0\} \oplus \{0\}, \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \{0\}, \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\}. \end{aligned}$$

The other inclusion is trivial. Now it is easy to see that

$$Alg\,Lat(T_1, T_2) = \left\{ \begin{bmatrix} a & b \\ 0 & s \end{bmatrix} \oplus \begin{bmatrix} c & d \\ 0 & t \end{bmatrix} : a, b, c, d, s, t \in \mathbb{C} \right\}.$$

Hence $\mathrm{AlgLat}(T_1, T_2)$ is larger than $\mathcal{W}(T_1, T_2)$.

We will finish this section with a proposition from [5], which will allow more substantial applications later.

Proposition 1.2.7. *Every one-dimensional operator space is reflexive.*

2. On The Reflexivity of N-Tuples of Operators on a Finite-Dimensional Space.

In their paper [35], J. Deddens and P. Fillmore characterized reflexive operators in terms of their Jordan Canonical Forms. In this section we will present the multi-operator extensions of their results. We would like to stress the role of rank-two operators.

The first step in the Deddens-Fillmore analysis is the reduction to the nilpotent case. A similar procedure is possible in the multi operator setting, but we have chosen to postpone the argument to Section 2.6 and concentrate on commuting families of nilpotents in the main discussion.

However, in this section we are mostly dealing with finite-dimensional case and we are oriented to the Hilbert space operators. Some part of the following section, especially part of Theorem 2.1 below, does not need the Hilbert space structure. So all operators can be considered on the vector space V. The definition of reflexivity itself does not need the Hilbert space structure and all subspaces in the finite-dimensional vector space are closed, so in the definition of reflexivity of the algebra $\mathcal{A}$ the elements of $Lat\mathcal{A}$ are allowed to be not necessarily closed. This is a *topological free* version of reflexive algebras. For finite-dimensional algebras they are equivalent. Since, in this section, we will mostly deal with finite-dimensional spaces we will use a small letter to denote an operator (linear transformation).

Analyzing the Deddens-Fillmore condition, as will be made in Section 2.1, can lead us to the connection between block sizes and operators of rank two.

Theorem 2.1. *Suppose $\mathcal{A}$ is an operator algebra generated by a commuting family of nilpotents. Then, in order for $\mathcal{A}$ to be reflexive, it is necessary that each rank-two member of $\mathcal{A}$ generates a one-dimensional ideal. If the underlying vector space is a finite-dimensional Hilbert space and the generators for $\mathcal{A}$ commute with each other's adjoints, then this condition is also sufficient.*

As will be seen, for the necessity in the above theorem, the underlying vector space need not even be finite-dimensional. The theorem is true in both versions of reflexivity, topological and topological free.

It is possible to apply Theorem 2.1 directly to concrete examples. Thus, one can see that the algebra

$$(*)\quad \left\{ \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ & \alpha & & \gamma \\ & & \alpha & \beta \\ & & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta & \epsilon \\ & \alpha & \beta \\ & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \gamma \\ & \alpha \end{pmatrix} : \alpha, \beta, \gamma, \delta, \epsilon, \in \mathbb{C} \right\}$$

is reflexive, but the algebra

$$(**)\left\{ \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ & \alpha & & \gamma \\ & & \alpha & \beta \\ & & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta & \epsilon \\ & \alpha & \beta \\ & & \alpha \end{pmatrix} \oplus \begin{pmatrix} \alpha & \beta \\ & \alpha \end{pmatrix} : \alpha, \beta, \gamma, \delta, \epsilon, \in \mathbb{C} \right\}$$

is not reflexive (missing entries are assumed to be zero). In Section 2.5 we will study such examples in more details in order to illustrate the full strength of Theorem 2.1. In the process, we present the original and more easily applicable version of the theorem. In order to simplify the notation, we concentrate on doubly commuting pairs (a, b) of nilpotents. For our simultaneous Jordan form, we consider direct sum decompositions of (a, b). As in the single operator case, the sizes of these direct summands provide a complete set of invariants for these pairs. We store this information in a finite "Jordan sequence" $(m_1, n_1), \cdots, (n_k, m_k)$; precise definitions are given in Section 2.5.

This leads to the following concrete version of Theorem 2.1 for pairs of operators. It reduces to the Deddens-Fillmore result when $b = 0$.

Theorem 2.2. *Suppose (a, b) is a doubly commuting pair of nilpotents acting on a finite-dimensional Hilbert space with Jordan sequence $(m_1, n_1), \cdots, (m_k, n_k)$. Then the algebra $\mathcal{A}(a, b)$ generated by a, b is reflexive if and only if for each index i,*

(1) *if $m_i \geq 2$, we call find $j \neq i$ with $m_j \geq m_i - 1$ and $n_j \geq n_i$, and*
(2) *if $n_i \geq 2$, we can find $j \neq i$ with $n_j \geq n_i - 1$ and $m_j \geq m_i$.*

2.1. The single operator case.

The present section is motivational, formal proofs being omitted as these results are consequences of Theorem 2.5.2. We begin by recalling the Deddens-Fillmole result. A nilpotent operator is said to be *simple* if its Jordan form consists of a single block.

Theorem 2.1.1. *Let a_1, a_2 be nilpotents of orders $m_1 \geq m_2$ respectively.*

(1) *If m_1 and m_2 differ by at most one, then $a_1 \oplus a_2$ is reflexive.*
(2) *If a_1 is simple and m_1 and m_2 differ by more than one, then $a_1 \oplus a_2$ is not reflexive.*

Let a be nilpotent. Apply the Jordan Canonical Form Theorem to write $a = \oplus_{i=1}^{k} a_i$, where the direct summands are simple. It is convenient to assume that $k \geq 2$; this can always be accomplished by including a direct summand acting on a zero-dimensional space (considered to have order zero) in the decomposition. Theorem 2.1.1 immediately tells us whether a is reflexive in terms of the orders $m_1, \cdots, m_k$ (not necessarily monotone) of the blocks $a_1, \cdots, a_k$. Indeed, it is clear that a is reflexive if and only if the following condition holds:

(0) The orders of the two largest blocks differ by at most one.

It will be useful to have several alternate versions of (0). We aim for characterizations which generalize naturally to the multi-operator case. One of the difficulties we face is that there is no natural ordering on pairs of natural numbers.

The following immediate restatement of (0) is the single-operator version of the condition in Theorem 2.2:

(1) For each i, if $m_i \geq 2$ then there is some $j \neq i$ with $m_j \geq m_i - 1$.

Next, suppose (0) holds, write m for the order of a, and consider the rank of a^{m-2}. Each block of order m in the Jordan decomposition of a contributes 2 to this rank, while each block of order $m-1$ contributes 1. Thus we must have rank $(a^{m-2}) > 2$. A fortiori, the ranks of smaller powers of a must also exceed 2. Moreover, the rank of any polynomial in a is determined by its term of lowest degree. In other words, the only members of $\mathcal{A}(a)$ which are allowed to have rank two are scalar multiples of a^{m-1}. Observing that no block of a^{m-1} can have rank two, we are led to the following version of (0):

(2) If $c = \oplus_{i=1}^{k} c_i \in \mathcal{A}(a)$ has rank two, then $c_i \neq 0$ for two values of i.

The following reformulation has the advantage of not depending on the decomposition:

(3) If $c \in \mathcal{A}(a)$ has rank two, then $c\mathcal{A}(a)$ is one-dimensional.

Complete proofs of the equivalence of $(0), (1), (2), (3)$ will follow from the multioperator case considered in Section 2.5.

2.2. The role of rank-two operators.

In this section, we prove necessity in Theorem 2.1 and present some related examples. There is no restriction on the dimension of the underlying vector space. We also recall two basic properties of nilpotent operators.

Lemma 2.2.1. *Let a be nilpotent and suppose c is a non-zero operator of finite rank commuting with a. Then rank$(a\,c) <$ rank(c).*

Proof. We know that $\mathrm{ran}(a\,c) \subseteq ran(c)$. If this inclusion was not proper, we would have $\mathrm{ran}(a^n\,c) = ran(c)$ for all n. But this is ruled out by the nilpotence of a.

Given x in V and ϕ in the dual space $V^{'}$, we write $x \cdot \phi$ for the operator on V defined by $(x \cdot \phi)z = \phi(z)x$ for $z \in V$. This is the zero operator if $x = 0$ or $\phi = 0$. Otherwise $x \cdot \phi$ has rank one, and it is clear that x spans its range and it has the same kernel as ϕ. Moreover, every rank-one operator has this form.

Lemma 2.2.2. *If b is a nilpotent commuting with the rank-one operator $x \cdot \phi$, then $bx = 0$.*

Proof. Applying Lemma 2.2.1, we have $(b\,x) \cdot \phi = b(x \cdot \phi) = 0$ and the conclusion follows since $\phi \neq 0$.

Consider the simplest non-reflexive algebra $\left\{ \left(\begin{smallmatrix} \lambda & \mu \\ 0 & \lambda \end{smallmatrix} \right) : \lambda, \mu \in \mathbb{C} \right\}$. Note that the identity operator, which is of rank two, does not generate a one dimensional ideal. The following proposition (the necessity in Theorem 2.1) shows that this is enough to prevent reflexivity.

Proposition 2.2.3. *Suppose $\mathcal{A}$ is a reflexive algebra generated by commuting nilpotents. Then each $c \in \mathcal{A}$ of rank two must generate a one-dimensional ideal.*

Proof. Assume that $c \in \mathcal{A}$ has rank two, but fails to generate a one-dimensional ideal. Thus, there is some $b \in \mathcal{A}$, such that $b\,c$ is independent of

c. Subtracting a multiple of the identity from b if necessary, we can assume that b is a nilpotent whence rank $(cb) = 1$ by Lemma 2.2.1. Choose $x \in V, \phi \in V'$ with $cb = x \cdot \phi$. Write $c = x \cdot \psi + w \cdot \xi$ for appropriate w, ψ, ξ. Since b commutes with $cb = x \cdot \phi$, we have $bx = 0$ by Lemma 2.2.2. Hence $bc = bw \cdot \xi = x \cdot \phi$. This forces ξ to be a scalar multiple of ϕ, so, changing w if necessary, we can write $c = x \cdot \psi + w \cdot \phi$.

We will complete the proof by showing that the rank-one operator $x \cdot \psi$ belongs to $Ref(\mathcal{A})$ but not to $\mathcal{A}$. For the first assertion, note that for $y \notin \ker \phi$, we have $(x \cdot \psi)y = \frac{\psi(y)}{\phi(y)} bcy$, while for $y \in \ker \phi$, we get $(x \cdot \psi)y = cy$.

Suppose, on the other hand, that $x \cdot \psi \in \mathcal{A}$. Then b commutes with $x \cdot \psi$. Moreover, since we already know that b commutes with c, we also learn that b commutes with $w \cdot \phi$. But then Lemma 2.2.2 yields $bx = bw = 0$ which leads to the contradiction $bc = 0$.

Example 2.2.4. The nilpotent matrices $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ generate the full algebra of two-by-two matrices. Since this is a reflexive algebra whose rank-two member $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ does not generate a one-dimensional ideal, it is not possible to drop the commutativity hypothesis in Proposition 2.2.3.

Example 2.2.5. To see that the condition in Proposition 2.2.3 does not guarantee reflexivity, fix a subspace $\mathcal{S}$ of $L(V)$ with dim $V \geq 2$ and take $\mathcal{A} = \left\{ \begin{pmatrix} \lambda I & a \\ 0 & \lambda I \end{pmatrix} : a \in \mathcal{S} \right\}$. Then the algebra $\mathcal{A}$ is commutative and each of its rank-two members generates a one-dimensional ideal. On the other hand, if $\mathcal{S}$ is not reflexive (for example, the algebra mentioned before Proposition 2.2.3), then neither is $\mathcal{A}$. See for example [5], [9].

For a more striking example, apply [5, Proposition 3.7] to get an operator space $\mathcal{T}$ which is not 3-reflexive and take $\mathcal{S}$ to be a three-fold copy of $\mathcal{T}$. Then the resulting algebra $\mathcal{A}$ fails to be reflexive even though it has *no* rank-two members.

2.3. Reflexivity of subdirect sums.

From now on to the end of Section 2, all underlying vector spaces are assumed to be finite-dimensional. It is easy to see that a full direct sum $\mathcal{S} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k$ of operator spaces is reflexive if and only if each of its direct summands is reflexive, but in general there is no relationship between the reflexivity of $\mathcal{S}$ and its various subspaces. This is unfortunate, since algebras generated by direct sums of operators are usually not full direct sums of algebras. Proposition 2.3.2 below provides a tool for dealing with this situation.

A vector $x \in V$ is called *separating* for the subspace $\mathcal{S} \subset L(V)$ if the map $s \to sx$ is injective on $\mathcal{S}$. It is easy to see that the existence of separating vectors survives the taking of direct sums. It follows that each singly generated algebra has a separating vector. In particular, existence of separating vectors does not guarantee reflexivity. We do, however, have the following basic result; see [5, Propositions 2.9, 3.2] for a proof.

Proposition 2.3.1. *Suppose a subspace $\mathcal{S}$ of $L(V)$ is reflexive and has a separating vector. Then every subspace of $\mathcal{S}$ is reflexive.*

The following proposition will be used to establish the sufficiency in Theorem 2.1. An operator $a = a_1 \oplus \cdots \oplus a_k$ in $L(V_1) \oplus \cdots \oplus L(V_k)$ is said to be *supported* on V_i if $a_j = 0$ for all $j \neq i$.

Proposition 2.3.2. *For each $i = 1, \cdots, k$, let $\mathcal{S}_i \subset L(V_i)$ be an operator space with a separating vector x_i. Suppose $\mathcal{T}$ is a subspace of $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k$, and for each i, write $\mathcal{T}_i = \{a \in \mathcal{T}\colon a$ is supported on $V_i\}$. Then $\mathcal{T}$ is reflexive if and only if each $\mathcal{T}_i$ is reflexive.*

Proof. We assume $k \geq 2$ to avoid trivialities. Note first that $x = x_1 \oplus \cdots \oplus x_k$ is a separating vector for $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k$. For the necessity, observe that x must also separate $\mathcal{T}$, whence the reflexivity of each $\mathcal{T}_i$ follows from that of $\mathcal{T}$ by Proposition 2.3.1.

For the sufficiency, suppose $c = c_1 \oplus \cdots \oplus c_k \in \mathit{Ref}\, \mathcal{T}$. Since $c\,x \in \mathcal{T}(x)$, by subtracting an appropriate member of $\mathcal{T}$ from c if necessary, we may as well assume that $c\,x = 0$. We will show that $c_1 = 0$ whence $c = 0$ by symmetry, completing the proof.

Write $\overline{c} = c_2 \oplus \cdots \oplus c_k$ and $\overline{x} = x_2 \oplus \cdots \oplus x_k$. Given $y \in V_1$, we must have $(c_1 \oplus \overline{c}(y \oplus \overline{x}) = (s_y \oplus t_y)(y \oplus \overline{x})$ for some operator $s_y \oplus t_y \in \mathcal{T}$. This is equivalent to the two conditions

$$0 = \overline{c}\,\overline{x} = t_y \overline{x} \quad \text{and} \quad c_1\, y = s_y\, y.$$

Since $\overline{x}$ is separating, we first see that $t_y = 0$, which means that $s_y \oplus 0 \in \mathcal{T}_1$. The arbitrariness of y thus yields $c_1 \oplus 0 \in \mathit{Ref}\, \mathcal{T}_1$. Since $\mathcal{T}_1$ is reflexive, we have $c_1 \oplus 0 \in \mathcal{T}_1$. Since $x_1 \oplus 0$ separates $\mathcal{T}_1$ and $c_1\, x_1 = 0$, we get $c_1 = 0$ as desired.

We conclude this section with a trivial instance of Proposition 2.3.2.

Corollary 2.3.3. *If a_1, a_2 are nilpotents of the same order, then $a_1 \oplus a_2$ is reflexive.*

Proof. Take $\mathcal{S}_i$ to be the algebra generated by a_i and $\mathcal{T}$ the algebra. generated by $a_1 \oplus a_2$. Apply Proposition 2.3.2, noting that $\mathcal{T}_i = \mathcal{T}_2 = \{0\}$.

2.4. Necessary and sufficient conditions for the reflexivity of doubly commuting N-tuples of nilpotents.

The first two propositions in this section record well-known facts; the second motivates our definition of simple N-tuple and is a special case of Proposition 2.4.4 below.

Proposition 2.4.1. *Let $\mathcal{A} \subset L(V)$ be a commutative algebra having a cyclic vector x. Then*

(1) *x is also a separating vector for $\mathcal{A}$, and*

(2) *for each $c \in \mathcal{A}$, the rank of c is equal to the dimension of the ideal $c\mathcal{A}$.*

Proof. To see (1), note that by the commutativity of $\mathcal{A}$, if the cyclic vector x belongs to the kernel of an operator in $\mathcal{A}$, the whole space V is contained in that kernel. For (2), observe that the map $a \to a\,x$ defines a vector space isomorphism between $\mathcal{A}$ and V; for each $c \in \mathcal{A}$ it maps the ideal $c\mathcal{A}$ to range(c).

Proposition 2.4.2. *Let $a \in L(V)$ be nilpotent. Then the following are equivalent:*

(1) *the trivial operators 0 and I are the only idempotents commuting with a,*

(2) *the Jordan form of a is a single block,*

(3) *$\mathcal{A}(a)$ has a cyclic vector.*

Suppose $a = (a_1, \cdots, a_N)$ and $b = (b_1, \cdots, b_N)$ are N-tuples of operators acting on vector spaces V and W, respectively. Then we say **a** is *similar* to **b** if there is an invertible operator $s \in L(V, W)$ satisfying $b_i = s a_i s^{-1}$ for $i = 1, \cdots, N$.

All operators in the remainder of this section act on finite-dimensional *Hilbert* spaces. Recall that an N-tuple $a = (a_1, \cdots, a_N)$ of operators is doubly commuting if $a_i a_j = a_j a_i$ and $a_i a_j^* = a_j^* a_i$ for each $i \neq j$. The condition is equivalent to requiring that the von Neumann algebras generated by the individual operators commute with each other.

An N-tuple $a = (a_1, \cdots, a_N)$ of doubly commuting nilpotents is called *simple* if there are no non-trivial idempotents commuting with all of them.

Example 2.4.3. Let $\tilde{a}_i$ act on a Hilbert space V_i for $i = 1, \cdots, N$ and form the tensor product space $V = V_1 \oplus \cdots \oplus V_N$. Define $a_i = I \oplus \cdots \oplus \tilde{a}_i \oplus \cdots \oplus I$. Then $a = (a_1, \cdots, a_N)$ is doubly commuting.

In order for $a = (a_1, \cdots, a_N)$ to be simple, it is necessary that the von Neumann algebras generated by the $\tilde{a}_i$ are factors. The condition fails to be sufficient even when $N = 1$, not only because commuting projections may fail to be central, but also because non-self-adjoint idempotents must be taken into account.

Proposition 2.4.4. *Let $a = (a_1, \cdots, a_N)$ be a doubly commuting N-tuple of nilpotents acting on a Hilbert space V. Then the following are equivalent:*

(1) *$a = (a_1, \cdots, a_N)$ is simple,*

(2) *$a = (a_1, \cdots, a_N)$ takes the form of Example 2.4.3 with each $\tilde{a}_i$ being simple,*

(3) *$\mathcal{A}(a_1, \cdots, a_N)$ has a cyclic vector.*

Proof. (1) $\Longrightarrow$ (2). The precise meaning of (2) involves a unitary map between the underlying Hilbert spaces. We argue by induction. If $N = 1$, there is nothing to prove. If $N \geq 2$, then the von Neumann algebra $\mathcal{N}(a_1)$ generated by a_1 must be a type I factor. Thus there are Hilbert spaces V_1 and K, and a unitary map $U : V \to V_1 \oplus K$ such that $U\mathcal{N}(a_1)U^{-1} = L(V_1) \oplus \mathbb{C}\, I_K$. There is no harm in suppressing U and assuming $\mathcal{N}(a_1) = L(V_1) \oplus \mathbb{C}\, I_k$, whence $a_1 = \tilde{a}_1 \oplus I_K$. By double commutativity, the von Neumann algebra $\mathcal{N}(a_2, \cdots, a_N)$ generated by $a_2, \cdots, a_N$ is contained in $I_{V_1} \oplus L(K)$. In particular, $\tilde{a}_1$ must be simple since $q \oplus I_K$ will commute with each a_i whenever q commutes with $\tilde{a}_1$. The decomposition is completed by applying the inductive hypothesis to $\mathcal{N}(a_2, \cdots, a_N)$.

(2) $\Longrightarrow$ (3). For each $i = 1, \cdots, N$, choose a cyclic vector x_i for $\mathcal{A}(\tilde{a}_i)$ and take $x = x_1 \oplus \cdots \oplus x_N$.

(3) $\Longrightarrow$ (1). Suppose q is an idempotent commuting with $a_1, \cdots, a_N$ and x is a cyclic vector for $\mathcal{A}(\mathbf{a})$. Then there is $c \in \mathcal{A}(\mathbf{a})$ with $q\,x = c\,x$. But then $\ker(q - c)$ contains all of $\mathcal{A}(\mathbf{a})x$, so $q = c$ belongs to $\mathcal{A}(\mathbf{a})$. Since 0 is the only operator which is simultaneously idempotent and nilpotent, we conclude that $q = 0$ or I, as desired.

Proposition 2.4.5. *Suppose $a = (a_1, \cdots, a_N)$ is a simple N-tuple of doubly commuting nilpotents and $c \in \mathcal{A}(\boldsymbol{a})$.*

(1) *All rank-one members of $\mathcal{A}(\boldsymbol{a})$ are scalar multiples of one other.*

(2) *If rank$(c) = 2$, then $c\mathcal{A}(a)$ is two-dimensional.*

(3) *If rank$(c) \geq 2$, then $c\mathcal{A}(a)$ contains a member of rank two.*

Proof. To establish (1), write n_i for the order of a_i, and set $a^k = a_1^{k_1} \cdots a_N^{k_N}$ for each N-tuple $k = (k_1, \cdots, k_N)$ of natural numbers. Suppose $c = \sum \lambda_k a^k$ has rank one. Lemma 2.2.1 tells us that $a_i c = 0$ for each i. But, in view of Proposition 2.4.4 (2), we know that the operators a^k for all $k = (k_1, \cdots, k_N)$ with $0 \leq k_i \leq n_i - 1$ are linearly independent. Thus, $\lambda_k = 0$ whenever $k_i \leq n_i - 2$. This forces c to be a scalar multiple of $a_1^{{n_1}^1} \cdots a_N^{n_N - 1}$.

Part (2) is a consequence of Propositions 2.4.4 and 2.4.1(2).

We prove (3) inductively. Given rank$(c) > 2$, Proposition 2.4.1(2) yields dim $c\mathcal{A}(\mathbf{a}) > 2$ as well. On the other hand, $\mathcal{A}(\mathbf{a})$ is spanned by its nilpotent members and I. Thus there are nilpotent members b, d of $\mathcal{A}(\mathbf{a})$ such that cb, cd are independent. By Part (1) of the present proposition, at least one of these, say cd, has rank greater than one. Lemma 2.2.1 thus makes it possible to apply the inductive hypothesis to cd.

In general, similarities destroy double commutativity. Note however that tensor products of similarities on the underlying spaces V_i preserve double commutativity of the operators a_i of Example 2.4.3. This will be important in the following proof.

Proposition 2.4.6. *Every N-tuple $a = (a_1, \cdots, a_N)$ of doubly commuting nilpotents is similar to an orthogonal direct sum of simple N-tuples.*

More precisely, there is a doubly indexed family $\{a_{ij} : i = 1, \cdots, N; j = 1, \cdots, k\}$ of nilpotents such that

(1) *$(a_{1j}, \cdots, a_{Nj})$ is a simple N-tuple for each fixed j, and*

(2) *the original operators a_i are simultaneously similar to the orthogonal direct sums $\oplus_{j=1}^{k} a_{ij}$,*

Proof. The von Neumann algebras $\mathcal{N}(a_1), \cdots, \mathcal{N}(a_N)$ commute, so any self-adjoint Projection in the center of one of them wi11 automatically commute with all of them. Doing a preliminary orthogonal decomposition we may thus assume all the $\mathcal{N}(a_i)$ to be factors. But then the proof of Proposition 2.4.4 ((1) $\Longrightarrow$ (2)) a11ows us to write $V = V_1 \oplus \cdots \oplus V_N$ and $a_i = I \oplus \cdots \oplus \tilde{a}_i \oplus \cdots \oplus I$ except that the $\tilde{a}_i$ need not be simple. Now the Jordan Canonical Form Theorem tells us that each $\tilde{a}_i$ is similar to an orthogonal direct sum of simple operators. Putting these similarities together, we can assume that the $\tilde{a}_i$ are themselves orthogonal direct sums of simple operators. The proof is completed by "splitting" these direct sums.

The sufficiency part of Theorem 2.1 can be stated as follows.

Theorem 2.4.7. *Suppose $a = (a_1, \cdots, a_N)$ is an N-tuple of doubly commuting nilpotents acting on a finite-dimensional Hilbert space V. If every* *rank-two member of $\mathcal{A}(\mathbf{a})$ generates a one-dimensional ideal, then $\mathcal{A}(\mathbf{a})$ is reflexive.*

Proof. Apply Proposition 2.4.6 to write $a_i = \oplus_{j=1}^{k} a_{ij}$ where, for each j, the N-tuple $(a_{1j}, \cdots, a_{Nj})$ acting on V_j is simple. For each j, take $\mathcal{S}_j$ to be $\mathcal{A}(a_{1j}, \cdots, a_{Nj})$ and set $\mathcal{T}= \{c \in \mathcal{A}(\mathbf{a})$: c is supported on $V_j\}$. Concentrating on $j = 1$ to simplify the notation, observe that every member of $\mathcal{T}_1$ takes the form $c_1 \oplus 0$ with c_1 belonging to the algebra $\mathcal{S}_1$. Applying Proposition 2.4.5 (2) and the hypothesis, we conclude that c_1 cannot have rank two. By Part (3) of the same Proposition, the rank of c_1 cannot exceed one. Finally Part (1) of the same Proposition implies that all such c_1 are scalar multiples of one another. By symmetry, each $\mathcal{T}_j$ is one-dimensional, and hence reflexive by Proposition 1.2.7. We complete the proof by applying Proposition 2.3.2 to $\mathcal{T} = \mathcal{A}(a_1, \cdots, a_N)$.

2.5. Jordan forms for doubly commuting pairs of nilpotents.

In the plesent section, we will prove Theorem 2.2 and discuss some of its applications. In order to simplify the notation we restrict attention to pairs of operators, but generalization to arbitrary N-tuples is routine. The *order* of a pair (a, b) of nilpotents is the pair of integers (order(a), order(b)).

We refer to the sequence of block sizes of a Jordan Canonical Form of an operator as a *Jordan sequence* for the operator; up to permutation, Jordan sequences provide a complete similarity invariant for single operators. Proposition 2.4.6 allows us to extend this notion to doubly commuting operator pairs.

Indeed, given a doubly commuting pair (a, b), apply Proposition 2.4.6 to obtain direct sums $\oplus_{i=1}^{k} a_i$, $\oplus_{i=1}^{k} b_i$ which are simultaneously similar to a, b such that for each i, the doubly commuting pair (a_i, b_i) is simple and acts on a Hilbert space V_i. Write (m_i, n_i) for the order of the simple pair (a_i, b_i). The finite sequence $(m_1, n_1), \cdots, (m_k, n_k)$ is referred to as a *Jordan sequence* of (a, b). Up to permutation, these sequences provide a complete similarity invariant for doubly commuting pairs.

Lemma 2.5.1. *Let (a, b) be a simple pair with order (m, n) and suppose $c \in \mathcal{A}\ (a, b)$. Then*

(1) *rank$(c) \leq 1$ if and only if c is a scalar multiple of $a^{m-1}b^{n-1}$,*
(2) *rank$(c) \leq 2$ if and only if c is a linear combination of $a^{m-2}b^{n-1}$, $a^{m-1}b^{n-2}$ and $a^{m-1}b^{n-1}$.*

Proof. (1) is a consequence of Propositions 2.4.5 and 2.4.1 (2).

For the sufficiency of (2), observe that if c takes this form, then ac and bc are both scalar multiples of $a^{m-1}b^{n-1}$, so $\dim(c\mathcal{A}(a,b)) \leq 2$. Thus $\mathrm{rank}(c) \leq 2$ by Proposition 2.4.1(2).

For the converse, suppose $c = \sum_{i,j=0}^{m-1,n-1} \lambda_{ij} a^i b^j$ has rank two. By Lemma 2.2.1, we have $\mathrm{rank}(ac) < rank(c) = 2$. Applying part (1), $ac = \sum_{i,j=0}^{m-1,n-1} \lambda_{ij}$ $a^{i+1}b^j = \alpha a^{m-1}b^{n-1}$ for some $\alpha \in \mathbb{C}$. Thus $\lambda_{ij} = 0$ for $i < m-2$ and $\lambda_{m-2,j} = 0$ for $j < n-1$. By symmetry, we have the desired form.

Lemma 2.5.1 admits a partial generalization in that every $c \in \mathcal{A}(a,b)$ of rank r or less must be a linear combination of $\{a^{m-i}b^{n-j} : 0 < i,\, 0 < j, i+j \leq r+1\}$. To see that the condition is not sufficient, note that if (a,b) is a simple pair of order (2,2) then $\mathcal{A}(a,b)$ does not contain any members of rank three.

When $b = 0$, the first three conditions of the following theorem reduce to the corresponding conditions of Section 2.2. The equivalence of (1) and (4) is Theorem 2.2.

Theorem 2.5.2. *Suppose (a,b) is a doubly commuting pair with corresponding Jordan sequence $(m_1,n_1),\cdots,(m_k,n_k)$. Then the following are equivalent:*

(1) *For each index i,*

if $m_i \geq 2$, we can find $j \neq i$ with $m_j \geq m_i - 1$ and $n_j \geq n_i$, and

if $n_i \geq 2$, we can find $j \neq i$ with $n_j \geq n_i - 1$ and $m_j \geq m_i$.

(2) *If $c = \oplus_{i=1}^k c_i \in \mathcal{A}(a,b)$ has rank 2, then $c_i \neq 0$ for two values of i.*

(3) *If $c \in \mathcal{A}(a,b)$ has rank 2, then $c\mathcal{A}(a,b)$ is one-dimensional.*

(4) *$\mathcal{A}(a,b)$ is reflexive.*

Proof. (1) $\Longrightarrow$ (2). Arguing contrapositively, assume for definiteness that $c = c_1 \oplus 0$ in $\mathcal{A}(a,b)$ has rank two and is supported on V_1. Write $c = \sum \lambda_{hl} a^h b^l$. On the one hand, $\sum \lambda_{hl} a_1^h b_1^l$ has rank two, so in view of Lemma 2.5.1(2), we may as well assume that $\lambda_{m_1-2,n_1-1} \neq 0$. On the other hand, for $j \neq 1$, the vanishing of c_j forces $m_j \leq m_1 - 2$ or $n_j \leq n_1 - 1$. This means that (1) fails for $i = 1$ and completes the proof.

(2) $\Longrightarrow$ (3). Suppose $c \in \mathcal{A}(a,b)$ has rank 2. By (2), we may assume $c = c_1 \oplus c_2 \oplus 0$ with c_1, c_2 of rank 1. By Lemma 2.2.1, we have $a_1\, c_1 = a_2\, c_2 = 0$, so $a\,c = 0$. Similarly, $bc = 0$. Thus $c\mathcal{A}(a,b)$ is one-dimensional.

(3) $\Longleftrightarrow$ (4). These are Theorems 2.4.7 and 2.2.3.

(3) $\Longrightarrow$ (1). Suppose $m_1 \geq 2$. By Lemma 2.5.1, the rank of the operator $a_1^{m_1-2}b_1^{n_1-1}$ is precisely two and hence by Proposition 2.4.1(2), it generates a two-dimensional ideal. The assumption (3) rules out the possibility that $a^{m_1-2}b^{n_1-1}$ be supported on V_1 . In other words, $m_j > m_1 - 2$ and $n_j > n_1 - 1$

for some $j \neq 1$. This establishes the first half of (1) when $i = 1$ and the rest follows by symmetry.

It is convenient to call the pair (m_i, n_i) *majorized* if Condition (1) of Theorem 2.5.2 is fulfilled for the index i. The discussion of examples is also facilitated by calling a Jordan sequence *reflexive* if the corresponding operator algebra is reflexive. As the first application of Theorem 2.5.2, we review the examples given in the introduction to Section 2.

Example 2.5.3. The algebra denoted by $(^*)$ is generated by a pair with Jordan sequence (2,2), (3,1), (1,2), while a generating pair for the algebra $(^{**})$ has Jordan sequence (2,2), (3,1), (2,1). It is easy to check that each term in the first sequence is majorized, but the term (2,2) has no majorant in the second sequence.

The next two results apply Theorems 2.1 and 2.2 to tensor products.

Proposition 2.5.4. *Suppose a, b are nilpotent Then $\mathcal{A}(a \oplus b)$ is reflexive.*

Proof. Suppose $c = p(a \oplus b) \in \mathcal{A}(a \oplus b)$ has rank two. Factor the polynomial p to obtain $p(X) = X^k q(X)$, with $q(0) \neq 0$. Then $q(a \oplus b)$ is invertible, so in fact $(a \oplus b)^k$ has rank two. But then $(\mathrm{rank}(a^k))(\mathrm{rank}(b^k)) = 2$. In particular, either a^k or b^k has rank one. In either case, $c(a \oplus b) = (a^{k+1} \oplus b^{k+1})q(a \oplus b) = 0$. Therefore, c generates a one-dimensional ideal and Theorem 2.1 applies.

The operators c, d appearing in the next result are not assumed to be simple.

Corollary 2.5.5. *Suppose c, d are nilpotent operators. If c and d are reflexive, then the algebra $\mathcal{A}(c \oplus I,\, I \oplus d)$ is reflexive. If the two largest members of the Jordan sequence of c or d are the same, then the algebra $\mathcal{A}(c \oplus I,\, I \oplus d)$ is reflexive. In all other cases, the algebra $\mathcal{A}(c \oplus I,\, I \oplus d)$ is not reflexive.*

Proof. The Jordan sequence of $(c \oplus I,\, I \oplus d)$ is the kartesian product of the Jordan sequences of c and d. Let $m_1 \geq m_2$ and $n_1 \geq n_2$ denote the two largest terms in the Jordan sequences of c and d, respectively. All elements of the Jordan sequence of $(c \oplus I,\, I \oplus d)$ except (m_1, n_1) are majorized. The remaining term (m_1, n_1) is majorized in precisely the following situations:

(1) $m_1 = m_2$ or

(2) $n_1 = n_2$ or

(3) $m_1 = m_2 + 1$ and $n_1 = n_2 + 1$.

These correspond to the cases listed in the statement of our corollary.

Example 2.5.6. In the single operator case, there are always at least three singleton sequences whose concatenations with a given Jordan sequence produce reflexive sequences. For example, the singleton sequence 5 can be lengthened to the reflexive sequellces 5, 4; 5, 5; and 5, 6. On the other hand, the only two-term reflexive extension of the Jordan sequence (5,7) is (5,7), (5,7). Even the three-term reflexive extensions of (5,7) are limited. For example, only the first of the following four extensions of (5,7) is reflexive:

$$(4,7),(5,7),(5,6); \quad (4,6),(5,7),(5,6); \quad (5,8),(5,7),(5,6); \quad (6,6),(5,7),(8,4).$$

Example 2.5.7. The Jordan sequence (1,10), (2, 9), (3, 8), (4, 7), (4, 6) represents a reflexive pair and it is minimal in the sense that none of its proper subsequences is reflexive. This contrasts with the single operator case, where discarding all but the two largest terms of a Jordan sequence does not affect reflexivity.

2.6. Generalizations and non-nilpotent case.

The first topic of this section is a Hilbert space free version of Theorems 2.1 and 2.2.

The reader is referred to [3] for the background in ring theory, in particular for the Wedderburn Structure Theory used below. We recall the relevant definitions here. A left module over a ring is said to be *simple* if it has no non-trivial submodules; it is *semisimple* if it can be expressed as a direct sum of simple modules. A ring is *simple* if it has no non-trivial two-sided ideals; it is *semisimple* if it is semisimple when regarded as a left module over itself.

We need the following well-known fact.

Proposition 2.6.1. *An operator algebra is semisimple if and only if it is similar to a von Neumann algebra.*

Proof. Let $\mathcal{B}$ be a subalgebra of $L(V)$. From the ring-theoretic point of view, the underlying vector space V is a (faithful, left) module over $\mathcal{B}$. It is clear from the definitions that simplicity and semisimplicity are invariant under similarity. Since we are restricting attention to finite-dimensional vector spaces, $\mathcal{B}$ is semisimple if and only if it is a direct sum of simple operator algebras.

It is easy to check that the full algebra $L(V)$ is simple. The proof of the sufficiency is completed by appealing to the known structure of von Neumann algebras as direct sums of factors.

For the converse, recall that the only finite-dimensional division algebra over the complex numbers is $\mathbb{C}$ itself. Suppose first that $\mathcal{B}$ is a simple operator algebra. The Wedderburn Structure Theorems then tell us that $\mathcal{B}$ is ring isomorphic to some full operator algebra $L(W)$; in fact, $\mathcal{B}$ is spatially isomorphic (i.e. similar) to $L(W) \oplus \mathbb{C}I_K$ for some auxiliary vector space K. The last algebra can be made into a von Neumann algebra by introducing an appropriate inner product on the underlying space $W \oplus K$. To complete the proof for semisimple, $\mathcal{B}$ apply the preceding construction to its direct summands, taking care to define the inner product to make its corresponding direct summands of the underlying space mutually orthogonal.

An N-tuple $a = (a_1, \cdots, a_N)$ of operators, acting on a common vector space, is called *semisimple* if the a_i belong to mutually commuting semisimple algebras.

A semisimple N-tuple $a = (a_1, \cdots, a_N)$ of nilpotents is called *simple* if only the trivial idempotents commute with all of them.

The following are immediate consequences of Proposition 2.6.1.

Corollary 2.6.2. *An N-tuple $a = (a_1, \cdots, a_N)$ of nilpotents is semisimple if and only if it is similar to a doubly commuting N-tuple.*

Proposition 2.6.3. *Theorems* 2.1, 2.2, 2.4.7*, and* 2.5.2 *remain valid when the assumption of double commutativity is replaced by semisimplicity.*

Our final generalization of Theorem 2.1 removes the hypothesis of nilpotence.

Theorem 2.6.4. *In order for a commutative operator algebra $\mathcal{A}$ to be reflexive, it is necessary that for each rank-two member c, there is an idempotent $q \in \mathcal{A}$ such that qc generates a one-dimensional ideal. If the underlying vector space is finite-dimensional and $\mathcal{A}$ has a set of generators belonging to mutually commuting semisimple algebras, then this condition is also sufficient.*

Proof. All properties mentioned in this theorem hold for a full direct sum of operator algebras if and only if they hold for each direct summand. Thus, we may as well assume that $\mathcal{A}$ contains only the trivial idempotents. In the latter situation, however, the theorem reduces to the Hilbert space free version of Theorem 2.1.

Remark 2.6.5. In Proposition 2.5.4, we showed that the tensor product of two nilpotent operators is always reflexive. As discussed in [50], the situation is much more complicated when the hypothesis of nilpotence is dropped. From the point of view of Theorem 2.6.4, the problem arises from the failure of idempotents in $\mathcal{A}(a \oplus b)$ corresponding to points in $\sigma(a \oplus b)$ to be simple tensor products of idempotents in $\mathcal{A}(a)$ and $\mathcal{A}(b)$.

3. Joint Spectra for N-tuples of Operators

This section and the following one deal with operators on infinite-dimensional separable Hilbert spaces.

3.1. Left and right spectra.

Before we state definitions, we would like to recall well-known facts. The first one can be found in [33] and the second one can be proved using similar arguments as in [40, Theorem 1.1] and [22, Lemma 2.3], where the equivalence was shown for one single operator.

Lemma 3.1.1. *If $T_1, \dots, T_N \in L(\mathcal{H})$ commute, then the following are equivalent:*

(1) *There exists $\delta > 0$ such that $\|T_1x\| + \dots + \|T_Nx\| \geq \delta\|x\|$ for all $x \in \mathcal{H}$.*

(2) *There exist $S_1, \dots, S_N \in L(\mathcal{H})$ Such that $S_1 T_1 + \dots + S_N T_N = I$.*

(3) *There is no sequence $\{x_n\} \subset \mathcal{H}$ with $\|x_n\| = 1$ such that $lim_{n\to\infty}\|T_ix_n\| = 0$ for $i = 1, \dots, N$.*

Lemma 3.1.2. *If $T_1, \dots, T_N \in L(\mathcal{H})$ commute, then the following are equivalent:*

(1) *There exists $\delta > 0$ such that $\|T_1x\| + \dots + \|T_Nx\| \geq \delta\|x\|$ for all x on the orthogonal complement of some finite-dimensional subspace.*

(2) *There exist $S_1, \dots, S_N \in L(\mathcal{H})$ such that $S_1 T_1 + \dots + S_N T_N - I$ is a projection on a finite-dimensional subspace.*

(3) *There exist $S_1, \dots, S_N$ such that $S_1 T_1 + \dots + S_N T_N - I$ is a compact operator.*

(4) *If P is a projection such that $T_1P, \dots, T_NP$ are compact, then P is finite-dimensional.*

(5) *There is no orthonormal sequence $\{x_n\}$ with $lim_{n\to\infty}\|T_ix_n = 0$ for $i = 1, \dots, N$.*

(6) *There is no sequence $\{x_n\}$ with $x_n \to 0$ weakly and $\|x_n\| = 1$ such that $lim_{n\to\infty}\|T_ix_n\| = 0$ for $i = 1, \dots, N$.*

For commuting operators $T_1, \dots, T_N$ on $\mathcal{H}$, we will call $T_1, \dots, T_N$ *joint left invertible* if there exist $S_1, \dots, S_N$ such that $S_1 T_1 + \cdots + S_N T_N = I$. Recall that $\lambda = (\lambda_1, \dots, \lambda_N)$ belongs to the *joint left spectrum* $\sigma_l(T_1, \dots, T_N)$ (sometimes called *joint approximate point spectrum*) if and only if $\lambda - T = (\lambda_1 - T_1, \dots, \lambda_N - T_N)$ is not left invertible. The negations of the conditions in Lemma 3.1.1 above give equivalent conditions for the left spectrum. If $N = 1$ then we obtain classical *approximate point spectrum* denoted by $\sigma_{ap}(T)$.

Denote by $C(\mathcal{H})$ the Calkin algebra and by π the quotient map $\pi : L(\mathcal{H}) \to C(\mathcal{H})$. Recall that the *joint left essential spectrum* $\sigma_{le}(T_1, \dots, T_N)$ of $T_1, \dots, T_N$ is defined as the joint left spectrum of $\pi(T_1), \dots, \pi(T_N)$. The negations of the conditions in Lemma 3.1.2 above give equivalent conditions for left essential spectrum. The most common definition is that $\lambda = (\lambda_1, \dots, \lambda_N) \in \sigma_{le}(T_1, \dots T_N)$, if and only if there exists an orthonormal sequence $\{x_n\}$ such that $\lim n \to \infty \|(T_i - \lambda_i)x_n\| = 0$, for $i = 1, \dots, N$.

We will call $T_1, \dots, T_N$ *joint right invertible* if there exist $S_1, \dots, S_N$ such that $T_1 S_1 + \ldots + T_N S_N = I$. Recall that $\lambda = (\lambda_1, \dots, \lambda_N) \in \sigma_r(T_1, \dots, T_N)$ if and only if $\lambda - T = (\lambda_1 - T_1, \dots, \lambda_N - T_N)$ is not right invertible. Similarly as above, the *joint right essential spectrum* $\sigma_{re}(T_1, \dots, T_N)$ of $T_1, \dots, T_N$ can be defined as the joint right spectrum of $\pi(T_1), \dots, \pi(T_N)$. Recall the well known equalities $\sigma_r(T_1, \dots, T_N) = \overline{\sigma_l(T_1^*, \dots, T_N^*)}$ and $\sigma_{re}(T_1, \dots, T_N) = \overline{\sigma_{le}(T_1^*, \dots, T_N^*)}$. The union $\sigma_l(T_1, \dots, T_N) \cup \sigma_r(T_1, \dots, T_N)$ is called Harte spectrum and denoted by $\sigma_H(T_1, \dots, T_N)$.

Let us prove the following result which we will need later.

Proposition 3.1.3. *If $T_1 \in L(\mathcal{H}_1)$, $T_2 \in L(\mathcal{H}_2)$, then*

$$\sigma_{le}(T_1) \times \sigma_{ap}(T_2) \cup \sigma_{ap}(T_1) \times \sigma_{le}(T_2) = \sigma_{le}(T_1 \oplus I_{\mathcal{H}_2}, I_{\mathcal{H}_1} \oplus T_2).$$

Proof. We begin the proof with the '$\subset$' inclusion. By symmetry, it is enough to show that if $\lambda_1 \in \sigma_{le}(T_1)$ and $\lambda_2 \in \sigma_{ap}(T_2)$ then $(\lambda_1, \lambda_2) \in \sigma_{le}(T_1 \oplus I_{\mathcal{H}_2}, I_{\mathcal{H}_1} \oplus T_2)$. But for λ_1, λ_2 as above, there exist orthonormal sequences $\{x_n^1\}$ and $\{x_n^2\}$ of unit vectors with $\lim_{n\to\infty} \|(T_i - \lambda_i)x_n^i\| = 0$ for $i = 1, 2$. Hence, the sequence $\{x_n^1 \oplus x_n^2\}$ is orthonormal and

$$\begin{aligned} &\|(T_1 \oplus I_{\mathcal{H}_2} - \lambda_1)\,(x_n^1 \oplus x_n^2)\| = \|((T_1 - \lambda_1) \oplus I_{\mathcal{H}_2})(x_n^1 \oplus x_n^2)\| \\ &\quad = \|(T_1 - \lambda_1)x_n^1 \oplus x_n^2\| = \|(T_1 - \lambda_1)x_n^1\| \to 0 \qquad (n \to \infty). \end{aligned}$$

In the same way we can prove that

$$\|(I \oplus T_2 - \lambda_2)\,(x_n^1 \oplus x_n^2)\| \to 0 \qquad (n \to \infty).$$

Thus $(\lambda_1, \lambda_2) \in \sigma_{le}(T_1 \oplus I_{\mathcal{H}_2}, I_{\mathcal{H}_1} \oplus T_2)$.

To prove '$\supset$', let us assume that (λ_1, λ_2) is not in the set on the left hand side. If $\lambda_1 \notin \sigma_{ap}(T_1)$, then there is $S_1 \in L(\mathcal{H})$ such that $S_1(\lambda_1 - T_1) - I = 0$. Thus $(S_1 \oplus I)(\lambda_1 - T_1 \oplus I) + 0(\lambda_2 - I \oplus T_2) - I \oplus I = 0$ is the projection on the subspace $\{0\}$. Hence, by Lemma 3.1.2, $(\lambda_1, \lambda_2) \notin \sigma_{le}(T_1 \oplus I_{\mathcal{H}_2}, I_{\mathcal{H}_1} \oplus T_2)$. If $\lambda_2 \notin \sigma_{ap}(T_2)$, we use the same argument. To finish the proof, let us assume that $\lambda_i \notin \sigma_{le}(T_i),\ i = 1, 2$. Then, by Lemma 2.3 of [40], there are $S_i \in L(\mathcal{H}_i),\ i = 1, 2$ such that $S_i(\lambda_i - T_i) - I_{\mathcal{H}_i} = P_i, i = 1, 2$, where P_i are projections on finite-dimensional subspaces of $\mathcal{H}_i,\ i = 1, 2$. Thus we have

$$\begin{aligned} &(S_1(\lambda_1 - T_1)) \oplus I_{\mathcal{H}_2} - I_{\mathcal{H}_1} \oplus I_{\mathcal{H}_2} = P_1 \oplus I_{\mathcal{H}_2} \\ \text{and} \quad &-P_1 \oplus (S_2(\lambda_2 - T_2)) + P_1 \oplus I_{\mathcal{H}_2} = -P_1 \oplus P_2. \end{aligned}$$

Hence

$$\begin{aligned} &(S_1 \oplus I_{\mathcal{H}_2})(\lambda_1 - T_1 \oplus I_{\mathcal{H}_2}) - I_{\mathcal{H}_1} \oplus I_{\mathcal{H}_2} = P_1 \oplus I_{\mathcal{H}_2} \\ \text{and} \quad &(-P_1 \oplus S_2)(\lambda_2 - I_{\mathcal{H}_1} \oplus T_2)) + P_1 \oplus I_{\mathcal{H}_2} = -P_1 \oplus P_2. \end{aligned}$$

Thus

$$(S_1 \oplus I_{\mathcal{H}_2})(\lambda_1 - T_1 \oplus I_{\mathcal{H}_2}) + (-P_1 \oplus S_2)(\lambda_2 - I_{\mathcal{H}_1} \oplus T_2)) - I_{\mathcal{H}_1} \oplus I_{\mathcal{H}_2} = -P_1 \oplus P_2.$$

But $-P_1 \oplus P_2$ is compact, since P_1, P_2 are finite-dimensional projections. So Lemma 3.1.2 implies that $(\lambda_1, \lambda_2) \notin \sigma_{le}(T_1 \oplus I_{\mathcal{H}_2}, I_{\mathcal{H}_1} \oplus T_2)$.

3.2. Taylor spectrum.

Recall from [72]–[73] that the Koszul (cochain) complex $K(T, \mathcal{H})$ for an N-tuple $T = (T_1, \cdots, T_N)$ of commuting operators in $L(\mathcal{H})$ with respect to $\mathcal{H}$ is given by

$$0 \to \Lambda^0(\mathcal{H}) \xrightarrow{\delta^0(T)} \Lambda^1(\mathcal{H}) \xrightarrow{\delta^1(T)} \cdots \dashrightarrow \xrightarrow{\delta^{N-1}(T)} \Lambda^N(\mathcal{H}) \to 0,$$

where $\Lambda^p(\mathcal{H})$ denotes the set of all p-forms with coefficients in $\mathcal{H}$ and the cochain mapping $\delta^p(T) : \Lambda^p(\mathcal{H}) \to \Lambda^{p+1}(\mathcal{H})$ is defined by

$$\delta^p(T) \sum_{|I|=p}{}' x_I s_I := \sum_{j=1}^{N} \sum_{|I|=p}{}' T_j x_I s_j \wedge s_I,$$

Where $\{s_1, \cdots, s_N\}$ is a fixed basis of $\Lambda^1(\mathbb{C})$ and $\sum_{|I|=p}{}'$ denotes that the sum is taken over all $I = (i_1, \cdots, i_p) \in \mathbb{N}^p$ with $1 \leq i_1 < \cdots < i_p \leq N,\ s_I :=$

$s_{i_1} \wedge \cdots \wedge s_{i_p}$. Let us notice that $\Lambda^p(\mathcal{H})$ can be endowed with the natural scalar product

$$\left({\sum_{|I|=p}}' \, x_I s_I, {\sum_{|I|=p}}' \, y_I s_I \right) := {\sum_{|I|=p}}' \, (x_I, y_I)$$

which gives us a canonical isomorphism with a direct sum of $\binom{N}{p}$ copies of $\mathcal{H}$. Following [72]–[73], λ belongs to the Taylor spectrum $\sigma(T) \subset \mathbb{C}^N$ if, by definition, the complex $K(\lambda - T, \mathcal{H})$ is not exact, and λ belongs to the Taylor essential spectrum $\sigma_e(T) \subset \mathbb{C}^N$ if, by definition, at least one of the cohomology groups $H^p(\lambda - T) := ker\, \delta^p(\lambda - T)/ran\, \delta^{p-1}(\lambda - T)$ has infinite-dimension.

It is known that $\sigma_l(T_1, \cdots, T_N) \cup \sigma_r(T_1, \cdots, T_N) \subset \sigma(T_1, \cdots, T_N)$, but these two sets are not always equal (see [32]). The same holds for the essential spectra: $\sigma_{le}(T_1, \cdots, T_N) \cup \sigma_{re}(T_1, \cdots T_N) \subset \sigma_e(T_1, \cdots, T_N)$. If a single operator T is considered, then $\sigma_{le}(T) \cup \sigma_{re}(T) = \sigma_e(T)$, the essential spectrum of T.

Following [1], we can decompose $\sigma_e(T) = \bigcup_{p=0}^{N} \sigma_e^p(T)$, where $\sigma_e^p(T)$ is the set of all $\lambda \in \mathbb{C}^N$ such that the induced mapping

$$\hat{\delta}^p(\lambda - T) : \Lambda^p(\mathcal{H})/\overline{ran\delta^{p-1}(\lambda - T)} \to \Lambda^{p+1}(\mathcal{H})$$

has non-closed range or infinite-dimensional kernel.

The points of $\sigma_e^p(T)$ have the following property:

Lemma 3.2.1. *Let $T = (T_1, \cdots, T_N)$ be an N-tuple of commuting operators. If λ is a point in $\sigma_e^p(T)$, then there exists an orthonormal sequence $\{\eta_n\}_{n=1}^{\infty}$ in $\Lambda^p(\mathcal{H})$ such that*

$$\delta^{p-1}(\lambda - T)\delta^{p-1}(\lambda - T)^* \eta_n + \delta^p(\lambda - T)^* \delta^p(\lambda - T)\eta_n \to 0 \tag{3.2.1}$$

for $n \to \infty$.

Proof. By the definition of $\sigma_e^p(T)$ and a standard argument, we can find an orthonormal sequence $\{\eta_n\}_{n=1}^{\infty}$ in $\Lambda^p(\mathcal{H}) \ominus$ ran $\delta^{p-1}(\lambda - T)$ such that $\delta^p(\lambda - T)\eta_n \to 0$. Obviously, this sequence satisfies (3.2.1).

We will need the following fact (see [31, Corollary 3.7]).

Lemma 3.2.2. *Let $T = (T_1, \cdots, T_N)$ be an N-tuple of almost doubly commuting operators. Then there is a compact operator K on $\Lambda^p(\mathcal{H})$ such that*

$$\delta^{p-1}(T)\delta^{p-1}(T)^* + \delta^p(T)^* \delta^p(T) = K + \bigoplus_F \sum_{j=1}^{N} T_F(j),$$

where the orthogonal sum runs over all functions $F : \{1, \cdots, N\} \rightarrow \{0, 1\}$, *card* $\{j : F(j) = 0\} = p$ *and* $T_F(j) = T_j T_j^*$ *for* $F(j) = 0$ *and* $T_F(j) = T_j^* T_j$ *for* $F(j) = 1$. *Moreover, if the* N*-tuple doubly commutes, then* $K = 0$.

4. Functional Calculus for N-tuples of Contractions

4.1. H$^\infty$-type algebras as dual algebras.

In what follows, $C(X)$ will stand for the algebra of all complex continuous functions on a compact set X, and $\mathcal{M}(X)$ for the set of all complex Borel measures on X. Recall that $\mathcal{M}(X)$ is a Banach space with the total variation norm. If E is a subset of $\mathcal{M}(X)$ then E^s will denote the set of all measures on X singular to every measure in E. A subset $\mathcal{B}$ of $\mathcal{M}(X)$ is a *band* (see [10]–[11]) if $\mathcal{B}^{ss} = \mathcal{B}$.

Let us note that any band is a closed subspace of $\mathcal{M}(X)$. It is also almost elementary that E^s is a band for every $E \subset \mathcal{M}(X)$ and $E^{ss} \supset E$.

It is easy to see that E^{ss} is the smallest band containing E, which we call the band generated by E. For further details on bands, we refer to [29, Section 20]. If A is a function algebra on X and $x \in X$, then $\nu \in \mathcal{M}(X)$ is called a *representing measure* of x if

$$\int u \, d\nu = u(x) \qquad for \ u \in A.$$

Let $K_1, \cdots, K_N$ be compact subsets of the complex plane $\mathbb{C}$ and $K = K_1 \times \cdots \times K_N$. Denote by $R(K)$ the uniform closure in $C(K)$ of the algebra of all rational functions with singularities off K, and by G_i $(i = 1, \cdots, N)$ the set of all non-peak points of $R(K_i)$. Let $G = G_1 \times \cdots \times G_N$. For a set $E \subset \mathbb{C}^N$, ∂E will stand for its topological boundary. We have

Lemma 4.1.1. *int* $G =$ *int* K.

Proof. If $z = (z_1, \cdots, z_N) \notin G$ then there exists i such that z_i is a peak-point of $R(K_i)$, and hence $z_i \in \partial K_i$. So $z \in \partial K$. Consequently, $G \supset$ *int* K. On the other hand, we also have $G \subset K$, which implies int $G \subset$ *int* K.

If μ is a positive measure on K, then by $H^\infty(\mu)$ we mean the weak-star closure of $R(K)$ in $L^\infty(\mu)$. $\|\cdot\|_\infty$ or $\|\cdot\|_{H^\infty(\mu)}$ will always denote the supnorm or the essential supnorm in function algebras. Denote by $\mathcal{M}_G$ the band of measures generated by all representing measures of points in G, by $B^\infty(G)$ the weak-star closure of $R(K)$ in the dual of $\mathcal{M}_G$ and by $\|\cdot\|_{B^\infty}$ the dual norm in $B^\infty(G)$. Many details on $B^\infty(G)$-type algebras can be found in [29]. We will denote by ϑ the volume measure on $\mathbb{C}^N$ restricted to G.

For any open set $\Omega \subset \mathbb{C}^N$, we will denote by $H^\infty(\Omega)$ the algebra of all bounded analytic functions on Ω.

We will consider three algebras: $H^\infty(\vartheta)$, $B^\infty(G)$ and $H^\infty(int\ K)$.

Observe that for $f \in B^\infty(G)$ and $z \in G$ we can define $f(z)$ as the value of f on a representing measure ν_z of z. By the weak-star density of $R(K)$ in $B^\infty(G)$, this value does not depend on the choice of the representing measure. So the elements of $B^\infty(G)$ can be regarded as functions on G and, by Lemma 4.1.1, also as functions on int K.

The first result we are interested in is

Proposition 4.1.2. *If $f \in B^\infty(G)$, then f is a bounded analytic function on int K. Thus $B^\infty(G) \subset H^\infty(int\ K)$.*

Proof. Let us consider an arbitrary point $z_0 \in intK$ and a small open polydisc Δ centerd at z_0 and included in $intK$. Denote by m the normalized Lebesgue measure on the Shilov boundary of Δ, and by P_z and C_z, respectively, the N-dimensional Poisson and Cauchy kernels for z. Then m is a representing measure for z_0 (with respect to the algebra $R(K)$). The measure $C_z\, dm$ is absolutely continuous with respect to m, and consequently is in $\mathcal{M}_G$.

On the other hand, every $u \in R(K)$ is analytic on Δ, so $u(z) = \int u\, C_z\, dm$ for $u \in R(K),\ z \in \Delta$, and by the weak-star density of $R(K)$ in $B^\infty(G)$, also $h(z) = \int h\, C_z\, dm$ for $h \in B^\infty(G),\ z \in \Delta$. Hence, by Cauchy theorem, h is analytic near z_0. Moreover, f is bounded since

$$\begin{aligned}
&\sup\{|f(z)| : z \in G\} \\
&= \sup\{| < f, \nu_z > | : z \in G,\ \nu_z \textit{ is a representing measure for } z\} \\
&\le \sup\{| < f, \nu > | : \nu \in \mathcal{M}_G\} = \|f\|_{B^\infty(G)}.
\end{aligned}$$

Lemma 4.1.3. *Let Ω be a bounded connected open set with the boundary of C^2 class. The space $H^\infty(\Omega)$ is isometrically embedded in $H^\infty(\vartheta)$.*

Proof. For any $f \in H^\infty(\Omega)$, by [48, Proposition 8.5.1] there is a sequence $\{f_n\} \subset R(\Omega)$ converging to f a.e. $[\vartheta]$ and $\|f_n\|_{H^\infty(\Omega)}$. Thus $\{f_n\} \subset R(\Omega) \subset H^\infty(\vartheta)$ converging weak-star to f in $H^\infty(\vartheta)$. The equality of the norms $\|f\|_{H^\infty(\Omega)} = \|f\|_{H^\infty(\vartheta)}$ is the consequence of the equality of the norms $\|f_n\|_{H^\infty(\Omega)} = \|f_n\|_{H^\infty(\vartheta)}$.

Now we state two lemmas which contain some results of Bekken: Corollaries 5.6 and 5.7 of [10]. The two-dimensional case, and their generalizations to an arbitrary N can be found in the last paragraph of [11].

Lemma 4.1.4. *If $f \in H^\infty(\vartheta)$, then there is a sequence $\{f_n\} \subset R(K)$ such that $\|f_n\|_{H^\infty(\vartheta)} \leq \|f\|_{H^\infty(\vartheta)}$ and $f_n \to f$ $[\vartheta]$ a.e.*

Lemma 4.1.5. *If μ is a measure in $\mathcal{M}_G$, then the inclusion map $H^\infty(\vartheta + |\mu|) \to H^\infty(\vartheta)$ is an isometric isomorphism.*

From the above two inclusions, we would like to construct identity isomorphism between $B^\infty(G)$ and $H^\infty(\vartheta)$.

The space $L^1(\vartheta)$ is a closed subspace of $\mathcal{M}_G$, so we can define a mapping Ψ: $B^\infty(G) \to H^\infty(\vartheta)$ such that for every $f \in B^\infty(G)$, $\Psi(f)$ is a functional on $L^1(\vartheta)$ defined as $< \Psi(f), g >=< f, gd\vartheta >$. Note that Ψ on $R(K)$ is the identity. As a consequence of Lemmas 4.1.4 and 4.1.5, we get

Proposition 4.1.6.

(1) *The mapping $\Psi : B^\infty(G) \to H^\infty(\vartheta)$ gives an isometric isomorphism between $B^\infty(G)$ and $H^\infty(\vartheta)$.*

(2) *This isomorphism is also a homeomorphism in the weak-star topologies.*

(3) *If $f \in B^\infty(G)$, then there is a sequence $\{f_n\} \subset R(K)$ such that $\|f_n\|_{H^\infty(\vartheta)} \leq \|f\|_{B^\infty(G)}$ and $f_n \to f$ weak-star in $(\mathcal{M}_G)^*$.*

Proof. By Lemma 4.1.5, the mapping Ψ is an isometry, and its weak-star continuity follows immediately from the definition.

To prove (3), let us take an arbitrary $f \in B^\infty(G)$. Then, by Lemma 4.1.4, there is a sequence $\{f_n\} \subset R(K)$ such that $\|f_n\|_{H^\infty(\vartheta)} \leq \|f\|_{H^\infty(\vartheta)} = \|f\|_{B^\infty(G)}$ and $f_n \to f[\vartheta]$ a.e. By Banach-Alaoglu theorem, the sequence $\{f_n\}$ has an adherent point $g \in B^\infty(G)$. Hence there is a subnet $\{f_{n_\alpha}\}$ converging to g. Thus, by the weak-star continuity of Ψ, a subnet of $\{\Psi(f_{n_\alpha})\}$ converges weak-star to $\Psi(g)$. Since $H^\infty(\vartheta) \subset L^\infty(\vartheta)$ and this space satisfies the sequence condition thus there is a sequence f_{n_k} such that $\{\Psi(f_{n_k})\}$ converges weak-star to $\Psi(g)$. Hence $\Psi(f) = \Psi(g)$. Since Ψ is an isometry, we obtain $f = g$ and (3) is proved.

By a similal argument as above, we show that Ψ is onto, and so we get (1). The statement (2) is a consequence of the following lemma (see [23], Theorem 2.7).

Lemma 4.1.7. *Let X and Y be Banach spaces and let Ψ be a continuous (in the weak-star topologies) linear map from X^* into Y^* with trivial kernel and norm closed range. Then $\Psi(X^*)$ is weak-star closed and Ψ is a weak-star homeomorphism of X^* onto $\Psi(X^*)$.*

The following remark is important for the remaining sections of the paper.

Remark 4.1.8. Lemma 4.1.3 shows that $H^\infty(\mathbb{D}^N)$ can be isometrically embedded in $H^\infty(\vartheta)$. Proposition 4.1.2 implies the inclusion $B^\infty(\mathbb{D}^N) \subset H^\infty(\mathbb{D}^N)$. Hence, Proposition 4.1.6 allows us to identify the algebra $B^\infty(\mathbb{D}^N)$ with $H^\infty(\mathbb{D}^N)$ and consider $H^\infty(\mathbb{D}^N)$ as a dual algebra.

After the identification in the above remark, the algebra $H^\infty(\mathbb{D}^N)$ will be the most interesting from our point of view and the supremum norm will be denoted by $\|\cdot\|_\infty$.

4.2. Representation of $\mathbf{A}(\mathbb{D}^N)$.

Recall that the algebra homomorphism $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ is a *representation* if $\|\Phi(f)\| \le \|f\|$ for $f \in A(\mathbb{D}^N)$. Now, let us consider a pair of commuting contractions T_1, T_2. For any polynomial p of two variables, we define the operator $p(T_1, T_2)$ in the natural way:

$$\Phi : \longmapsto p(T_1, T_2). \tag{4.2.1}$$

By Ando's Theorem ([71], Theorem I.6.4) the pair T_1, T_2 has a unitary dilation. More precisely, there is a space $\mathcal{K} \supset \mathcal{H}$ and a pair of unitary operators $U_1, U_2 \in L(\mathcal{K})$ such that

$$T_1^n\, T_2^m\; x = P_{\mathcal{H}}\; U_1^n\, U_2^m\; x \qquad for \quad x \in \mathcal{H},\; n, m = 0, 1, 2, \ldots$$

The pair U_1, U_2 has a spectral measure E on the two-dimensional torus $\mathbb{T}^2$ (see [12] for a product of spectral measures) such that

$$U_1^n\, U_2^m = \int_{\mathbb{T}^2} z_1^n\, z_2^m\, dE(z_1, z_2) \quad for \quad n, m = 0, 1, 2, \ldots .$$

Hence, for any polynomial p of two-variables, and $x, y \in \mathcal{H}$ we have

$$\|(p(T_1, T_2)x, y)| = |(p(U_1, U_2)x, y)| = |\int_{\mathbb{T}^2} p(z_1, z_2) dE(z_1, z_2)| \le \|p\|_\infty\, \|x\|\, \|y\|.$$

Thus we have the von Neumann inequality $\|p(T_1, T_2)\| \le \|p\|_\infty$. Since the polynomials are dense in the bidisc algebra $A(\mathbb{D}^2)$, by the von Neumann inequality, we can extend Φ to $A(\mathbb{D}^2)$. Then we also have

$$\|h(T_1, T_2)\| \le \|h\|_\infty \quad for \quad h \in A(\mathbb{D}^2). \tag{4.2.2}$$

The multiplication property $\Phi(uv) = \Phi(u)\Phi(v)$ is easy to see for two polynomials, and by (4.2.2) it is also fulfilled for any two functions in $A(\mathbb{D}^2)$. Hence Φ is a representation. (Since the polynomials are weak-star dense in

$H^\infty(\mathbb{D}^2)$, the multiplication property can also be proved directly for $H^\infty(\mathbb{D}^2)$, as in Proposition 4.3.3 below.)

Now, let us consider an N-tuple $T = (T_1, \dots, T_N) \subset L(\mathcal{H})$ of doubly commuting (i.e. $T_iT_j = T_jT_i$, $T_iT_j^* = T_j^*T_i$ for $i \neq j$) contractions. It has unitary dilations by [71]. Thus, the representation $\Phi : H^\infty(\mathbb{D}^N) \to L(\mathcal{H})$ can be constructed as above, and (4.2.2) is also fulfilled for a doubly commuting N-tuple $T = (T_1, \dots, T_N)$.

Let $T = (T_1, \dots, T_N) \subset L(\mathcal{H})^N$ be a commuting N-tuple and $K \subset \mathbb{C}^N$. A compact set K is called a (common) *spectral set* for $T = (T_1, \dots, T_N)$ if for every rational function u of N variables with singularities off K there exists a naturally defined operator $u(T_1, \cdots, T_N)$ such that

$$\|u(T_1, \dots, T_N)\| \leq \sup\{|u(\lambda_1, \dots, \lambda_n)| : (\lambda_1, \dots, \lambda_N) \in K\}.$$

Let us assume that $\mathbb{D}^N$ is a spectral set for a given N-tuple of c.n.u. (completely non-unitary) contractions $T = (T_1, \dots, T_N)$. It is known that $R(\mathbb{D}^N)$ is equal to the polydisc algebra $A(\mathbb{D}^N)$. We define a representation

$$\Phi : A(\mathbb{D}^N) \ni u \longmapsto u(T_1, \dots, T_N) \in L(\mathcal{H}).$$

Now we will work on extending the above representations.

4.3. Extension to H^∞-type algebras.

Having the representation $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$, it is a consequence of standard techniques that for every $x, y \in \mathcal{H}$ there exists a complex, Borel, regular measure $\mu_{x,y}$ on $\overline{\mathbb{D}}^N$ such that $(\Phi(u)x, y) = \int u \, d\mu_{x,y}$ for $u \in A(\mathbb{D}^N)$. We say that ϕ is *absolutely continuous* (*a.c.*) if it has a system of *elementary measure* $\{\mu_{x,y}\}_{x,y \in \mathcal{H}}$ such that each element of the system is absolytely continuous with respect to some (positive) representing measure ν_a, $z \in \mathbb{D}^N$. By [41, VI.1.2, II.7.5], the above definition is equivalent to that one, which uses the terminology of bands of measures. Namely, we can say that Φ is absolutely continuous if it has a system of elementary measures $\{\mu_{x,y} : x, y \in \mathcal{H}\}$ which belong to $\mathcal{M}_G$. We will say that the N-tuple $T = (T_1, \cdots, T_N)$ is *absolutely continuous* (*a.c.*) if the representation generated by T (assuming that it exists) is a.c.

Having the representation constructed above $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ and using [44, Proposition] (see also [44, Sec. 2], [45, Sec. 3]), we can decompose Φ and $\mathcal{H}$ into orthogonal sums as follows

$$\Phi = \bigoplus_{i=1}^{M} \Phi_i, \qquad \mathcal{H} = \bigoplus_{i=1}^{M} \mathcal{H}_i, \tag{4.3.1}$$

where $\Phi_i \, (i = 0, \cdots, M)$ is the restriction of Φ to the subspace $\mathcal{H}_i$ which reduces all the values of Φ. Moreover, we get

(1) Φ_0 is an absolutely continuous representation,

(2) $T_i|_{\mathcal{H}_i}$ $(i = 1, \cdots, N)$ is a unitary operator with singular spectral measure,

(3) $T_{k_i}|_{\mathcal{H}_i}$, $T_{l_i}|\mathcal{H}_i$ $(i = N, \ldots, M)$ are unitary operators for some k_i, $l_i (k_i \neq l_i)$.

So we get

Lemma 4.3.1. *If T_1 is c.n.u. and T_2 is a.c., then the pair $\{T_1, T_2\}$ is a.c.*

Lemma 4.3.2. *If the N-tuple $T = (T_1, \ldots, T_N)$ is c.n.u., then it is a.c.*

Now we show an extension property of absolutely continuous representations.

Proposition 4.3.3. *If $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ is an absolutely continuous representation, then*

(1) *Φ can be extended to a homomorphism, denoted also by Φ, of the algebra $H^\infty(\mathbb{D}^N)$ into $L(\mathcal{H})$ such that*

$$\|\Phi(h)\| \leq \|h\|_\infty \qquad \textit{for} \quad h \in H^\infty(\mathbb{D}^N), \textit{ and} \tag{4.3.2}$$

(2) *Φ is continuous with respect to the weak-star topologies.*

Proof. (1) By Section 4.1, the algebra $H^\infty(\mathbb{D}^2)$ can be identified with $B^\infty(\mathbb{D}^N)$, the closure of polynomials in $(\mathcal{M}_{\mathbb{D}^N})^*$. Thus, for every $h \in H^\infty(\mathbb{D}^N)$ and $x, y \in \mathcal{H}$, we choose an elementary measure $\mu_{x,y} \in \mathcal{M}_{\mathbb{D}^N}$ and put

$$(\Phi(h)x, y) := \langle h, \, \mu_{x,y} \rangle \,,$$

where $\langle h, \, \mu_{x,y} \rangle$ denotes the value of the functional h on $\mu_{x,y}$. We can ask whether Φ is well-defined. If we take another elementary measure $\mu'_{x,y}$, then for $\mu \in A(\mathbb{D}^N)$ we have

$$\langle u, \mu_{x,y} - \mu'_{x,y} \rangle = \langle u, \mu_{x,y} \rangle - \langle u, \mu'_{x,y} \rangle = (\Phi(u)x, y) - (\Phi(u)x, y) = 0.$$

Let $\{u_k\}$ be a net in $A(\mathbb{D}^N)$ converging to $h \in H^\infty(\mathbb{D}^2)$ in the weak-star topology. Then

$$\langle h, \mu_{x,y} - \mu'_{x,y} \rangle = \lim_k \langle u_k, \mu_{x,y} - \mu'_{x,y} \rangle = 0 \,.$$

Hence the extension Φ is well-defined and linear, and we have

$$|(\Phi(h)x, y)| = |\langle h, \mu_{x,y} \rangle| \leq \|h\|_\infty \|\mu_{x,y}\| \leq \|h\|_\infty \|x\| \, \|y\| \,,$$

which gives the inequality

$$\|\Phi(h)\| \le \|h\|_\infty \qquad for\ h \in H^\infty(\mathbb{D}^N)\,.$$

We show first the multiplicativity of Φ for $u \in A(\mathbb{D}^N),\ h \in H^\infty(\mathbb{D}^N)$. Take a net $\{v_k\} \subset A(\mathbb{D}^N)$ which converges weak-star in $(\mathcal{M}_{\mathbb{D}^N})^*$ to h. Then

$$\begin{aligned}(\Phi(uh)x, y) &= \langle uh, \mu_{x,y}\rangle = \langle h, u\mu_{x,y}\rangle = \lim\langle v_k, u\mu_{x,y}\rangle \\ &= \lim\langle v_k u, \mu_{x,y}\rangle = \lim(\Phi(v_k u)x, y) = \lim(\Phi(v_k)\Phi(u)x, y) \\ &= \lim\langle v_k, \mu_{\Phi(u)x,y}\rangle = \langle h, \mu_{\Phi(u)x,y}\rangle = (\Phi(h)\Phi(u)x, y).\end{aligned}$$

We get the general case by repeating the procedure for $h, g \in H^\infty(\mathbb{D}^2)$.

(2) since $L(\mathcal{H})$ is the dual of $\mathcal{C}_1(\mathcal{H})$, it is enough to check whether the functional $H^\infty(\mathbb{D}^2) \ni h \longmapsto tr(\Phi(h)C)$ is weak-star continuous for every $C \in \mathcal{C}_1(\mathcal{H})$. Let $\{\lambda_i\}_{i=1}^\infty$ be the set of singular numbers of C, and $\{y_i\}_{i=1}^\infty$ be the corresponding orthonormal basis in $\mathcal{H}$ (see VI.17 of [61]). Then, there is a collection of elementary measures $\{\mu_{Cy_i,y_i}\} \subset \mathcal{M}_{\mathbb{D}^N}$ such that

$$\begin{aligned} tr(\Phi(h)C) &= \sum_{i=1}^\infty (\Phi(h)Cy_i, y_i) = \sum_{i=1}^\infty \int h\, d\mu_{Cy_i,y_i}\,, \\ and \qquad & \|\mu_{Cy_i,y_i}\| \le \|Cy_i\|\,\|y_i\| = \|Cy_i\|\,. \end{aligned}$$

Since $C \in \mathcal{C}_1(\mathcal{H})$, by VI.17 and VI.20 of [41] we have

$$\sum_{i=1}^\infty \|\mu_{Cy_i,y_i}\| \le \sum_{i=1}^\infty \|Cy_i\| = \sum_{i=1}^\infty |\lambda_i| < \infty\,.$$

Hence $\mu := \sum_{i=1}^\infty \mu_{Cy_i,y_i}$ is a measure of bounded variation and $\mu \in \mathcal{M}_{\mathbb{D}^N}$, since $\mu_{Cy_i,y_i} \in \mathcal{M}_{\mathbb{D}^N}$ for $i = 1, 2, \cdots$. It implies that the mapping

$$H^\infty(\mathbb{D}^2) \ni h \longmapsto \int h\, d\mu$$

is a weak-star continuous functional. By (4.3.3), we have $tr(\Phi(h)C) = \int h\, d\mu$ which completes the proof.

Let us sum up our functional calculus results.

Theorem 4.3.4. *Assume that $T = (T_1, \ldots, T_N) \subset L(\mathcal{H})$ is an absolutely continuous N-tuple of commuting contractions and that*

(i) *$T = (T_1, T_2)$ is a pair of contractions ($N = 2$) or*
(ii) *$T = (T_1, \ldots, T_N)$ is a doubly commuting N-tuple or*

(iii) $\mathbb{D}^N$ *is a spectral set for* $T = (T_1, \ldots, T_N)$.

Then there is an algebra homomorphism $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{A}(T)$ *such that*

(1) $\Phi(1) = I$ *and* $\Phi(p_i) = T_i$ *for* $i = 1, \ldots, N$, *where* $p_i(z_1, \ldots, z_N) = z_i$,
(2) $\|\Phi(h)\| \le \|h\|_\infty$ *for all* $h \in H^\infty(\mathbb{D}^N)$,
(3) Φ *is weak-star continuous,*
(4) *the range of* Φ *is weak-star dense in* $\mathcal{A}(T)$, *and*
(5) *if* Φ *is an isometry, then it is a weak-star homeomorphism onto* $\mathcal{A}(T)$.

Proof. We only need to show (4) and (5). To see (4), it is enough to notice that polynomials in T are weak-star dense in $\mathcal{A}(T)$. The claim (5) is a consequence of [16, Theorem 2.7]

Having the N-tuple $T = (T_1, \ldots, T_N)$ of commuting contractions and the representation $\Phi : A(\mathbb{D}^N) \to \mathcal{A}(T)$ constructed in Section 4.2, we can define the adjoint representation Φ^* of $A(\mathbb{D}^N)$ generated by $T^* = (T_1^*, \ldots, T_N^*)$ Moreover, we have

Lemma 4.3.5.

(1) *If the representation* Φ *is a.c., then* Φ^* *is a.c. and we can extend* Φ^* *to* $H^\infty(\mathbb{D}^N)$,
(2) *If* $T = (T_1, \ldots, T_N)$ *is an a.c.* N*-tuple of commuting contractions and* $f \in H^\infty(\mathbb{D}^N)$, *then for any vectors* x, y *we have*

$(f(T_1, \ldots, T_N)y, x) = (y, f^\sim(T_1^*, \ldots, T_N^*)x)$, *where* $f^\sim(z) = \overline{f(\overline{z})}$ *for* $z \in \mathbb{D}^N$.

Proof. Let $u \in A(\mathbb{D}^N)$, and let $\mu_{y,x}$ be an elementary measure of Φ for $y, x \in \mathcal{H}$ absolutely continuous with respect to a representing measure ν_z for some $z \in \mathbb{D}^N$. It is obvious that $u^\sim \in A(\mathbb{D}^N)$.

For a complex measure μ on $\overline{\mathbb{D}}^N$, denote by $\mu^\sim$ the Borel measure $\mu^\sim(\cdot) = \overline{\mu(\Pi(\cdot))}$, where $\Pi : z \to \overline{z}$ is a homeomorphism of $\overline{\mathbb{D}}^N$ onto itself. Then, for $\mu \in A(\mathbb{D}^N)$ we have

$$(u(T_1^*, \ldots, T_N^*)x, y) = (x, u(T_1^*, \ldots, T_N^*)^* y) = (x, u^\sim(T_1, \ldots, T_N)y)$$

$$= \overline{(u^\sim(T_1, \ldots, T_N)y, x)} = \overline{\int u^\sim d\mu_{y,x}} = \int \overline{u^\sim} d\,\overline{\mu_{y,x}} = \int u\, d\mu_{y,x}^\sim .$$

So $\eta_{x,y} := \mu_{y,x}^\sim$ is an elementary measure of Φ^* for vectors $x, y \in \mathcal{H}$. Since $\mu_{y,x}$ is absolutely continuous with respect to ν_z the measure $\eta_{x,y}$ is absolutely continuous with respect to $\nu_z^\sim$, and an easy calculation shows that $\nu_z^\sim$ is a representing measure for $\overline{z} \in \mathbb{D}^N$, which finishes the proof of (1).

Next, to see (2), by (1) we can extend Φ^* to $H^\infty(\mathbb{D}^N)$ and can easily see that $f^\sim \in H^\infty(\mathbb{D}^N)$. Let, as in (1), $\mu_{y,x}$ and $\mu_{y,x}^\sim$ be elementary measures of Φ and Φ^*, respectively. Notice that $\sigma_z := \nu_z + \nu_z^\sim$ is positive and symmetric with respect to the adjoint. Let $d\mu_{y,x} = h_{y,x} d\sigma_z$. Then $d\mu_{y,x}^\sim = h_{y,x}^\sim d\sigma_z$. Hence, the following finishes the proof:

$$\begin{aligned}(f(T_1, \ldots, T_N)y, x) &= \int f\, d\mu_{y,x} = \int f(\lambda) h_{y,x}(\lambda) d\sigma_z(\lambda) \\ &= \int f(\overline{\lambda}) h_{y,x}(\overline{\lambda}) d\sigma_z(\overline{\lambda}) = \overline{\int f^\sim(\lambda) h_{y,x}^\sim(\lambda) d\sigma_z(\lambda)} \\ &= \overline{\int f^\sim d\mu_{y,x}^\sim} = \overline{(f^\sim(T_1^*, \ldots, T_N^*)x, y)} = (y, f^\sim(T_1^*, \ldots, T_N^*)x).\end{aligned}$$

Now we will try to find some conditions when the functional calculus is isometric. Recall that a set E contained in the closed unit polydisc $\overline{\mathbb{D}}^N$ is dominating for the algebra $H^\infty(\mathbb{D}^N)$ of all bounded analytic functions on $\mathbb{D}^N$ if for all $h \in H^\infty(\mathbb{D}^N)$ we have $\|h\|_\infty = \sup_{z \in E \cap \mathbb{D}^N} |h(z)|$.

We will assume the dominancy of some type of the spectra. In various situations we will assume that various spectra are dominating for the algebra $H^\infty(\mathbb{D}^N)$: Taylor spectrum $\sigma(T)$, Taylor essential spectrum $\sigma_e(T)$, left essential spectrum $\sigma_{le}(T)$, and right essential spectrum $\sigma_{re}(T)$.

The following known idea gives an isometric functional calculus in various situations.

Lemma 4.3.6. *Suppose that the assumptions of Theorem* 4.3.4 *are satisfied. Assume also that $E \subset \sigma(T)$ is dominating for $H^\infty(\mathbb{D}^N)$. If $f \in H^\infty(\mathbb{D}^N)$, then $|f\|_\infty \leq \|f(T)\|$.*

Proof. Let $\lambda \in E \cap \mathbb{D}^N \subset \sigma(T) \cap \mathbb{D}^N$. Then, by [65], $f(\lambda) \in \sigma(f(T))$ and $\|f\|_\infty \leq r(f(T)) \leq \|f(T)\|$, which completes the proof of the lemma.

5. Dual Algebras and its Application to Invariant Subspace Problem and Reflexivity

We shall also need the language of dual algebras. Recall that $L(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$, where $\mathcal{C}_1(\mathcal{H})$ is the ideal of trace-class operators and the duality is given by the form $< T, S >:= tr(TS)$ for $T \in L(\mathcal{H})$, $S \in \mathcal{C}_1(\mathcal{H})$. Hence every ultraweakly closed subalgebra, $\mathcal{A}$ of $L(\mathcal{H})$ is a dual Banach space with predual space $\mathcal{Q}_\mathcal{A} \cong \mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{A}$ via $< T, [S] >:= tr(TS)$ for $T \in \mathcal{A}$, $[S] \in$

$\mathcal{C}_1(\mathcal{H})/^{\perp}\mathcal{A}$. (Usually we will write $\mathcal{Q}$ instead of $\mathcal{Q}_{\mathcal{A}}$.) Thus, for a rank-one operator $x \otimes y(z \longmapsto (z,y)x)$, we have $< T, [x \otimes y] >= (Tx, y)$.

5.1. The role of rank-one operators.

We start by recalling a well-known result (see [16, Prop. 2.5]).

Proposition 5.1.1. *Let X, Y be Banach spaces. A linear mapping $\Phi : X^* \to Y^*$ is continuous in weak-star topology in both spaces if and only if there exists a map $\phi : Y \to X$ such that $< y, \Phi\alpha >=< \phi(y), \alpha >$ for all $y \in Y$ and $\alpha \in X^*$. Moreover $\phi = \Phi^*$.*

Let the assumptions of Theorem 4.3.4 be fulfilled and assume that our functional calculus $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{A}(T)$ is isometric. Since Φ is weak-star continuous and onto $\mathcal{A}(T)$, there is the mapping $\phi : \mathcal{C}_1(\mathcal{H})/^{\perp}\mathcal{A}(T) \to L^1(\mathbb{D}^N)/^{\perp}H^\infty(\mathbb{D}^N)$. Let P_λ be a reproducing kernel for a point $\lambda \in \mathbb{D}^N$. Then $P_\lambda \in L^1(\mathbb{D}^N)$ and $< h, P_\lambda >= h(\lambda)$ for $h \in H^\infty(\mathbb{D}^N)$. We will denote by $[C_\lambda] = \phi^{-1}([P_\lambda])$. Moreover, $< h(T), [C_\lambda] >=< h, \phi^{-1}([P_\lambda]) >= h(\lambda)$ for $\lambda \in \mathbb{D}^N$ and $h \in H^\infty(\mathbb{D}^N)$.

To show the role of rank-one operators we will restrict ourselves to a very simple situation. We will consider only a single operator instead of N-tuple and we will aim in showing the existence of non-trivial invariant subspaces instead of the reflexivity.

Let us assume that $\Phi : H^\infty \to \mathcal{A}(T)$ is an isometry and a weak-star homeomorphism. Thus we have a mapping $\phi : \mathcal{C}_1(\mathcal{H})/^{\perp}\mathcal{A}(T) \to L^1/^{\perp}H^\infty$. Assume also that for any $[L] \in \mathcal{Q}$ there are $x, y \in \mathcal{H}$ such that $[L] = [x \otimes y]$. We will show that this gives an invaliant subspace for T. Let us consider $[C_0] = \phi^{-1}([P_0])$ and assume that $[C_0] = [x \otimes y]$ for some $x, y \in \mathcal{H}$. Thus, for any $h \in \mathcal{H}^\infty$, we have

$$h(0) =< h, [P_0] >=< h(T), [C_0] >=< h(T), x \otimes y >= (h(T)x, y).$$

If $h(\lambda) \equiv 1$, then $1 = (x, y)$ and $x \neq 0,\ y \neq 0$. Now take $h(\lambda) = \lambda g(\lambda)$ for $g \in H^\infty$. Then $0 = (Tg(T)x, y)$, which means that $y \perp M := span\{T^n Tx, n \geq 0\}$. Thus $M \neq \mathcal{H}$ since $y \neq 0$, and if ker $T = \{0\}$, then $M \neq \{0\}$. If ker $T \neq \{0\}$, then ker T is a non-trivial invariant subspace or T is a zero operator. Hence ker T or M is a non-trivial invariant subspace.

5.2. Approximation in predual spaces.

In this section we state some sufficient approximation conditions to show the reflexivity. If $\Omega \subset \mathbb{C}^N$, then $\overline{aco}\Omega$ denotes the closure of the absolutely convex hull of Ω.

Lemma 5.2.1. *Assume that E is dominating for $H^\infty(\mathbb{D}^N)$. Then $\overline{aco}\{[C_\lambda]_{\mathcal{Q}}$: $\lambda \in E \cap \mathbb{D}^N\}$ contains the closed unit ball about the origin in $\mathcal{Q}$.*

Proof. If $f \in H^\infty(\mathbb{D}^N)$, then

$$\|\Phi(f)\| \leq \|f\|_\infty = \sup_{\lambda \in E} |f(\lambda)| = \sup_{\lambda \in E} \langle \Phi(f), [C_\lambda] \rangle.$$

Now the result follows from the next proposition (see [16, Proposition 2.8].

Proposition 5.2.2. *Let X be a complex Banach space, and let E be a subset of the closed unit ball $\mathcal{B}$ of X such that for all ϕ in X^*, $\|\phi\| = \sup_{x \in E} < x, \phi >$. Then the closure of the absolutely convex hull of E is the entire unit ball $\mathcal{B}$.*

We shall need the well-known fact from [17] that every dual algebra with property $X_{0,1}$ is reflexive. Recall that $\mathcal{A}$ has property $X_{0,1}$ if the unit ball of $\mathcal{Q}_\mathcal{A}$ is contained in $\chi_{0,1}$, the set of all those $[L] \in \mathcal{Q}_\mathcal{A}$ such that there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty, \subset \mathcal{H}$ with $\|x_n\| \leq 1, \|y_n\| \leq 1$ for all n, which fulfill the conditions:

$$\lim_{n\to\infty} \|x_n \otimes y_n] - [L]\|_{\mathcal{Q}} = 0, \tag{5.2.1}$$

$$\lim_{n\to\infty} \|[x_n \otimes w]\|_{\mathcal{Q}} = 0 \qquad \textit{for all } w \in \mathcal{H}, \tag{5.2.2}$$

$$\textit{and} \qquad \lim_{n\to\infty} \|[w \otimes y_n]\|_{\mathcal{Q}} = 0 \qquad \textit{for all } w \in \mathcal{H}, \tag{5.2.3}$$

since $\chi_{0,1}$ is absolutely convex and closed (see [17]), it suffices to show that $\chi_{0,1}$, contains all $[C_\lambda], \lambda \in \sigma_e(T) \cap \mathbb{D}^N$, in order to prove the reflexivity of $\mathcal{A}(T)$.

6. Reflexivity of Isometries

6.1. Wold-type model for pairs of doubly commuting isometries and partial results.

In [69, Theorem 3]), it was shown that for any pair $\{V_1, V_2\} \subset L(\mathcal{H})$ of doubly commuting isometries, there are subspaces $H_{uu}, H_{us}, H_{su}, H_{ss}$ such that

(1) $H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss}$, where all summands reduce V_1 and V_2,
(2) $V_1|_{H_{uu}}$ and $V_2|_{H_{uu}}$ are unitary operators,
(3) $V_1|_{H_{us}}$ is a unitary operator, $V_2|_{H_{us}}$ is a shift,
(4) $V_1|_{H_{su}}$ is a shift, $V_2|_{H_{su}}$ is a unitary operator, and
(5) $V_1|_{H_{ss}}$, $V_2|_{H_{ss}}$ are shifts.

The above decomposition we will call the Wold-type decomposition. Using the above decomposition the following partial result was shown in [56].

Theorem 6.1.1. *Every pair $\{V_1, V_2\}$ of doubly commuting isometries on a Hilbert space $\mathcal{H}$ is reflexive and has property $\mathbb{A}_1(1)$.*

The above results can be easily extended to N-tuples.

6.2. General case.

Let us state a result from [13].

Theorem 6.2.1. *Every commuting family of isometries $V = (V_\alpha)_{\alpha \in J}$ is reflexive and has property $\mathbb{A}_1(1)$.*

For the proof, we need some function theory results. Let $(\Omega, \sum, \mu)$ be a measure space with $\mu(\Omega) = 1$ and $\mathcal{F}$ be a Hilbert space. For $x, y \in L^2(\mu, \mathcal{F})$, we can define $x \cdot y \in L^1(\mu)$ by $(x \cdot y)(\omega) = \langle x(\omega), y(\omega) \rangle$. We can ask whether for any $f \in L^1(\mu)$ there are $x \in \mathcal{H} \subset L^2(\mu, \mathcal{F})$, $y \in L^2(\mu, \mathcal{F})$ such that $f = x \cdot y$. This is always possible if $\mathcal{H} = L^2(\mu, \mathcal{F})$ unless $\mathcal{F} = \{0\}$. For another example, consider L^1 on $\mathbb{T}$. If $f \in L^1(\mu)$, $f \geq 0$ and $\log f \in L^1$, then $f = |g|^2$ for $g \in H^2$.

We will say that $\mathcal{H} \subset L^2(\mu, \mathcal{F})$ has the *approximate factorization propert*, if for all $h \in L^1(\mu)$ with $h \geq 0$ and for all $\varepsilon > 0$ there exists $x \in \mathcal{H} \subset L^2(\mu, \mathcal{F})$ such that $|h - x \cdot x| < \varepsilon$. The proof of the above theorem is based on the following function theory result from [13].

Theorem 6.2.2. *Assume that $\mathcal{H} \subset L^2(\mu, \mathcal{F})$ has the approximate factorization property. Then for all $h \in L^1(\mu)$ and for all $\varepsilon > 0$ there exists $x \in \mathcal{H}$ such that $\|x(\omega)\|^2 \geq h(\omega)$ a.e. and $\|x\|^2 \leq \|h\|_1 + \varepsilon$.*

Corollary 6.2.3. *Assume that $\mathcal{H} \subset L^2(\mu, \mathcal{F})$ has the approximate factorization property. Then for every $f \in L^1(\mu)$ and every $\varepsilon > 0$, there exists $x \in \mathcal{H}$ and $y \in L^2(\mu, \mathcal{F})$ such that $x \cdot y = f$ and $\|x\| \cdot \|y\| < \|f\|_1 + \varepsilon$.*

Proof. Assume first that $\|f\| = 1$. We are applying the above theorem with $h = |f|$. Thus there is $x \in \mathcal{H}$ such that $\|x(\omega)\|^2 \geq f(\omega)$ and $\|x\|^2 \geq \|f\|_1 + \varepsilon$.

Let us define $y \in L^2(\mu, \mathcal{F})$ by $y(\omega) = \frac{f(\omega)x(\omega)}{\|x(\omega)\|^2}$ for ω such that $x(\omega) \neq 0$ and $y(\omega) = 0$ otherwise. Then $\|y\|^2 = \int \frac{|f(\omega)^2\|x(\omega)\|^2}{\|x(\omega)\|^4}\, d\mu \leq \int \frac{\|x(\omega)\|^4}{\|x(\omega)\|^4}\, d\mu \leq 1$ and $\|x\|\,\|y\| \leq \sqrt{\|f\|_1 + \varepsilon} \leq \|f\|_1 + \varepsilon$. Moreover, $(x \cdot y)(\omega) = \langle x(\omega), y(\omega)\rangle = \langle x(\omega), \frac{f(\omega)x(\omega)}{\|x(\omega)\|^2}\rangle = f(\omega)$ if $x(\omega) \neq 0$. If $x(\omega) = 0$ then $f(\omega) = 0$ too.

The next step in the proof will be a construction of an extension of $V = (V_\alpha)_{\alpha \in J}$ to a set of unitary operators. We can assume that $\mathbf{V}$ is a semigroup. We define a relation $\rho \subset (V \times \mathcal{H})^2$ such that $(V_\alpha, h)\rho(V_\beta, k)$ if and only if $V_\alpha k = V_\beta h$. It is easy to see that ρ is an equivalence relation. Then $V \times \mathcal{H}/\rho$ is a prehilbert space with operations: $[V_\alpha, h] + [V_\beta, k] = [V_\alpha V_\beta, V_\beta h + V_\alpha k], \lambda[V_\alpha, h] = [V_\alpha, \lambda h]$ and scalar product $([V_\alpha, h], [V_\beta, k]) = (V_\beta h, V_\alpha k)$, which gives a norm $\|[V_\alpha, h]\| = \|h\|$.

Let $\mathcal{K}$ be an completion of $V \times \mathcal{H}/\rho$ and note that we can embed $\mathcal{H}$ in $\mathcal{K}$ isometrically by $h \longmapsto [V_\alpha, V_\alpha h]$ (note that the embedding does not depend on V_α). Let us take T commuting with $V = (V_\alpha)_{\alpha\in J}$. Then we can define $\tilde{T} \in L(\mathcal{K})$ by $\tilde{T}[V_\alpha, h] = [V_\alpha, Th]$. Since T commutes with $\mathbf{V}$, $\tilde{T}$ does not depend on any particular choice of the representative from $[V_\alpha, h]$. Hence we can construct $\tilde{V}_\alpha$ for all $V_\alpha \in V$ and denote by $\tilde{V} = (\tilde{V}_\alpha)_{\alpha\in J}$. Moreover, since T commutes with $V = (V_\alpha)_{\alpha\in J}$, $\tilde{T}$ commutes with $\tilde{V} = (\tilde{V}_\alpha)_{\alpha\in J}$.

We can easily check that $\tilde{V}_\beta$ is a unitary operator and $\tilde{V}_\beta^*[V_\alpha, h] = [V_\beta V\alpha, h]$. The extension is minimal: since $[V_\beta, k] = \tilde{V}_\beta^*[V_\alpha, V_\alpha k]$, $\tilde{V}_\beta^*[V_\alpha, V_\alpha k]$ are linearly dense in $\mathcal{K}$ and hence $\cup V_\beta^*\mathcal{H}$ is dense in $\mathcal{K}$.

The important step in the proof is the following result from [43].

Theorem 6.2.4. *Assume that* $\boldsymbol{V}$ *is a family of commuting isometries. If* $T \in AlgLat(V)$ *then* $T \in V'$. *Moreover the operator* $\tilde{T}$ *(which is defined since* $T \in V'$*) is in the double commutant* $\tilde{V}''$ *of* $\tilde{V}$.

Theorem 6.2.5. *If* $\mathcal{B} = \{T \in L(\mathcal{H}) : T \in V'$ *and* $\tilde{T} \in \tilde{V}''\}$, *then* $\mathcal{B}$ *has property* $\mathbb{A}_1(1)$.

Proof of Theorem 6.2.1. It is obvious that $V \subset AlgLat(V)$. Moreover, by Theorem 6.2.4, $AlgLat(V) \subset \mathcal{B}$. Thus $AlgLat(V)$ is reflexive and has property $\mathbb{A}_1(1)$. Hence $\mathcal{W}(\mathbf{V})$, as a subalgebra, is reflexive and has property $\mathbb{A}_1(1)$ by [42, Proposition 2.5].

Proof of Theorem 6.2.5. For simplicity, we assume that the underlying space is separable. We know that $\bigcup_{\alpha\in I} \tilde{V}_\alpha^*\mathcal{H}$ is dense in $\mathcal{K}$. By the spectral theory, there is a measure space $(\Omega, \sum, \mu)$ $(\Omega = \mathbb{T}^I)$ and a measurable function f_α with $|f_\alpha| = 1$ a.e. on the set $\Omega = \sigma_1 \supset \sigma_2 \supset \cdots$ such that there is a unitary operator $U : \mathcal{K} \to \oplus_{j\geq1} L^2(\mu|\sigma_j)$ with $U\tilde{V}_\alpha U^* = M_{f_\alpha}$. The space $\oplus_{j\geq1} L^2(\mu|\sigma_j)$ can be identified with $L^2(\mu, \mathcal{F})$, whele $\mathcal{F}$ is a separable Hilbert space.

We would like to prove that $\mathcal{H}$ has the approximate factorization property. Assume that $\mathcal{K} = L^2(\mu,\mathcal{F})$ and that $h \in L^1(\mu)$ with $h \geq 0$. Then there is $y \in \mathcal{K}$ (say, in $L^2(\mu|\sigma_1)$) such that $h = y \cdot y$ (if $h \in L^1(\mu)$, $h \geq 0$, then $\sqrt{h} \in L^2(\mu)$ and $h = \sqrt{h} \cdot \sqrt{h}$) and $\|y\|_2 = \|h\|_1$. There is $y_1 = [V_\alpha, k]$ such that $\|y - y_1\| \leq \varepsilon$ and $\|y_1\| \leq \|y\| + 1$. Then $x = \tilde{V}_\alpha[V_\alpha, k] = [V_\alpha, V_\alpha k] \in \mathcal{H}$. Since $\tilde{V}_\alpha$ is the multiplication operator by the function f_α with $|f_\alpha| = 1$, thus $\tilde{V}_\alpha y_1 \cdot \tilde{V}_\alpha y_1 = y_1 \cdot y_1$. We have

$$\begin{aligned}\|h - x \cdot x\|_1 &= \|h - V_\alpha y_1 \cdot V_\alpha y_1\|_1 = \|h - y_1 \cdot y_1\|_1\\ &\leq \|h - y \cdot y\|_1 + \|y \cdot y - y_1 \cdot y_1\|_1 \leq \|y \cdot y - y \cdot y_1\|_1 + \|y \cdot y_1 - y_1 \cdot y_1\|_1\\ &\leq \|y\|_2\ \|y - y_1\|_2 + \|y_1\|_1\ \|y - y_1\|_2 \leq (\|h\|_1 + \|h\|_1 + 1)\varepsilon.\end{aligned}$$

Thus $\mathcal{H}$ has the approximate factorization property.

Now we can start the main part of the proof. Take $\varepsilon > 0$ and let $\phi : \mathcal{B} \to \mathbb{C}$ be weak-star continuous. Then ϕ can be represented as $\phi(T) = \sum\limits_{n=0}^{\infty}(Tx_n, y_n)$ with $\sum\limits_{o=0}^{\infty} \|x_n\|\,\|y_n\| < \|\phi\| + \frac{\varepsilon}{2}$ for some $x_n, y_n \in \mathcal{H} \subset \mathcal{K} = L^2(\mu,\mathcal{F})$. If we define $f = \sum\limits_{n=0}^{\infty} x_n \cdot y_n \in L^1(\mu)$, then $\|f\|_1 \leq \|\phi\| + \frac{\varepsilon}{2}$. By Corollary 6.2.3, there is $x \in \mathcal{H}$ and $z \in \mathcal{K}$ such that $x \cdot z = f$ and $\|x\|\ \|z\| \leq \|f_1\| + \frac{\varepsilon}{2} < \|\phi\| + \varepsilon$. Define $y = P_{\mathcal{H}} z$. If $T \in \mathcal{B}$, then $\tilde{T} \in \tilde{V}''$ and $T = M_u|_{\mathcal{H}}$ (M_u, multiplication operator). Then

$$\begin{aligned}\phi(T) &= \sum_{n=0}^{\infty}(ux_n, y_n) = \sum_{n=0}^{\infty}\int (u(\omega)x(\omega), y(\omega))d\mu(\omega)\\ &= \int \sum_{n=0}^{\infty} u(\omega)(x(\omega), y(\omega))d\mu(\omega) = \int u(\omega)f(\omega)d\mu(\omega)\\ &= \int u(\omega)(x(\omega), z(\omega))d\mu(\omega) = \int (u(\omega)x(\omega), z(\omega))d\mu(\omega)\\ &= (ux, z) = (Tx, z) = (Tx, y).\end{aligned}$$

Hence $\phi(T) = (Tx, y)$.

7. Reflexivity of Jointly Quasinormal Operators and Spherical Isometries

An operator T is called *quasinormal* if T commutes with T^*T. W. Wogen [75] proved that individual quasinormal operators are reflexive. Following A. Lubin [52], we call a commutative family $\mathcal{S}$ of operators *jointly quasinormal* if S and T^*T commute for any $S, T \in \mathcal{S}$. Commutative families of normal operators or isometries are automatically jointly quasinormal, as are doubly commuting families of quasinormal operators. Example 7.2 below exhibits a commutative pair of quasinormal operators which is not jointly quasinormal.

Theorem 7.1. *Every jointly quasinormal family $\mathcal{S}$ of operators is reflexive and has property $\mathbb{A}_1(1)$.*

Proof. Since the underlying Hilbert space is separable, we may as well assume the family $\mathcal{S}$ to be countable. Write $\mathcal{Z}$ for the commutative von Neumann algebra generated by $\{T^*T : T \in \mathcal{S}\}$. By direct integral theory (see [70]), $\mathcal{Z}$ is the diagonal, algebra corresponding to a direct integral decomposition of the underlying Hilbert space $\mathcal{H} = \int_\Lambda^\oplus H(\lambda)d\mu(\lambda)$. Here μ is a finite regular Borel measure on Λ and we have $\mathcal{Z} = \int_\Lambda^\oplus \mathcal{Z}(\lambda)d\mu(\lambda)$, where each $\mathcal{Z}(\lambda)$ consists of scalar multiples of $I_{H(\lambda)}$. For the simplicity of notation, we assume that $H(\lambda) \equiv H$ and the corresponding field of measurable vectors is the constant field. Each $T \in \mathcal{S}$ is decomposable, and by our choice of $\mathcal{Z}$, we know that $T^*(\lambda)T(\lambda)$ i.s a scalar multiple of the identity for almost all λ. Discarding a set of measure zero if necessary, we can assume $T(\lambda)$ to be a scalar multiple of an isometry for each $T \in \mathcal{S}$ and $\lambda \in \Lambda$.

For each fixed λ, the algebra $\mathcal{W}(T(\lambda) : T \in \mathcal{S})$ is generated by a family of commuting isometries. By Theorem 6.2.1, it is reflexive and has property $\mathbb{A}_1(1)$. Hence, by [6, Proposition 5.6], $\mathcal{W}(\mathcal{Z} \cup \mathcal{S})$ is reflexive, and by [42, Theorem 3.6], has property $\mathbb{A}_1(1)$. Thus its subalgebra $\mathcal{W}(\mathcal{S})$ is reflexive and has property $\mathbb{A}_1(1)$ by [42, Proposition 2.5].

Example 7.2. Write U for the forward bilateral shift relative to an orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$. Denote by P and Q the projections onto the spaces spanned by $\{e_n : n \geq 0\}$ and $\{e_n : n \leq 0\}$, respectively. Set $S = UP$ and $T = U^*Q$. The quasinormality of S and T follows from the fact that P and Q are invariant under U and U^* respectively. It is easy to check that $ST = 0 = TS$. On the other hand, while $T^*TS = QUP = 0$, we have $ST^*T = UPQ \neq 0$. Thus $\{S, T\}$ is a commuting pair of quasinormal operators which is not jointly quasinormal.

An N-tuple $V = (V_1, \ldots, V_N)$ is called a *spherical isometry* if $\sum_{i=1}^N V_i^* V_i = I$. One consequence of Theorem 7.1 is the following

Proposition 7.3. *Every N-tuple $V = (V_1, \cdots, V_N)$ of doubly commuting spherical isometries is reflexive.*

Proof. First, we will show that each V_i, $i = 1, \cdots, N$, is quasinormal. This follows from the following equalities

$$\begin{aligned} V_iV_i^*V_i &= V_i(I-\sum_{j\neq i}V_j^*V_j) = V_i - \sum_{j\neq i}V_iV_j^*V_j \\ &= V_i - \sum_{j\neq i}V_j^*V_jV_i = (I-\sum_{j\neq i}V_j^*V_j)V_i = V_i^*V_iV_i\,. \end{aligned}$$

Now, since V is a doubly commuting set of quasinormal operators, it is also jointly quasinormal and the conclusion follows from Theorem 7.1.

8. Reflexivity and Invariant Subspace Results for Contractions

The Dual Algebra Technique has had great achievements in showing the existence of non-trivial invariant subspace or reflexivity for a single operator. Recall some of them [14], [16–17], [23–28], [54], [63–64], [74]. The most striking result we will discuss is in Section 9. Let us present the following two results which give the inspiration for the results for an N-tuple of contractions presented in this section. Instead of presenting them in the full strength, we show them in the form so that the theorems below can be seen as their N-tuple generalizations.

Theorem [63]. *Let T be a C_0 contraction. If the intersection of the left essential spectrum with the open disc $\sigma_{le}(T)\cap\mathbb{D}$ is dominating for H^∞, then $\mathcal{W}(T)$ is reflexive.*

The next is a stronger one.

Theorem [15]. *Let T be a completely non-unitary contraction. If the intersection of the essential spectrum with the open disc $\sigma_e(T)\cap\mathbb{D}$ is dominating for H^∞, then $\mathcal{W}(T)$ is reflexive.*

8.1. Results with dominancy of left and right spectra.

We start with the following

Theorem 8.1.1. *Let $\{T_1,T_2\}\subset L(\mathcal{H})$ be a pair of commuting contractions. Assume also that $T_1\in C_0$. If the intersection of the left spectrum with the open bidisc $\sigma_l(T_1,T_2)\cap\mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$, then $\{T_1,T_2\}$ has a common non-trivial invariant subspace.*

If T_2 is not a.c., we can decompose T_2 for an absolutely Continuous part and a singular unitary part. If the subspace on which T_2 is a singular unitary operator is not equal to $\{0\}$, then since this subspace is hyperinvariant for

T_2 (unless the space $\mathcal{H}$ is one-dimensional), we have a non-trivial invariant subspace for the pair $\{T_1, T_2\}$. Thus we can assume that T_2 is a.c.

Assume now that $\sigma_l(T_1,T_2)\backslash\sigma_{le}(T_,T_2) \neq \emptyset$ and take $\lambda = (\lambda_1, \lambda_2) \in \sigma_l(T_1,T_2) \backslash\sigma_{le}(T_1,T_2)$. Then, by Lemmas 3.1.1 and 3.1.2, there is a sequence x_n in some finite-dimensional subspace $\mathcal{F}$ of $\mathcal{H}$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty}\|(\lambda_i - T_i)x_n\| = 0$ for $i = 1, 2$. Since the ball in the finite-dimensional space is compact, there is $x \in \mathcal{H}$ with $\|x\| = 1$ and $\|(\lambda_i - T_i)x\| = 0$ for $i = 1, 2$. Then ker $(\lambda_i - T_i)$ is a non-trivial hyperinvariant subspace for T_i for $i = 1, 2$, or $T_1 = \lambda_1 I$ and $T_2 = \lambda_2 I$. Thus the pair $\{T_1, T_2\}$ has a common non-trivial invariant subspace unless the space $\mathcal{H}$ is one-dimensional.

Hence we can also assume that $\sigma_l(T_1,T_2) = \sigma_{le}(T_1,T_2)$. In this case, Theorem 8.1.1 is reduced to the following reflexivity result.

Theorem 8.1.2. *Let $\{T_1, T_2\} \subset L(\mathcal{H})$ be a pair of commuting contractions. Assume also that $T_1 \in C_{0\cdot}$ and T_2 is absolutely continuous. If the intersection of the left essential spectrum with the open bidisc $\sigma_{le}(T_1,T_2) \cap \mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$, then the algebra $\mathcal{W}(T_1,T_2)$ is reflexive.*

Now, let us discuss the related results. First, Theorem 8.1.1, for $N = 2$, generalizes the following

Theorem [4]. *Let $T = (T_1, \cdots, T_N)$ be an N-tuple of C_{00} contractions and $\mathbb{D}^N$ be a spectral set for T. If the intersection of the left essential spectrum with the open polydisc $\sigma_{le}(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $T = (T_1, \cdots, T_N)$ has a common non-trivial invariant subspace.*

By symmetry Theorem 8.1.2 can be stated in the following way.

Theorem 8.1.2'. *Let $\{T_1, T_2\} \subset L(\mathcal{H})$ be a pair of commuting contractions. Assume also that $T_1 \in C_{\cdot 0}$ and T_2 is absolutely continuous. If $\sigma_{re}(T_1,T_2) \cap \mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$, then the algebra $\mathcal{W}(T_1,T_2)$ is reflexive.*

Thus the above results overlap with

Theorem [39]. *Let $T = (T_1, \cdots, T_N)$ be an N-tuple of commuting completely non-unitary contractions and $\mathbb{D}^N$ be a spectral set for T. Assume also that there are $i, j \in \{1, \cdots, N\}$ such that $T_i \in C_{0\cdot}$ and $T_j \in C_{\cdot 0}$. If the intersection of the Harte spectrum with the open polydisc $\sigma_H(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $\mathcal{W}(T)$ is reflexive.*

In the end, let us quote the related result with dominancy of the left essential spectrum, which will find its generalization in Section 8.2.

Theorem [47]. *Let $T = (T_1, \ldots, T_N)$ be an N-tuple of doubly commuting completely non-unitary contractions. If the intersection of the left essential spectrum with the open polydisc $\sigma_{le}(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $\mathcal{W}(T)$ is reflexive.*

Now, we can start

Proof of Theorem 8.1.2. We can construct the representation $\Phi : A(\mathbb{D}^2) \to L(\mathcal{H})$ generated by $T = (T_1, T_2)$ as in Section 4.2. Moreover, by Lemma 4.3.1, T is a.c., and thus we can extend Φ to $H_\infty(\mathbb{D}^2)$. By Lemma 4.3.5, since $\sigma_{le}(T)$ is dominating for $H^\infty(\mathbb{D}^2)$, we see that Φ is an isometry and a weak-star homeomorphism. Hence, as it was mentioned in Section 5.2, to show reflexivity of $\mathcal{A}(T_1, T_2)$ it is enough to check (5.2.1)–(5.2.3).

The first two lemmas are true not only for $N = 2$ but also for an arbitrary N and since they might be of independent interest we present them as follows.

Lemma 8.1.3. *Assume $T = (T_1, \ldots, T_N)$ generates an a.c. isometric representation. Let $\lambda = (\lambda_1, \ldots, \lambda_N) \in \sigma_{le}(T_1, \cdots, T_N) \cap \mathbb{D}^N$ and let $\{x_n\}$ be an orthonormal sequence with $\lim_{n\to\infty} \|(T_i - \lambda_i)x_n\| = 0$ for all $i = 1, \ldots, N$. Then*

$$\lim_{n\to\infty} \|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} = 0.$$

Proof. The Hahn-Banach theorem implies that there is $f_n \in H^\infty(\mathbb{D}^N)$ such that $\|f_n(T)\| = \|f_n\|_\infty = 1$ and $\|[x_n \otimes x_n] - [C_\lambda]\|\mathcal{Q}= \langle f_n(T),\ [x_n \otimes x_n] - [C_\lambda]\rangle$. The Gleason property for a polydisc shows that there are g_i^n such that $f_n(z) = f_n(\lambda) + \sum_{i=1}^N (z_i - \lambda_i) g_i^n(z)$ and $\|g_i^n\|_\infty \le M_\lambda \|f_n\|_\infty = M_\lambda$ for $z \in \mathbb{D}^N$. Hence

$$\begin{aligned}
\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} &= \langle f_n(\lambda) + \sum_{i=1}^N g_i^n(T)(T_i - \lambda_i), [x_n \otimes x_n] - [C_\lambda]\rangle \\
&= \langle f_n(\lambda), [x_n \otimes x_n]\rangle - \langle f_n(\lambda), [C_\lambda]\rangle \\
&+ \sum_{i=1}^N \langle g_i^n(T)(T_i - \lambda_i), [x_n \otimes x_n]\rangle + \sum_{i=1}^N \langle g_i^n(T)(T_i - \lambda_i), [C_\lambda]\rangle \\
&= f_n(\lambda)(x_n, x_n) - f_n(\lambda) + \sum_{i=1}^N (g_i^n(T)(T_i - \lambda_i)x_n, x_n) \\
&+ \sum_{i=1}^N g_i^n(\lambda)(\lambda_i - \lambda_i) \le \sum_{i=1}^N \|g_i^n(T)\| \, \|(T_i - \lambda_i)x_n\| \\
&\le M_\lambda \sum_{i=1}^N \|(T_i - \lambda_i)x_n\| \to 0.
\end{aligned}$$

Thus, the proof is completed.

Lemma 8.1.4. *Assume $T = (T_1, \ldots, T_N)$ generates an a.c. isometric representation. Let $\lambda = (\lambda_, \ldots, \lambda_N) \in \sigma_{le}(T_1, \ldots, T_N) \cap \mathbb{D}^N$ and let x_n be an orthonormal sequence such that $lim_{n\to\infty}\|(T_i - \lambda_i)x_n\| = 0$ for all $i = 1, \ldots, N$. Then $lim_{n\to\infty}\|[x_n \otimes y]\|_{\mathcal{Q}} = 0$ for any fixed $y \in \mathcal{H}$.*

Proof. Via Hahn-Banach theorem, there is $f_n \in H^\infty(\mathbb{D}^2)$ such that $\|f_n(T)\| = \|f_n\|_\infty = 1$ and $\|[x_n \otimes y]\|_{\mathcal{Q}} = \langle f_n(T), x_n \otimes y\rangle = |(f_n(T)x_n, y)|$. As above, the Gleason property shows that there are g_i^n such that $f_n(z) = f_n(\lambda) + \sum_{i=1}^N (z_i - \lambda_i)g_i^n(z)$ and $\|g_i^n\|_\infty \leq M_\lambda \|f_n\|_\infty = M_\lambda$ for $z \in \mathbb{D}^N$. Now,

$$
\begin{aligned}
\|[x_n \otimes y]\|_{\mathcal{Q}} &= f_n(\lambda)(x_n, y) + \sum_{i=1}^N (g_i^n(T)(T_i - \lambda_i)x_n, y) \\
&\leq \|f_n\|_\infty (x_n, y) + \sum_{i=1}^N \|g_i^n(T)\| \, \|T^i - \lambda_i)x_n\| \, \|y\| \\
&\leq |(x_n, y)| + \sum_{i=1}^N \|g_i^n\|_\infty \|(T_i - \lambda_i)x_n\| \, \|y\| \\
&\leq |(x_n, y)| + M_\lambda \|y\| \sum_{i=1}^N \|(T_i - \lambda_i)x_n\| \to 0
\end{aligned}
$$

since x_n is an orthonormal sequence and $\|(T_i - \lambda_i)x_n\| \to 0$ for $i = 1, 2, \ldots, N$.

It is an easy observation that in [63, Lemma 3.4] the assumption c.n.u. is not essential, only the absolute continuity of T is needed. Moreover, weaker assumptions for the sequence $\{x_n\}$ are sufficient. So we have

Lemma 8.1.5. *Assume that $T \in L(\mathcal{H})$ is an a.c. contraction generating all isometric functional calculus. If $\{x_n\}$ is a sequence since that $x_n \to 0$ weakly, $\|x_n\| = 1$ and $\|(T - \lambda)x_n\| \to 0$, then $\|[y \otimes x_n]\|_{\mathcal{Q}_T} \to 0$ for all $y \in \mathcal{H}$.*

The last approximation lemma will be proved for $N = 2$

Lemma 8.1.6. *Assume that T_1, T_2 generate an a.c. isometric representation and $T_1^n \to 0$ strongly. If $\{x_n\}$ is a sequence such that $x_n \to 0$ weakly, $\|x_n\| = 1$ and $\|(T_2 - \lambda_2)x_n\| \to 0$, then $\|[y \otimes x_n]\|_{\mathcal{Q}} \to 0$ for all $y \in \mathcal{H}$.*

Proof. By [71, Theorem II.2.1], choose $V_1 \in L(\mathcal{K})$ a minimal isometric dilation of T_1^*. Then V_1 is a unilateral shift of certain multiplicity and $T_1 = V_1^*|_{\mathcal{H}}$. On the other hand, by the commutant lifting theorem of Sz-Nagy and Foias (see [71], [55, p. 484]), there is an operator W_2 (not necessarily an isometry) preserving the norm of T_2^* such that the pair $\{V_1, W_2\}$ dilates $\{T_1^*, T_2^*\}$. Let $\varepsilon > 0$. Choose $M > 0$ such that $\|(I - P_{ker V_1^{*M}})y\| \leq \frac{\varepsilon}{3}$ and denote $y_1 = (P_{ker V_1^{*M}} y,\ y_2 = (I - P_{ker V_1^{*M}})y$. By Hahn-Banach theorem, for each n, there is $f_n \in H^\infty(\mathbb{D}^2)$ such that

$$\|[y \otimes x_n]\|_{\mathcal{Q}} = \langle f_n(T_1, T_2), [y \otimes x_n]\rangle = |(f_n(T_1, T_2)y, x_n)|,\ \|f_n\| = 1.$$

For each n we can decompose f_n as follows

$$f_n(z_1, z_2) = \sum_{k=0}^{M-1} a_{nk}(z_2) z_1^k + z_1^M\, h_n(z_1, z_2).$$

The functions a_{nk} are measurable. Moreover, since $a_{nk}(z_2)$ is the k-th Fourier coefficient of $f_n(\cdot, z_2)$, we have $|a_{nk}(z_2)| \leq 1$ for $z_2 \in \mathbb{D}$. Thus $\|a_{nk}\| \leq 1$ and consequently $\|h_n\| \leq M + 1$. It is easy to check that the negative Fourier coefficients of every a_{nk} vanish. Hence, for every n, k, we get $a_{nk} \in H^\infty(\mathbb{D})$ and $h_n \in H^\infty(\mathbb{D}^2)$.

Applying the Lebesgue-type decomposition (4.3.1) to space $\mathcal{K}$, we get via an easy calculation on elementary measures, $\mathcal{H} \subset \mathcal{K}_0$. So, by the minimality of V_1, we have $\mathcal{K} = \mathcal{K}_0$ and hence the pair $\{V_1, W_2\}$ is a.c. By Lemma 4.3.5, $\{V_1^*, W_2^*\}$ is also a.c. By [45, Proposition], the representation generated by T_2 is a.c. and the same is true for W_2^*.

The above facts give us the existence of the functional calculus for all the above mentioned pairs and single operators. So, applying Lemma 4.3.5 to the pairs $\{T_1, T_2\}$ and $\{V_1^*, W_2^*\}$, we have

$$\begin{aligned}
\|[y \otimes x_n]\|_{\mathcal{Q}} &= |(f_n(T_1, T_2)y, x_n)| = |(y, f_n^{\sim}(T_1^*, T_2^*)x_n)| \\
&\leq |(y, f_n^{\sim}(V_1, W_2)x_n)| = |(f_n(V_1^*, W_2^*)y, x_n)| \\
&\leq |(f_n(V_1^*, W_2^*)y_1, x_n)| + |(f_n(V_1^*, W_2^*)y_2, x_n)| \\
&\leq \sum_{k=0}^{M-1} |(a_{nk}(W_2^*)V_1^{*k} y_1, x_n)| \\
&\quad + |(V_1^{*M}\, h_n(V_1^*, W_2^*)y_1, x_n)| + \|f_n\|\ \|y_2\|\ \|x_n\|.
\end{aligned} \tag{8.1.1}$$

The vector y_1 is defined such that $V_1^{*M} y_1 = 0$, and thus the second component is equal to 0. Observe that for all k,

$$|(a_{nk}(W_2^*)V_1^{*k} y_1, x_n)| \leq \|[V_1^{*k} y_1 \otimes x_n]\|_{\mathcal{Q}_{W_2^*}}.$$

On the other hand, by [71, Theorem II.2.3], W_2^* is an extension of T_2. Thus not only $\|(W_2^* - \lambda_2)x_n\| \to 0$, but also W_2^* generates an isometric representation, since T_2 does. Hence, Lemma 8.1.5 shows that $\|[V_1^{*k}\, y_1 \oplus x_n]\|_{\mathcal{Q}_{W_2^*}} \to 0 (n \to \infty)$. Hence we can choose n_0 such that for all $n > n_0$ we have $\|[V_1^{*k}\, y_1 \oplus x_n]\|_{\mathcal{Q}_{W_2^*}} \leq \frac{\varepsilon}{3M}$ for $k = 0, 1, \ldots, M-1$. Now come back to the estimation of $\|[y \oplus x_n]\|_{\mathcal{Q}}$. Using (8.1.1) and the estimation of $\|y_2\|$, we obtain for $n > n_0$,

$$\|[y \oplus x_n]\|_{\mathcal{Q}} \leq \sum_{k=0}^{M-1} \|[V_1^{*k}\, y_1 \oplus x_n]\|_{\mathcal{Q}_{W_2^*}} + \|f_n\|\; \|y_2\|\; \|x_n\| \leq \varepsilon.$$

The proof of the lemma is finished.

The next part of this section deals with pairs of operators which extend the case of doubly commuting operators. For any pair $\{T_1, T_2\}$ of commuting contractions we can construct a minimal isometric dilation $\{V_1, V_2\}$ of the pair $\{T_1^*, T_2^*\}$ (see [71]). One can see that V_i^* is a coisometric extension of $T_i,\ i = 1, 2$. We call $\{V_1^*, V_2^*\}$ *a joint coisometric extension* of $\{T_1, T_2\}$. It can be seen that it is minimal in the standard meaning; for the details, see [53]. We say that a pair $\{T_1, T_2\} \subset L(\mathcal{H})$ is *diagonally extendable* if there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a minimal joint coisometric extension $\{B_1, B_2\} \subset L(\mathcal{H})$ of $\{T_1, T_2\}$ such that, for either $j = 1$ or $j = 2$, if $\mathcal{K}$ is decomposed as $\mathcal{K} = \mathcal{S}_j \oplus \mathcal{R}_j$, relative to which the matrix for B_j has the form

$$B_j = \begin{pmatrix} S_j^* & 0 \\ 0 & R_j \end{pmatrix},$$

where $S_j^* \in L(\mathcal{S}_j)$ is a (unilateral) backward shift and $R_j \in L(\mathcal{R}_j)$ is a unitary operator, then the matrix for B_k, for $k \neq j$, relative to the decomposition $\mathcal{K} = \mathcal{S}_j \oplus \mathcal{R}_j$ has the form

$$B_k = \begin{pmatrix} A_s & 0 \\ 0 & A_r \end{pmatrix}$$

for some $A_s \in L(\mathcal{S}_j)$ and $A_r \in L(\mathcal{R}_j)$.

Let us recall the result of [53, Theorem 2.5] and [68, Lemma 1] as

Proposition 8.1.7. *With the above notation, a pair $\{T_1, T_2\} \subset L(\mathcal{H})$ is diagonally extendable if any of the following conditions holds*

(a) *R_1 has no part of uniform multiplicity $\aleph_0$,*

(b) *R_2 has no part of uniform multiplicity $\aleph_0$,*

(c) *T_1 and T_2 doubly commute.*

Now we will present the following

Theorem 8.1.8. *Let $\{T_1, T_2\} \subset L(\mathcal{H})$ be a pair of commuting contractions. Assume also that $\{T_1, T_2\} \subset L(\mathcal{H})$ is diagonally extendable. If $\sigma_l(T_1, T_2) \cap \mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$, then the pair $\{T_1, T_2\}$ has a common non-trivial invariant subspace.*

We can reduce this result to Theorem 8.1.9 below just as Theorem 8.1.1 to Theorem 8.1.2.

Theorem 8.1.9. *Let $\{T_1, T_2\} \subset L(\mathcal{H})$ be a pair of a.c. commuting contractions. Assume also that $\{T_1, T_2\} \subset L(\mathcal{H})$ is diagonally extendable. If $\sigma_{le}(T_1, T_2) \cap \mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$, then the algebra $\mathcal{W}(T_1, T_2)$ is reflexive.*

For pairs of operators, the above theorem is a generalization of Theorem [47] quoted in the beginning of the section. The proof is based on the same considerations as the proof of Theorem 8.1.2. using Lemma 8.1.10 below instead of Lemma 8.1.6.

Lemma 8.1.10. *Let $\{T_1, T_2\}$ be a diagonally extendable pair of contractions generating an a.c. isometric representation. If $\{x_n\}$ is a sequence such that $x_n \to 0$ weakly, $\|x_n\| = 1$ and $\|(T_i - \lambda_i)x_n\| \to 0$ for $i = 1, 2$, then $\|[y \otimes x_n]\|_{\mathcal{Q}} \to 0$ for all $y \in \mathcal{H}$.*

Proof. Without loss of generality, we assume that the space $\mathcal{K} \supset \mathcal{H}$ and operator $B_i \in L(\mathcal{H})$ extends T_i for $i = 1, 2$. Moreover, B_1 is a coisometry, $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$ and

$$B_1 = \begin{pmatrix} S_1^* & 0 \\ 0 & R_1 \end{pmatrix},$$

where $S_1^* \in L(\mathcal{S}_1)$ is a (unilateral) backward shift and $R_1 \in L(\mathcal{R}_1)$ is a unitary operator, and B_2 has the form

$$B_2 = \begin{pmatrix} A_s & 0 \\ 0 & A_r \end{pmatrix},$$

for some $A_s \in L(\mathcal{S}_1)$ and $A_r \in L(\mathcal{R}_1)$.

By the decomposition (4.3.1) and the minimality of $\{B_1, B_2\}$, we conclude similarly as in the proof of Lemma 8.1.6 that $\{B_1, B_2\}$ is an a.c. pair. It also generates an isometric functional as an extension of the pair $\{T_1, T_2\}$. Thus we have

$$\begin{aligned}\|[y\otimes x_n]\|_{\mathcal{Q}} &= \sup\{|(h(B_1,B_2)y,x_n)| : h\in H^\infty(\mathbb{D}^2), \|h\|_\infty = 1\}\\ &= \sup\{|(h(B_1,B_2)y,x_n)| : h\in H^\infty(\mathbb{D}^2), \|h\|_\infty = 1\}\\ &= \|[y\otimes x_n]\|_{\mathcal{Q}_{B_1,B_2}}.\end{aligned}$$

So we need to show that the last tends to zero.

For a contraction T and $\lambda\in\mathbb{D}$, let T^λ denote the operator $(T-\lambda)(I-\overline{\lambda}T)^{-1}$. An easy calculation based on [71] shows that the decomposition $\mathcal{K}=\mathcal{S}_1\oplus\mathcal{R}_1$ is also a decomposition of $(B_1^*)^{\lambda_1}$ into a pure isometry and a unitary part. Let us note that $\mathcal{S}_1,\ \mathcal{R}_1$ reduce $B_2^{\lambda_2}$. Moreover, $B_i^{\lambda_i}$ is a coisometric extension of $T_i^{\lambda_i}$ for $i=1,2$. Let $x_n=x_n^s\oplus x_n^r,\ y=y^s\oplus y^r$ with respect to the decomposition $\mathcal{K}=\mathcal{S}_1\oplus\mathcal{R}_1$. Since $\|(T_1-\lambda_1)x_n\|\to 0$, we have $\|T_1^{\lambda_g}x_n\|\to 0$. One can easily see that

$$\begin{aligned}\|x_n^r\| &\le \|x_n - P_{KerB_1^{\lambda_1}}x_n = \|(I-P_{KerB_1^{\lambda_1}})x_n\|\\ &= \|(B_1^*)^{\lambda_1}B_1^{\lambda_1}x_n\| = \|T_1^{\lambda_1}x_n\|\to 0.\end{aligned}\tag{8.1.2}$$

Thus,

$$\begin{aligned}\|[y\otimes x_n]\|_{\mathcal{Q}_{B_1,B_2}} &= \sup\{|(h(B_1,B_2)y,x_n)| : h\in H^\infty(\mathbb{D}^2), \|h\|_\infty = 1\}\\ &\le \sup\{|(h(S_1^*,A_s)y^s,x_n^s)| : h\in H^\infty(\mathbb{D}^2), \|h\|_\infty = 1\}\\ &\quad+\sup\{|(h(R_1,A_r)y^r,x_n^r)| : h\in H^\infty(\mathbb{D}^2), \|h\|_\infty = 1\}\\ &\le \|[y^s\otimes x_n^s]\|_{\mathcal{Q}_{S_1^*,A_s}} + \|y^r\|\ \|x_n^r\|.\end{aligned}$$

The second component converges to 0. To prove that the first one converges to 0, it is enough to show that the pair $\{S_1^*,A_s\}$ and the sequence $z_n=\frac{x_n^s}{\|x_n^s\|}$ fulfills the assumptions of Lemma 8.1.6. The sequence of operators $\{S_1^{*n}\}$ converges strongly to 0 since S_1^* is a backward shift. Since B_2 is an extension of T_2, thus $\|(B_2-\lambda_2)x_n\|\to 0$. Since $\|(B_2-\lambda_2)x_n\|^2=\|(A_s-\lambda_2)x_n^s\|^2+\|(A_r-\lambda_2)x_n^r\|^2$, thus $\|(A_s-\lambda_2)x_n^s\|\to 0$ and $\|(A_s-\lambda_2)z_n\|\to 0$, since $\|x_n^s\|\to 1$. Hence, we can finish the proof of Lemma 8.1.6 and Theorem 8.1.9 with the following

Lemma 8.1.11. *The pair $\{S_1^*,A_s\}$ generates an a.c. isometric representation.*

Proof. To see that the pair $\{S_1^*,A_s\}$ generates an a.c. representation, it is enough to notice that it is a restriction of the pair $\{B_1,B_2\}$, which is an a.c. pair, and then make an easy calculation on elementary measures. Now we prove that $\sigma_{le}(T_1,T_2)\cap\mathbb{D}^{\ 2}\subset\sigma_{le}(S_1^*,A_s)\cap\mathbb{D}^2$. Let $(\lambda_1,\lambda_2)\in\sigma_{le}(T_1,T_2)\cap\mathbb{D}^2$, and $\{x_n\}$ be an orthonormal sequence such that $\lim_{n\to\infty}\|(T_i-\lambda_i)\|=0$, for $i=1,2$. Hence, $\lim_{n\to\infty}\|(B_i-\lambda_i)x_n\|=0$, for $i=1,2$, too. Let $x_n=x_n^s\oplus x_n^r$

with respect to the decomposition $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$. We can prove as above that $\|(A_s - \lambda_2)x_n^s\| \to 0$ and in the same way that $\|(S_1^* - \lambda_1)x_n^s\| \to 0$. As in (8.1.2), it can be show that $\|x_n^r\| \to 0$, and hence $\|x_n^s\| \to 1$. Let $z_n = \frac{x_n^s}{\|x_n^s\|}$. It is easy to check that $\|(A_s - \lambda_2)z_n\| \to 0$ and $\|(S_1^* - \lambda_1)z_n\| \to 0$. Thus $(\lambda_1, \lambda_2) \in \sigma_{le}(S_1^*, A_s) \cap \mathbb{D}^2$ by Lemma 4.3.5. Hence, if $\sigma_{le}(T_1, T_2) \cap \mathbb{D}^2$ is dominating for $H^\infty(\mathbb{D}^2)$, then so is $\sigma_{le}(S_1^*, A_s) \cap \mathbb{D}^2$. Hence, by Theorems 4.3.4 and 4.3.5, the proof of the lemma is finished.

8.2. Results with dominancy of Taylor spectrum.

Our main result of this section will be the following

Theorem 8.2.1. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of doubly commuting contractions. If the intersection of the Taylor spectrum with the open polydisc $\sigma(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $T = (T_1, \dots, T_N)$ has a common non-trivial invariant subspace.*

As we shall see, because of the double commutativity, it will be sufficient in the proof to consider the case that the Taylor spectrum of T coincides with the essential Taylor spectrum. This allows us to reduce the proof of Theorem 8.2.1 to the following

Theorem 8.2.2. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of doubly commuting completely non-unitary contractions. If the intersection of the Taylor essential spectrum with the open polydisc $\sigma_e(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $T = (T_1, \dots, T_N)$ is reflexive.*

Of course, the reflexivity is a much stronger property than the existence of a non-trivial common invariant subspace. Theorem 8.2.2 is a generalization of the reflexivity result for a single contraction case (Theorem [15]). It also improves Theorem [47] mentioned in the beginning of Section 8.1.

Let us quote also two other related results

Theorem [18]. *Let $\{T_1, T_2\}$ be a pair of commuting contractions and assume that it generates a weak-star continuous isometric functional calculus $\Phi : H^\infty(\mathbb{D}^2) \to L(\mathcal{H})$. Then (5.2.1) is fulfilled. (If T_1, T_2 are of C_{00} class, then $\mathcal{A}\ (T_1, T_2)$ is reflexive.)*

Theorem [1]. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of commuting completely non-unitary contractions and $\mathbb{D}^N$ be a spectral set for T. Assume also that $T = (T_1, \dots, T_N)$ is of C_{00} class. If the intersection of the Taylor essential* *spectrum with the open polydisc $\sigma_e(T)\cap\mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $T=(T_1,\ldots,T_N)$ has a common non-trivial invariant subspace.*

To reduce Theorem 8.2.1 to Theorem 8.2.2, let us note first that if one of the contractions $T_1,\ldots,T_N$, say T_{i_0}, has a non-trivial unitary part, then from the formula for the subspace $\mathcal{H}_{u\,i_0}$ on which the contraction T_{i_0} is unitary (cf. [71, Theorem I.3.2]), we see that $\mathcal{H}_{u\,i_0}$ is invariant for all operators S doubly commuting with T_{i_0} and $Lat(T)$ will be non-trivial. Hence we may assume that $T_1,\ldots,T_N$ are all c.n.u. Also, because of the proposition below we may assume that $\sigma(T)=\sigma_e(T)$ and we are in the situation of Theorem 8.2.2.

Proposition 8.2.3. *Let $T=(T_1,\ldots,T_N)$ be an N-tuple of doubly commuting operators in $L(\mathcal{H})$. If $\sigma(T)\backslash\sigma_e(T)\neq\emptyset$, then T_{i_0} has a non-trivial hyperinvariant subspace for some $i_0\in\{1,\ldots,N\}$.*

Since the space $\mathcal{H}$ is infinite-dimensional, the proposition is a consequence of the following

Lemma 8.2.4. *Let $T=(T_1,\ldots,T_N)$ be an N-tuple of doubly commuting operators in $L(\mathcal{H})$. If, for some $\lambda\in\mathbb{C}^N$ and some $p\in\{1,\ldots,N\}$, we have*

$$ker\ \delta^p(\lambda-T)\cap ran\ \delta^{p-1}(\lambda-T)^\perp\neq\{0\},$$

then either there is $i_0\in\{1,\ldots,N\}$ such that the operator T_{i_0} has a non-trivial hyperinvariant subspace or the tuple consists of scalar operators.

Proof. By our assumptions, there exists some $0\neq\omega\in\Lambda^p(\mathcal{H})$ with $\delta^p(\lambda-T)\omega=0=\delta^{p-1}(\lambda-T)^*\omega$. Hence

$$\delta^{p-1}(\lambda-T)\delta^{p-1}(\lambda-T)^*\omega+\delta^p(\lambda-T)^*\delta^p(\lambda-T)\omega=0.$$

By Lemma 3.2.2, there are disjoint sets $\mathcal{S}$, $\mathcal{T}$ with $\mathcal{S}\cup\mathcal{T}=\{1,\ldots,N\}$ and a vector $0\neq x\in\mathcal{H}$ such that

$$\sum_{i\in\mathcal{S}}(\lambda_i-T_i)^*(\lambda_i-T_i)x+\sum_{k\in\mathcal{T}}(\lambda_k-T_k)\,(\lambda_k-T_k)^*x=0.$$

Hence $x\in ker\,(\lambda_i-T_i)$ for $i\in\mathcal{S}$ and $x\in ker\,(\lambda_k-T_k)^*$ for $k\in\mathcal{T}$. From this we obtain a non-trivial hyperinvariant subspace or $T_j=\lambda_j$ for all $j\in\{1,\ldots,N\}$.

The next lemma will be a preparation to the proof of Theorem 8.2.2.

Lemma 8.2.5. *Assume that $T=(T_1,\ldots,T_N)$ is an N-tuple of doubly commuting completely non-unitary contractions. Let $\lambda\in\sigma_e(T)\cap\mathbb{D}^N$.*

Then there are disjoint sets $\mathcal{S}, \mathcal{T}$ such that $\mathcal{S} \cap \mathcal{T} = \{1, \ldots, N\}$, and $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{D}^N$ and sequence $\{x_n\}$ with $x_n \to 0$ weakly, $\|x_n\| = 1$ for all n such that $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$.

Proof. By Lemma 3.2.1, there is a number $p \in \{1, \ldots, N\}$ and an orthonormal sequence $\{\eta_n\}_{n=1}^{\infty}$ in $\Lambda^p(\mathcal{H})$ such that (3.2.1) holds. Passing, if necessary, to some subsequence, we may assume that for some $I = (i_1, \ldots, i_p) \in \mathbb{N}^p$ the coefficients x_n of $s_{i_1} \wedge \cdots \wedge s_{i_p}$ in η_n satisfy $\|x_n\| \geq \alpha$ for all $n \in \mathbb{N}$ and some $\alpha > 0$. By Lemma 3.2.2, there are disjoint sets $\mathcal{S}$, $\mathcal{T}$ with $\mathcal{S} \cup \mathcal{T} = \{1, \ldots, N\}$ such that

$$\sum_{i \in \mathcal{S}} (\lambda_i - T_i)^*(\lambda_i - T_i)x_n + \sum_{k \in \mathcal{T}} (\lambda_k - T_k)(\lambda_k - T_k)^* x_n \to 0.$$

Taking the scalar product with x_n, we get $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_k^* - \overline{\lambda}_k)x_n\| \to 0$ for all $k \in \mathcal{T}$. Since the sequence $\{\eta_n\}_{n=1}^{\infty}$ is orthonormal, thus $x_n \to 0$ weakly. Moreover, since the numbers $\|x\|$ are bounded below, we can assume without loss of generality that $\|x_n\| = 1$.

Proof of Theorem 8.2.2. We can construct the representation $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ generated by $T = (T_1, \ldots, T_N)$ as in Section 4.2. Moreover, by Lemma 4.3.2 T is a.c., since each T_i is c.n.u. Thus we can extend Φ to $H^\infty(\mathbb{D}^N)$. By Lemma 4.3.5, since $\sigma_e(T)$ is dominating for $H^\infty(\mathbb{D}^N)$ we can see that Φ is an isometry and a weak-star homeomorphism. Hence, as it was mentioned in Section 5.2, to show the reflexivity of $T = (T_1, \cdots, T_N)$ it is enough to check the approximation properties (5.2.1)–(5.2.3).

We will start with the approximation of the point evaluation.

Lemma 8.2.6. *Let $T = (T_1, \ldots, T_N)$ be an a.c. N-tuple of commuting contractions, $\mathcal{S}$, $\mathcal{T}$ be disjoint sets such that $\mathcal{S} \cup \mathcal{T} = \{1, \ldots, N\}$, and $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{D}^N$. Assume thht $x_n \to 0$ weakly and $\|x_n\| = 1$ for all n. If $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$, then $\lim_{n\to\infty} \|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} = 0$.*

Proof. By the Hahn-Banach theorem, for each n, there exists some $f_n \in H^\infty(\mathbb{D}^N)$ such that $\|f_n(T)\| = \|f_n\| = 1$ and $\|[x_n \oplus x_n] - [C_\lambda]\|_{\mathcal{Q}} = | < f_n(T), [x_n \oplus x_n] - [C_\lambda] > |$. Since the polydisc has the Gleason property, there are $g_i^n \in H^\infty(\mathbb{D}^N)$ satisfying $\|g_i^n\| \leq M_\lambda$ for $i = K + 1, \cdots, N$ and

$f_n(z) = f_n(\lambda) - \sum_{i=1}^N (z_i - \lambda_i) g_i^n(z)$, where $z = (z_1, \dots, z_N) \in \mathbb{D}^N$. Hence

$$\begin{aligned}
\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} &= \left| < f_n(\lambda) + \sum_{i=1}^N (T_i - \lambda_i) g_i^n(T), [x_n \otimes x_n] - [C_\lambda] > \right| \\
&= \left| \left(\sum_{i=1}^N (T_i - \lambda_i) g_i^n(T) x_n, x_n \right) \right| \\
&\le \sum_{i \in \mathcal{S}} |(g_i^n(T)(T_i - \lambda_i) x_n, x_n)| + \sum_{i \in \mathcal{T}} |(g_i^n(T) x_n, (T_i^* - \overline{\lambda}_i) x_n)| \\
&\le \sum_{i \in \mathcal{S}} \|(g_i^n(T)\| \, \|(T_i - \lambda_i) x_n\| + \sum_{i \in \mathcal{T}} \|(g_i^n(T)\| \, \|(T_i^* - \overline{\lambda}_i) x_n)\| \\
&\le M_\lambda \left(\sum_{i \in \mathcal{S}} \|(T_i - \lambda_i) x_n\| + \sum_{i \in \mathcal{T}} \|(T_i^* - \overline{\lambda}_i) x_n\| \right).
\end{aligned}$$

Thus, the proof is finished since $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$.

Next, the approximate orthogonality (5.2.2) will be shown.

Lemma 8.2.7. *Let $T = (T_1, \dots, T_N)$ be an a.c. N-tuple of commuting contractions. Let $\mathcal{S}$, $\mathcal{T}$ be disjoint sets such that $\mathcal{S} \cup \mathcal{T} = \{1, \dots, N\}$. Assume that $\{T_i : i \in \mathcal{S}\}$ is doubly commuting. Let $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}^N$ and $x_n \to 0$ weakly, $\|x_n\| = 1$ for all n. If $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$, then $\lim_{n\to\infty} \|[y \otimes x_n]\|_{\mathcal{Q}} = 0$ for all $y \in \mathcal{H}$.*

Proof. Without loss of generality, we can assume that $\mathcal{S} = \{1, \cdots, K\}$. There can be found some functions $f_n \in H^\infty(\mathbb{D}^N)$ such that $\|f_n(T)\| = \|f_n\| = 1$ and $\|y \otimes x_n]\|_{\mathcal{Q}} = |(f_n(T)y, x_n)|$. Notice that

$$\begin{aligned}
\|[y \otimes x_n]\|_{\mathcal{Q}} \le\ & |(f_n(T_1, \dots, T_K, \lambda_{K+1}, \dots, \lambda_N) y, x_n)| \\
& + |((f_n(T_1, \dots, T_N) - f_n(T_1, \dots, T_K, \lambda_{K+1}, \dots, \lambda_N)) y, x_n)|.
\end{aligned}$$

By an obvious modification of the Gleason property for polydomains, there are functions $g_{K+1}^n, \dots, g_N^n \in H^\infty(\mathbb{D}^N)$ such that $\|g_i^n\| \le M_\lambda$ for $i = K+1, \dots, N$ and

$$f_n(z) - f_n(z_1, \dots, z_K, \lambda_{K+1}, \dots, \lambda_N) = \sum_{i=K+1}^N (\lambda_i - z_i) g_i^n(z),$$

where $z = (z_1, \dots, z_N)$. The point λ being fixed, let us denote by $h_n \in H^\infty(\mathbb{D}^K)$ the function $h_n(z_1, \dots, z_K) = f_n(z_1, \dots, z_K, \lambda_{K+1}, \dots, \lambda_N)$ and write $T = (T_1, \dots, T_K)$. Obviously, we have a natural functional calculus for $\tilde{T} =$

$(T_1, \dots, T_K)$. Hence, for any $\varepsilon > 0$, we have

$$\begin{aligned}\|[y \otimes x_n]\|_{\mathcal{Q}} &\leq |(f_n(T_1, \dots, T_K, \lambda_{K+1}, \dots, \lambda_N)y, x_n)| \\ &\quad + \left| \left(\sum_{i=K+1}^{N} (\lambda_i - T_i) g_i^n(T) y, x_n \right) \right| \\ &\leq |(h_n(\tilde{T})y, x_n)| + \sum_{i=K+1}^{N} \|g_i^n(T)\| \, \|y\| \, \|(\overline{\lambda}_i - T_i^*)x_n\| \\ &\leq |(h_n(\tilde{T}y, x_n)| + \varepsilon\end{aligned}$$

for n sufficiently large, since $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for $i = K+1, \dots, N$.

By the same construction as in [68, p.1234], we can construct a doubly commuting K-tuple of isometries $V = (V_1, \dots, V_K) \subset L(\mathcal{K})$, which is a minimal isometric dilation of the K-tuple $T^* = (T_1^*, \dots, T_K)$. Then $V^* = (V_1^*, \dots, V_K^*)$ is an extension of $\tilde{T} = (T_1, \dots, T_K)$. By the minimality and the decomposition (4.3.1), the K-tuple $V = (V_1, \dots, V_K)$ is a.c. and so is V^* by Lemma, 4.3.5. Moreover, by Theorem 4.3.4 we can construct a functional calculus for each of them.

For any contraction A and $\mu \in \mathbb{D}$, we will denote by A^μ the operator $(A - \mu)(I - \overline{\mu}A)^{-1}$. Let $\mathcal{K} = \mathcal{K}_s^i \oplus \mathcal{K}_u^i$ be the decomposition of V_i into a unilateral shift and a unitary part. The decomposition coincides with the decomposition of $V_i^{\lambda_i}$ (see [71, Proposition I.4.3 and its proof]). Moreover, $(V^\lambda)^* = ((V_1^{\lambda_1})^*, \dots, (V_K^{\lambda_K})^*)$ is an extensioll of $\tilde{T}^\lambda = (T_1^{\lambda_1}, \dots, T_K^{\lambda_K})$. Since $\|(T_i - \lambda_i)x_n\| \to 0$ for $i = 1, \dots, K$, we have $\|T_i^{\lambda_i} x_n\| \to 0$ for $i = 1, \dots, K$.

By the double commutativity of $V = (V_1, \dots, V_K)$ and [69, Theorem 3], we can write $\mathcal{K} = \mathcal{K}_s \oplus \mathcal{K}_r$, where $V_i^s = V_i|_{\mathcal{K}_s}$ is a shift operator for all $i = 1, \dots, K$, and $V_{i_0}|_{\mathcal{K}_r}$ is a unitary operator for some $i_0 \in \{1, \dots, K\}$. Moreover, $\mathcal{K}_s$ and $\mathcal{K}_r$. reduce V_i. We will denote $V_i^r = V_i|_{\mathcal{K}r}$. Form $x_n = x_n^s \oplus x_n^r$, $y = y^s \oplus y^r$ with respect to this orthogonal decomposition. Let P_i denote the projection onto $\ker(V_i^{\lambda_i})^*$. By the double commutativity, one can see that P_i and P_j commute for $i, j = 1, \dots, K$. We also have $P_1 \cdots P_K \mathcal{H} \subset \mathcal{K}_s$. Thus

$$\begin{aligned}\|x_n^r\| \leq \|x_n - P_1 \cdots P_K x_n\| &\leq \sum_{i=1}^{K} \|P_1 \cdots P_{i-1}(x_n - P_i x_n)\| \\ \leq \sum_{i=1}^{K} \|x_n - P_i x_n\| &= \sum_{i=1}^{K} \|V_i^{\lambda_i}(V_i^{\lambda_i})^* x_n\| \\ = \sum_{i=1}^{K} \|(V_i^{\lambda_i})^* x_n\| &= \sum_{i=1}^{K} \|T_i^{\lambda_i} x_n\| \to 0.\end{aligned}$$

There is also a natural functional calculus for $V_s^* = (V_1^{s*}, \dots, V_K^{s*})$ and for $V_r^* = (V_1^{r*}, \dots, V_K^{r*})$, since V_s^* and V_s^* are the restrictions of V^*. Hence, because of $\|x_n^r\| \to 0$, we have

$$\begin{aligned}\|(h_n(\tilde{T})y, x_n)\| &= |(h_n(V^*)y, x_n)| \\ &\leq |(h_n(V_s^*)y^s, x_n^s)| + |(h_n(V_r^*)y^r, x_n^r)| \\ &\leq |(h_n(V_s^*)y^s, x_n^s)| + \|h_n\|\ \|y^r\|\ \|x_n^r\| \\ &\leq |(h_n(V_s^*)y^s, x_n^s)| + \varepsilon\end{aligned}$$

for n sufficiently large, since $\|x_n^r\| \to 0$, we have $\|x_n^s\| \to 1$ and hence we may assume that $\|x_n^s\| = 1$.

Now we know that V_i^{s*} is a unilateral shift for $i = 1, \cdots, K$. For $M \in \mathbb{N}$, let R_M be the orthogonal projection onto the space $\bigvee_{i=1}^K \operatorname{ran} V_i^{sM}$. Using the obvious extension of [69, Theorem 1] from pairs to K-tuples of doubly commuting shifts, there is a sufficiently large M such that $\|R_M\, y^s\| \leq \frac{\varepsilon}{2}$. Let $y_1 = (I - R_M)y^s$ and $y_2 = R_M\, y^s$. Then $V_i^{s*M}\, y_1 = 0$ for $i = 1, \ldots, K$. We can write

$$h_n(z) = \sum_{|I|=0}^{M-1} a_I^n z^I + \sum_{i=1}^{L} z_i^M\, q_i^n(z),$$
$$\textit{where } z = (z_1, \ldots, z_K),\ a_I^n \in \mathbb{C},\ q_i^n \in H^\infty(\mathbb{D}^K).$$

Moreover, we can estimate that $|a_I^n| \leq 1$.

Since $x_n^s \to 0$ weakly, we have, for n sufficiently large, $|(V_1^{*I} y_1, x_n^s)| \leq \frac{\varepsilon}{2M^K}$ for all I such that $|I| \leq M - 1$. Hence

$$\begin{aligned}|(h_n(V_s^*)y^s, x_n^s)| &\leq |(h_n(V_s^*)y_s, x_n^s)| + \sum_{|I|=0}^{M-1} |a_I^n|\ |(V_s^{*I} y_1, x_n^s)| \\ &\quad + \sum_{i=1}^{K} |(q_i^n(V_s^*) V_i^{s*M}\, y_1, x_n^s)| \\ &\leq \|h_n\|\ \|y_s\|\ \|x_n^s\| + \sum_{i=1}^{K} \frac{\varepsilon}{2M^K} + 0 \leq \varepsilon.\end{aligned}$$

Thus, for n sufficiently large, $\|[y \otimes x_n]\|_{\mathcal{Q}} \leq 3\varepsilon$.

The second orthogonality condition (5.2.3) turns out to be symmetric to the previous one.

Lemma 8.2.8. *Let $T = (T_1, \ldots, T_N)$ be an a.c. N-tuple of commuting contractions. Let $\mathcal{S}'$, $\mathcal{T}'$ be disjoint sets such that $\mathcal{S}' \cup \mathcal{T}' = \{1, \ldots, N\}$. Assume that $\{T_i : i \in \mathcal{T}'\}$ is doubly commuting. Let $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{D}^N)$ and $x_n \to 0$ weakly, $\|x_n\| = 1$ for all n. If $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}'$ and $\|(T_i^* - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{T}'$, then* $\lim_{n\to\infty} \|[x_n \otimes y]\|_{\mathcal{Q}} = 0$ *for all $y \in \mathcal{H}$.*

Proof. Since the set $\{T_i^* : i \in \mathcal{T}'\}$ is also doubly commuting, we can apply Lemma 8.2.7 to $\mathcal{S} = \mathcal{T}'$ and $\mathcal{T} = \mathcal{S}'$. Hence we get $\|[x_n \otimes y]\|_{\mathcal{Q}} = |[y \otimes x_n]\|_{\mathcal{Q}(\mathcal{A}(T^*))} \to 0$.

8.3. Application to weighted shifts.

Let R denote a weighted shift on l^2_+. Put $Re_n = s_n e_{n+1}$, where $(e_n)_{n=1}^{\infty}$ is an orthonormal basis of l^2_+. Following [62], let

$$i(R) = \lim_{n\to\infty} \inf_k |s_{k+1}\cdots s_{k+n}|^{\frac{1}{n}} \quad \textit{and} \quad r(R) = \lim_{n\to\infty} \sup_k |s_{k+1}\cdots s_{k+n}|^{\frac{1}{n}}.$$

We need the following result from [62].

Lemma 8.3.1.

(1) *For any positive numbers ε, M, there are integers k, n, both greater than M, such that $|s_{k+1}\cdots s_{k+n}|^{\frac{1}{n}} \geq r(R) - \varepsilon$.*

(2) *If no s_n vanishes and ε, M are positive numbers, then there are integers m, p, both greater than M, such that $|s_{m+1}\cdots s_{m+p}|^{\frac{1}{p}} \leq i(R) + \varepsilon$.*

Note the following

Proposition 8.3.2.

(1) *If no s_n vanishes, then $\sigma_{ap}(R) = \{c : i(R) \leq |c| \leq r(R)\} = \sigma_{le}(R)$.*

(2) *If finitely many s_n vanish, then $\sigma_{ap}(R) - \{0\} = \sigma_{le}(R) = \sigma_{le}(R')$, where R' is the shift with weights s_{k+1}, $s_{k+2}, \ldots$, and s_k is the last zero weight of R.*

(3) *If infinitely many s_n vanish, then $\sigma_{ap}(R) = \{c : |c| \leq r(R)\} = \sigma_{le}(R)$.*

Proof. In the following proof we develop the ideas used in [62], where the first equality of (1) was shown. Let us note that both $\sigma_{ap}(R)$ and $\sigma_{le}(R)$ have the circular symmetry [67, Corollary 2, p.52]. Hence, if $i(R) = r(R)$, then the nonemptiness of $\sigma_{le}(R)$ shows the needed equality.

Since $\sigma_{ap}(R) \supset \sigma_{le}(R)$ and $\sigma_{le}(R)$ is closed, to finish the proof of the first equality in the case when $i(R) < r(R)$, it is enough to show that $i(R) < c < r(R)$ implies $c \in \sigma_{le}(R)$.

Take a, b with $i(R) < a < c < b < r(R)$. An orthogonal sequence $\{x^l\}$, $x^l = (x^l_r)$, such that

$$\|Rx^l - cx^l\| \leq \frac{1}{l}\,\|x^l\|\,\|R\| \quad \textit{for} \quad l \in \mathbb{N} \tag{8.3.1}$$

will be constructed by induction. Let $x_1^1 = 1,\ x_r^1 = 0$ for $r > 1$ and $k(1) = n(1) = p(1) = m(1) = 1$. Let us assume that vectors x^j, and positive integers $k(j)$, $n(j)$, $m(j)$, $p(j)$ are defined for all $j < l$.

By Lemma 8.3.1 (1), we can choose $n(l)$, $k(l)$ such that $(\frac{c}{b})^{n(l)} < \frac{1}{l},\ k(l) > m(l-1) + p(l-1)$, and $|s_{k(l)+1} \cdots s_{k(l)+n(l)}|^{\frac{1}{n(l)}} > b$. Now, by Lemma 8.3.1 (2), we can choose $p(l)$, $m(l)$ such that $(\frac{a}{c})^{p(l)} < \frac{1}{l},\ m(l) > k(l) + n(l)$, and $|s_{m(l)+1} \cdots s_{m(l)+p(l)}|^{\frac{1}{p(l)}} < a$.

Define $x^l = (x_r^l)$ as follows:

$$
\begin{aligned}
&x_{k(l)+1}^l = 1,\\
&x_r^l = \frac{s_{k(l)+1} \cdots s_{r-1}}{c^{r-k-1}} \quad if \quad k(l) + 2 \le r \le m(l) + p(l) + 1,\\
&x_r^l = 0 \quad if \quad r < k(l) + 1 \quad or \quad r > m(l) + p(l) + 1.
\end{aligned}
$$

The vectors x^l are mutually orthogonal, since $k(l) > m(l-1) + p(l-1)$. Some calculation as in [62, p.350] shows that (8.3.1) is fulfilled. Hence $c \in \sigma_{le}(R)$.

If finitely many s_n vanish, then R is the orthogonal sum of R' and a nilpotent operator defined on a finite-dimensional space. Hence 0 is not in $\sigma_{le}(R)$ and thus $\sigma_{le}(R) = \sigma_{le}(R')$.

Now we show the last statement. Suppose that infinitely many s_n vanish. It is easy to see that $0 \in \sigma_{le}(R)$. As above, it is enough to show that each c with $0 < c < r(R)$ is in $\sigma_{le}(R)$. A similar construction as before yields an orthogonal sequence $\{x^l\}$, $x^l = (x_r^l)$ such that

$$
\|Rx^l - cx^l\| \le \frac{1}{l}\, c\, \|x^l\| \quad for \quad l \in \mathbb{N}. \tag{8.3.2}
$$

Let $x_1^1 = 1,\ x_r^1 = 0$ for $r > 1$ and $k(1) = n(1) = m(1) = 1$. Let l be any positive integer and assume that vectors x^j and positive integers $k(j)$, $n(j)$, $m(j)$ are defined for all $j < 1$. By Lemma 8.3.1 (1), we can choose $n(l)$, $k(l)$ such that $(\frac{c}{b})^{n(l)} < \frac{1}{l}, k(l) > m(l-1)$, and $|s_{k(l)+1} \cdots s_{k(l)+n(l)}|^{\frac{1}{n(l)}} b$. Let $m(l)$ be the first index greater then $k(l) + n(l)$ such that $s_{m(l)} = 0$. Define $x^l = (x_r^l)$ as follows:

$$
\begin{aligned}
&x_{k(l)+1}^l = 1,\\
&x_r^l = \frac{s_{k(l)+1} \cdots s_{r-1}}{c^{r-k-1}} \quad if \quad k(l) + 2 \le r \le m(l),\\
&x_r^l = 0 \quad if \quad r < k(l) + 1 \quad or \quad r > m(l).
\end{aligned}
$$

The vectors x^l are mutually orthogonal, since $k(l) > m(l-1)$. Calculating as in [62, p.350] one can show that (8.3.2) is fulfilled. Hence $c \in \sigma_{le}(R)$.

Let us conclude with the following example.

Example 8.3.3. Let R_1, R_2 be weighted shifts such that $i(R_j) < r(R_j) = 1$, $(j = 1, 2)$. [Lemma 7, 62] shows the possibility of a construction of such shifts. We may let $R_1 = R_2 = R$, where R has weights $s_n = \frac{1}{2}$ if $2(2^k - 1) < n < 2(2^k - 1) + 2^k$ for $k = 1, 2, \ldots$, and $s_n = 1$ otherwise. It is easy to see that $i(R) = \frac{1}{2}$, $r(R) = 1$. By Propositions 3.1.3 and 8.3.2, we have $\sigma_{le}(R_1 \oplus I, I \oplus R_2) = \{(\lambda_1, \lambda_2) : i(R_j) < \lambda_j < r(R_j) = 1, j = 1, 2\}$. Hence, the assumption of Theorem 8.2.2 is fulfilled for the pair $R_1 \oplus I$, $I \oplus R_2$.

9. Questions and Open Problems.

In Section 2, we considered the finite-dimensional case. In fact, we completely characterized the reflexive algebras generated by N-tuples of doubly commuting linear transformations on finite-dimensional Hilbert spaces. A natural further generalization will be to drop the double commutativity. The condition 2.5.2.(1) seems to be not suitable, since it depends on the Jordan sequence, whose definition is based on the double commutativity. In Theorems 2.1 and 2.6.4, we proved that a necessary condition for the reflexivity of an algebra generated by commuting linear transformations is that each rank-two member generates a one-dimensional ideal. This condition is also sufficient in the doubly commuting case. Thus our first conjecture will be

Conjecture 9.1. *Suppose $\mathcal{A}$ is an operator algebra in a finite-dimensional Hilbert space generated by a commuting family of (nilpotent) linear transformations. Then $\mathcal{A}$ is reflexive if and only if each rank-two member of $\mathcal{A}$ generates a one-dimensional ideal.*

We can also search for different conditions which completely characterize reflexive families of linear transformations.

We can also restrict ourselves to the nilpotent case in an infinite-dimensional Hilbert space and ask

Conjecture 9.2. *Suppose $\mathcal{A}$ is an operator algebra in a Hilbert space generated by a commuting family of nilpotents. Then $\mathcal{A}$ is reflexive if and only if each rank-two member of $\mathcal{A}$ generates a one-dimensional ideal.*

The next questions concern Theorem 7.1. It is unknown whether we can assume only that each operator is quasinormal instead of joint quasinormality.

Conjecture 9.3. *Every family $\mathcal{S}$ of commuting quasinormal operators is reflexive and has property $\mathbb{A}_1(1)$.*

Now we can try to drop the double commutativity assumption in Proposition 7.3.

Conjecture 9.4. *Every N-tuple $V = (V_1, \dots, V_N)$ of commuting spherical isometries is reflexive.*

The next set of questions concerns the Dual Algebra Technique. Let us recall the most striking results for a single operator.

Theorem [24]. *Let T be a contraction. If $\mathbb{T} \subset \sigma(T)$, then T has a non-trivial invariant subspace.*

Theorem [25]. *Let T be an a.c. contraction. If $\mathbb{T} \subset \sigma_e(T)$, then T is reflexive.*

Thus the natural conjectures for N-tuples should be:

Conjecture 9.5. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of commuting contractions. If $\mathbb{T}^N \subset \sigma(T)$, then T has a common non-trivial invariant subspace.*

Conjecture 9.6. *Let $T = (T_1, \dots, T_N)$ be an a.c. N-tuple of commuting contractions. If $\mathbb{T}^N \subset \sigma_e(T)$, then $\mathcal{W}(T)$ is reflexive.*

These problems seem to be out of reach nowadays. Hence, let us state the following easier problems which are also not known. We want to drop the double commutativity from Theorems 8.2.1 and 8.2.2.

Conjecture 9.7. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of commuting contractions. If the intersection of the Taylor spectrum with the open polydisc $\sigma(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then T has a common non-trivial invariant subspace.*

Conjecture 9.7'. *We can ask about the reflexivity of $T = (T_1, \dots, T_N)$ assuming the dominancy of Taylor essential spectrum $\sigma_e(T)$.*

The answers for the above are not known even if we consider Harte spectrum $\sigma_H(T)$ instead of Taylor spectrum $\sigma(T)$.

The above conjectures were stated for a polydisc. The unit ball $\mathbb{B}^N$ is also a natural generalization of the unit disc $\mathbb{D}$. We have also a notion of the spherical contraction. Namely, a commuting N-tuple $T = (T_1, \dots, T_N)$ is called a *spherical contraction* if $\sum_{i=1}^N \|T_1 x\|^2 \le \|x\|^2$ for all vector x. Hence, all the above conjectures given for N-tuples of contractions can be stated for spherical contractions. For example, we have

Conjecture 9.8. *Let $T = (T_1, \dots, T_N)$ be a spherical contracction. If the intersection of the Taylor spectrum with the open unit ball $\sigma(T) \cap \mathbb{B}^N$ is dominating for $H^\infty(\mathbb{B}^N)$, then T has a common non-trivial invariant subspace.*

We will finish this section by recalling the famous result from [54] that every subnormal operator is reflexive. It is a generalization of [21] that every subnormal operator has a non-trivial invariant subspace. In [76], the existence of common non-trivial invariant subspaces was shown for jointly subnormal family. Thus we can state.

Conjecture 9.9. *Jointly subnormal families are reflexive.*

References

1. E. Albrecht, B. Chevreau, Invariant subspaces for certain representations of $H^\infty(G)$, Functional Analysis, Editors: Bierstedt, Pietsch, *Ruess, Vogt, Marcel Dekker, Inc., New York*, 1993, pp. 293–305.
2. E. Albrecht, M. Ptak, Invariant subspaces for doubly Commuting contractions with rich Taylor spectrum, preprint.
3. F. W. Anderson, K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York, 1974.
4. C. Apostol, Functional calculus and invariant subspaces, *J. Operator Theory* **4** (1980), 159–190.
5. E. A. Azoff, On finite-rank operators and preannilhilators, *Mem. Amer. Math. Soc.* **357** (1986).
6. E. A. Azoff, C. K. Fong, F. Gilfeather, A reduction theory for non-self-adjoint operator algebras, *Trans. Amer. Math. Soc.* **224** (1976), 351-366.
7. E. A. Azoff, M. Ptak, On the rank-two linear transformations and reflexivity, *J. London Math. Soc.* **53** (1996), 383–396.
8. E. A. Azoff, M. Ptak, Jointly Quasinormal families are reflexive, *Acta Sci. Math.* (Szeged) **61** (1995), 545–547.
9. E. A. Azoff, H. A. Shehada, Literal embeddings of linear spaces of operators, *Indiana Univ. Math. J.* **42** (1993), 571–589.
10. O. B. Bekken, Product of algebras on plane sets, U.C.L.A. Thesis (1972).
11. O. B. Bekken, Rational approximation on product sets, *Trans. Amer. Math. Soc.* **191** (1974), 301–316.
12. S. K. Berberian, Notes on spectral theory, D. Van Nostrand Company, Inc., Princeton, 1966.
13. H. Bercovici, A factorization theorem with applications to invariant subspaces and reflexivity of isometries, *Math. Research Letters*, **1** (1994), 511–518.
14. H. Bercovici, B. Chevreau, C. Foiaş, C. Pearcy, Dilation theory and systems of simultaneous equations in the predual of an operator algebra, II, *Math. Z.* **189** (1984), 97–103.

15. H. Bercovici, C. Foiaş, J. Langsam, C. Pearcy, (BCP)-Operators are reflexive, Michigan *Math. J.* **29** (1982), 371-379.

16. H. Bercovici, C. Foiaş, C. Pearcy, Dilation theory and systems of simultaneous equations in the predual of an operator algebra, I, *Michigan Math. J.* **30** (1983), 335–354.

17. H. Bercovici, C. Foiaş, C. Pearcy, Dual algebras with applications to invariant subspaces and dilation theory, CBMS-NSF Lecture Notes no. 56, AMS, Providence, 1985.

18. H. Bercovici, W. S. Li, Isometric functional calculus on the bidisc, *Bull. London Math. Soc.* **25** (1993), 582–590.

19. H. Bercovici, D. Westwood, The factorization of functions in the polydisc, *Houston J. Math.* **18** (1992), 1–6.

20. E. Breim, A. M. Davie, B. K. Øksendal, Functional calculus for commuting contractions, *J. London Math. Soc.* **7** (1973), 709–718.

21. S. Brown, Some invariant subspaces for subnormal operators, Integral Equations Operator Theory **1** (1978), 310–330.

22. A. Brown, C. Pearcy, Compact restrictions of operators, *Acta Sci. Math.* (Szeged) **33** (1971), 271-282.

23. S. Brown, B. Chevreau, C. Pearcy, Contractions with rich spectrum have invariant subspaces, *J. Operator Theory*, **1** (1979), 123–136.

24. S. Brown, B. Chevreau, C. Pearcy, On the structure of contraction operators, II, *J. Func. Anal.* **76** (1988), 30–55.

25. B. Chevreau, G. Exner, C. Pearcy, On the structure of contraction operators, III, *Michigan Math J.* **36** (1989), 29–62.

26. B. Chevreau, C. Pearcy, On the structure of contraction operators with applications to invariant subspces, *J. Func. Anal.* **67** (1986), 36O–379.

27. B. Chevreau, C. Pearcy, On the structure of contraction operators, I, *J. Func. Anal.* **76** (1988), 1–29.

28. B. Chevreau, C. Pearcy, A. L. Shields, Finitely connected domains G, representation of $H^{\infty}(G)$, and invariant subspaces, *J. Operator Theory* **6** (1981), 375–405.

29. B. J. Cole, T. W. Gamelin, Tight Uniform Algebras of Analytic Functions, *J. Func. Anal.* **46** (1982), 158–220.

30. R. Curto, Fredholm and invertible n-tuples of operators. The deformation problem, *Trans. Amer. Math. Soc.* **266** (1981), 129–159.

31. R. Curto, Joint hyponormality: a bridge between hyponormality and subnormality, *Proc. Symp. Pure Math.* **51** (1990), 61–91.

32. R. Curto, Applications of several complex variables to multiparameter spectral theory, Survey Or recent result in operator theory, V. II, Editors: J. B. Conway, B. B. Morrel, Longman, London, 1988, pp. 25–90.

33. A. T. Dash, Joint spectra, *Studia Math.* **45** (1973), 225–237.

34. J. A. Deddens, Every isometry is reflexive, *Proc. Amer. Math. Soc.* **28** (1971), 509–511.

35. J. A. Deddens, P. A. Fillmore, Reflexive linear transformations, Linear Algebra *Appl.* **10** (1975), 89–93.

36. L. Ding, Separating vectors and reflexivity, *Linear Algebra Appl.* **174** (1992), 37–52.

37. L. Ding, An algebraic reflexivity result, Houston *J. Math.* **19** (1993), 533–540.

38. J. Eschmeier, Representations of $H^\infty(G)$ and invariant subspaces, *Math. Ann.* **298** (1994), 167–186.

39. J. Eschmeier, C_{00}-representations of $H^\infty(G)$ with dominating Harte spectrum, Preprint.

40. P. A. Fillmore, J. G. Stampfli, J. P. Williams, On the essential numerical range, the essential spectrum, and a problem of Halmos, *Acta Sci. Math.* (Szeged) **32** (1972), 179–192.

41. T. W. Gamelin, Uniform Algebras, Prentice Hall, Englewood Clifs, N. J., 1969.

42. D. Hadwin, E. A. Nordgren, Subalgebras of reflexive algebras, *J. Operator Theory*, **7** (1982), 3–23.

43. K. Horák, V. Müller, On commuting isometries, *Czechoslovak Math. J.* **43(118)** (1993), 373–382.

44. M. Kosiek, Lebesgue-Type Decomposition of a Pair of Commuting Hilbert Space Operators, *Bull. Acad. Polon. Sai., Ser. Sci. Math*, **27** (1979), 583–589.

45. M. Kosiek, Representation generated by a finite number of Hilbert space operators, *Ann. Polon. Math.* **44** (1984), 309–315.

46. M. Kosiek, A. Octavio, M. Ptak, On the reflexivity of pairs of contractions, *Proc. Amer. Math. Soc.* **123** (1995), 1229–1236.

47. M. Kosiek, M. Ptak, Reflexivity of N-tuples of contractions with rich joint left essential spectrum, Integral Equations Operator Theory **13** (1990), 395–420.

48. S. G. Krantz, Function theory of several complex variables, John Wiley and Sons, New York, 1982.

49. W. S. Li, J. McCarthy, Reflexivity of isometries, preprint.

50. W. S. Li, E. Strouse, Reflexivity of tensor products of linear transformations, *Proc. Amer. Math. Soc.* **123** (1995), 2021–2030.

51. A. I. Loginov, V. I. Sulman, On hereditary and intermediate reflexivity of W^* algebras, *Soviet Math. Dokl.* **14** (1973), 1473–1476.

52. A. Lubin, Weighted shifts and commuting normal extentions, *J. Austral. Math. Soc.* (Ser. A) **27** (1979), 17–26.

53. A. Octavio, Coisometric extension and functional calculus for pairs of commuting contractions, (to appear).

54. R. Olin, J. Thomson, Algebras of subnormal operators, *J. Func. Anal.* **37** (1980), 271–301.

55. S. Parrot, Unitary dilation for commuting contractions, *Pacific J. Math.* **34** (1970), 481–490.

56. M. Ptak, On the reflexivity of pairs of isometries and of tensor products for some reflexive algebras, *Studia Math.* **37** (1986), 47–55.

57. M. Ptak, Reflexivity of pairs of shifts, *Proc. Amer. Math. Soc.* **109** (1990), 409–415.

58. M. Ptak, Reflexivity of multiplication operators in certain domains in $\mathbb{C}^N$, *Bull. Polish Acad. Sci., Math.* **37** (1989), 217-220.

59. M. Ptak, The algebra generated by pair of operator weighted shifts, *Ann. Polon. Math.* **62** (1995), 97–110.

60. H. Radjavi, P. Rosenthal, Invariant Subspaces, Springer-Verlag, New York, 1973.

61. M. Reed, B. Simon, Methods of Modern Mathematical Physics I, Functional Analysis, Academic Press, New York, 1972.

62. W. Ridge, Approximate Point Spectrum of a Weighted Shift, *Trans. Amer. Math. Soc.* **147** (1970), 349-356.

63. G. Robel, On the structure of (BCP)-operators and related algebras I, *J. Operator Theory*, **12** (1984), 23-45.

64. G. Robel, On the structure of (BCP)-operators and related algebras II, *J. Operator Theory*, **12** (1984), 235–245.

65. K. Rudol, On spectral mapping theorem, *J. Math. Anal. Appl.* **97** (1983), 131–193.

66. D. Sarason, Invariant subspaces and unstarred operator algebras, *Pacific J. Math.* **17** (1966), 511-517.

67. A. L. Shields, Weighted shift operators and analytic function theory, *Math. Surveys*, **13** (1974), 49–128.

68. M. Słociński, Isometric dilation of doubly commuting contractions and related models, *Bull. Acad. Sci. Pol.* **25** (1977), 1233–1242.

69. M. Słociński, On the Wold-type decomposition of a pair of commuting isometries, *Ann. Polon. Math.* **37** (1980), 255–262.

70. J. T. Schwartz, W^*-algebras, Gordon and Breach, New York, 1967.

71. B. Sz.-Nagy, C. Foiaş, Harmonic analysis of operators on Hilbert space, North Holland, Amsterdam, 1970.

72. J. L. Taylor, A joint spectrum of several commuting operators, *J. Func. Anal.* **6** (1970), 172–191.

73. J. L. Taylor, The analytic functional calculus for several commuting operators, *Acta. Math.* **125** (1970), 1–38.

74. D. Westwood, On C_{00} contractions with dominating spectrum, *J. Func. Anal.* **66** (1982), 96–104.

75. W. R. Wogen, Quasinormal operators are reflexive, *Bull. London Math. Soc.* **(2)31** (1979), 19–22.

76. K. Yan, Invariant subspaces for jointly subnormal systems, Preprint.

Marek Ptak, Institute of Mathematics, University of Agriculture,
Ul. Królewska 6, 30-045 Kraków, Poland
E-mail: e-mail rmptak@cyf-kr.edu.pl