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EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS FOR A NONLINEAR WAVE EQUATION

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Abstract. The initial boundary value problem for a Kirchhoff type plate equation in a bounded domain is considered. We prove the existence of global solutions by the similar arguments as in [11]. We derive the blow-up properties of solutions by energy method. Moreover, the estimates of the lifespan of solutions are also given.

1. Introduction

In this paper we consider the initial boundary value problem for the following nonlinear wave equation :

$$u_{tt} + \alpha\Delta^2 u - M(\|\nabla u\|_2^2)\Delta u = f(u), \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.2}$$

and boundary condition

$$u(x,t) = \frac{\partial}{\partial \nu} u(x,t) = 0, x \in \partial\Omega, t \geq 0, \tag{1.3}$$

where $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ and $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$ so that Divergence theorem can be applied. Here $\alpha > 0$, f is a nonlinear function like $f(u) = |u|^{p-2}\, u$, $p > 2$, $M(s)$ is a positive locally Lipschitz function like $M(s) = m_0 + bs^\gamma$, $m_0 > 0$, $b \geq 0$, $\gamma \geq 1$ and $s \geq 0$, ν is the normal

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unit vector pointing toward the exterior of Ω and $\frac{\partial}{\partial\nu}$ denotes the normal derivative on $\partial\Omega$.

First, we mention some of the known results related to the problem (1.1). When $f \equiv 0,$ Woinowsky-Krieger [26] first proposed the problem $(1.1) - (1.3)$ in the one-dimensional case as a model to describe the dynamic buckling of a hinged extensible beam under an axial force. The derivation of this model also can be found in [10, 9, 23]. Dickey [9] considered (1.1) with hinged boundary condition and the existence of solution was established. Later, Ball [2] extended the work of Dickey to both the cases of hinged ends and that of clamped ends, and he obtained the existence of weak solutions for (1.1) by using the technique of Lions [18]. For the general space dimension N, Mederiors [20] considered the problem (1.1) with $f \equiv 0$ in abstract framework. When the influence of the internal damping is considered, the problem (1.1) was treated by Brito [3] and Biler [5] for the linear damping case. On the other hand, for the nonlinear damping case, Komémou-Patcheu [17], Vasconcellos [24] and Aassila [1] investigated the problem (1.1) with $f \equiv 0.$ Recently, Cavalcanti et. al. [7] considered the problem (1.1) with nonlinear damping and internal force for general domains, and obtain the global existence of weak solutions. Concerning the nonexistence of global solutions, Kirane et. al. [15] and Can [6] studied the blow-up properties of (1.1) with a dynamic boundary condition in the case that $M \equiv 0.$ Later, Guedda and Labani [12] discussed the nonexistence result of the problem (1.1) for the nontrival function M.

When $\alpha \equiv 0$ in the equation (1.1), it is Kirchhoff equation which has been modeled in describing the nonlinear vibrations of an elastic string. Kirchhoff [16] was the first one to study the oscillations of stretched strings and plates. In this direction, there has been a large literatures concerning the existence and nonexistence of global solutions and some properties of solutions with initial and null Dirchlet boundary conditions [13, 14, 21, 27].

In this paper, we shall discuss the existence, uniqueness, global existence and blow-up properties of solutions for the problem $(1.1) - (1.3)$ in a bounded domain Ω in $\mathbb{R}^N$. The content of this paper is organized as follows. In section 2, we give some lemmas and assumptions which will be used later. In section 3, we first use Galerkin approximation method to study the existence of the linear problem $(3.1) - (3.3)$. Then, we obtain the local existence of regular solutions for the problem $(1.1) - (1.3)$ by using contraction mapping principle, and the uniqueness of solution is also given. By using density arguments, we derive the local existence of weak solution in Theorem 3.3. In section 4, we first define an energy function $E(t)$ in (4.7) and show that it is a constant function of t. Then, we obtain Theorem 4.4, which shows global existence of solutions under some restrictions on the initial data. In the last section, the blow-up properties of local solution for the problem $(1.1) - (1.3)$ with small positive initial energy are obtained by using the direct method [19]. Moreover, the estimates for the blow-up time T^* are also given. In this way, we can extend the result of [2] to nonzero external force term $f(u)$ and to more general $M(s)$, and the result of [20] to nonzero external force term $f(u)$.

2. Preliminary Results

In this section, we shall give some lemmas and assumptions which will be used throughout this work.

Lemma 2.1. (*Sobolev-Poincaré inequality* [22]). *If* $2 \le p \le \frac{2N}{[N-2m]^+}$, *then*

$$\|u\|_p \le B_1 \left\| (-\Delta)^{\frac{m}{2}} u \right\|_2, \text{ for } u \in D\left((-\Delta)^{\frac{m}{2}} \right),$$

holds with some constant B_1, *where* we *put* $[a]^+ = \max\{0, a\}$, $\frac{1}{[a]^+} = \infty$ if $[a]^+ = 0$ *and denote* $\|\cdot\|_p$ *to be the norm of* $L^p(\Omega)$.

Lemma 2.2. [19]. *Let* $\delta > 0$ *and* $B(t) \in C^2(0, \infty)$ *be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \ge 0. \tag{2.1}$$

If

$$B'(0) > r_2 B(0), \tag{2.2}$$

with $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$, *then*

$$B'(t) > 0,$$

for $t > 0$.

Lemma 2.3. [19]. *If* $J(t)$ *is a nonincreasing function on* $[t_0, \infty)$, $t_0 \ge 0$ *and satisfies the differential inequality*

$$J'(t)^2 \ge a + bJ(t)^{2+\frac{1}{\delta}} \text{ for } t \ge t_0, \tag{2.3}$$

where $a > 0$ *and* $b \in \mathbb{R}$, *then there exists a finite time* T^* *such that*

$$\lim_{t \to T^{*-}} J(t) = 0$$

and the upper bound of T^* *is estimated respectively by the following cases* :

(i) *If* $b < 0$ *and* $J(t_0) < \min\left\{1, \sqrt{\frac{a}{-b}}\right\}$ *then*

$$T^* \le t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}.$$

(ii) *If* $b = 0$, *then*

$$T^* \le t_0 + \frac{J(t_0)}{\sqrt{a}}.$$

(iii) *If* $b > 0$, *then*

$$T^* \le \frac{J(t_0)}{\sqrt{a}}$$

or

$$T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\},$$

where $c = (\frac{b}{a})^{\frac{\delta}{2+\delta}}$.

Now, we state the hypothesis on f :

(A1) $f(0) = 0$ and there is a positive constant k_1 such that

$$|f(u) - f(\upsilon)| \le k_1 |u - \upsilon| \left(|u|^{p-2} + |\upsilon|^{p-2}\right),$$

for $u, \upsilon \in \mathbb{R}$ and $2 < p \le \frac{2(N-3)}{N-4}$; $(2 < p,\ \text{if } N \le 4)$.

3. Local Existence

In this section, we shall discuss the local existence of solutions for wave equations $(1.1) - (1.3)$ by using contraction mapping principle.

An important tool in the proof of local existence Theorem 3.2 is based on studying the following linear problem :

(3.1) $$u_{tt} + \alpha\Delta^2 u - \mu(t)\Delta u = f_1(x,t) \text{ on } \Omega \times (0,T),$$

with initial conditions

(3.2) $$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega,$$

and Dirichlet boundary condition

(3.3) $$u(x,t) = \frac{\partial}{\partial \nu} u(x,t) = 0,\ x \in \partial\Omega,\ t > 0.$$

Here, $T > 0$, f_1 is some fixed forcing term on $\Omega \times (0,T)$, and μ is a positive locally Lipschitz function on $[0,\infty)$ with $\mu(t) \ge m_0 > 0$ for $t \ge 0$.

Lemma 3.1. *Suppose that* $u_0 \in U$, $u_1 \in H_0^1(\Omega) \cap H^2(\Omega)$ *and* $f_1 \in W^{1,2}(0,T; L^2(\Omega))$. *Then the problem* $(3.1) - (3.3)$ *admits a unique solution* u *such that*

$$u \in L^\infty(0,T;U), u_t \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))$$

and

$$u_{tt} \in L^\infty(0,T;L^2(\Omega)),$$

where

$$U = \left\{u \in H_0^2(\Omega); \Delta^2 u \in L^2(\Omega)\right\}.$$

Proof. Let $(w_n)_{n\in N}$ be a basis in U and let V_n be the space generated by $w_1,\cdots,w_n$, $n = 1,2,\cdots$. Let us consider

$$u_n(t) = \sum_{i=1}^{n} r_{in}(t) w_i$$

to be the solution of the following approximate problem corresponding to $(3.1)-(3.3)$

$$\begin{aligned}&\int_\Omega u_n''(t) w dx + \alpha \int_\Omega \Delta u_n(t)\Delta w dx + \mu(t)\int_\Omega \nabla u_n(t)\cdot\nabla w dx \\ &= \int_\Omega f_1(x,t) w dx \text{ for } w \in V_n,\end{aligned} \tag{3.4}$$

with initial conditions

$$u_n(0) = u_{0n} \equiv \sum_{i=1}^{n} p_{in} w_i \to u_0 \text{ in } U, \tag{3.5}$$

and

$$u_n'(0) = u_{1n} \equiv \sum_{i=1}^{n} q_{in} w_i \to u_1 \text{ in } H_0^1(\Omega)\cap H^2(\Omega), \tag{3.6}$$

where $p_{in} = \int_\Omega u_0 w_i dx$, $q_{in} = \int_\Omega u_1 w_i dx$ and $u' = \frac{\partial u}{\partial t}$.
By standard methods in differential equations [8], we prove the existence of solutions to $(3.4)-(3.6)$ on some interval $[0,t_n), 0 < t_n < T$. In order to extend the solution of $(3.4)-(3.6)$ to the whole interval $[0,T]$, we need the following a priori estimates.

Step 1. Setting $w = 2u_n'(t)$ in (3.4), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\left\|u_n'(t)\right\|_2^2 + \alpha\left\|\Delta u_n\right\|_2^2 + \mu(t)\left\|\nabla u_n(t)\right\|_2^2\right) \\ &= 2\int_\Omega f_1(x,t) u_n'(t) dx + \mu'(t)\left\|\nabla u_n(t)\right\|_2^2.\end{aligned} \tag{3.7}$$

Note that by Hölder inequality and Young's inequality, we have

$$2\left|\int_\Omega f_1(x,t) u_n'(t) dx\right| \le \left\|f_1\right\|_2^2 + \left\|u_n'(t)\right\|_2^2. \tag{3.8}$$

Then, by integrating (3.7) over $(0,t)$ and using (3.8), we obtain

$$\begin{aligned}&\left\|u_n'(t)\right\|_2^2+\alpha\left\|\Delta u_n\right\|_2^2+\mu(t)\left\|\nabla u_n(t)\right\|_2^2\\ &\le c_1+\int_0^t\left(1+\frac{|\mu'(s)|}{\mu(s)}\right)\left[\left\|u_n'(s)\right\|_2^2+\mu(s)\left\|\nabla u_n(s)\right\|_2^2\right]dt,\end{aligned}\tag{3.9}$$

where $c_1=\|u_{1n}\|_2^2+\alpha\|\Delta u_{0n}\|_2^2+\mu(0)\|\nabla u_{0n}\|_2^2+\int_0^T\|f_1\|_2^2\,dt.$
We observe that conditions (3.5) and (3.6) and the assumption of f_1 implies that c_1 is bounded. Thus, by employing Gronwall's Lemma, we see that

$$\left\|u_n'(t)\right\|_2^2+\alpha\left\|\Delta u_n\right\|_2^2+\mu(t)\left\|\nabla u_n(t)\right\|_2^2\le L_1,\tag{3.10}$$

for $t\in[0,T]$ and L_1 is a positive constant independent of $n\in N$.

Step 2. To estimate $u_n''(0)$ in L^2-norm, we let $t=0$ in (3.4) and put $w=2u_n''(0)$, we deduce that

$$\left\|u_n''(0)\right\|_2^2\le\left\|u_n''(0)\right\|_2\left[\alpha\left\|\Delta^2u_{0n}\right\|_2+\mu(0)\left\|\Delta u_{0n}\right\|_2+\|f_1\|_2\right].$$

Thus, using (3.5) and (3.6), there exists a positive constant L_2 independent of $n\in N$ such that

$$\left\|u_n''(0)\right\|_2\le L_2.\tag{3.11}$$

Next, we are going to give an upper bound for $\|u_n''(t)\|_2$.

Step 3. Taking the derivative of (3.4) with respect to t and setting $w=2u_n''(t)$, we have

$$\begin{aligned}&\frac{d}{dt}\left(\left\|u_n''(t)\right\|_2^2+\alpha\left\|\Delta u_n'(t)\right\|_2^2+\mu(t)\left\|\nabla u_n'(t)\right\|_2^2\right)\\ &=-2\mu'(t)\int_\Omega\nabla u_n(t)\cdot\nabla u_n''(t)dx+\mu'(t)\left\|\nabla u_n'(t)\right\|_2^2+2\int_\Omega f_1'(x,t)u_n''(t)dx.\end{aligned}\tag{3.12}$$

By Hölder inequality and Young's inequality, we note that

$$2\left|\mu'(t)\int_\Omega\nabla u_n(t)\cdot\nabla u_n''(t)dx\right|\le M_1^2\left(\left\|\Delta u_n'\right\|_2^2+\left\|u_n''\right\|_2^2\right)\tag{3.13}$$

and we also get

$$2\left|\int_\Omega f_1'(x,t)u_n''(t)dx\right|\le\left\|f_1'\right\|_2^2+\left\|u_n''\right\|_2^2,\tag{3.14}$$

where $M_1=\sup\limits_{0\le t\le T}|\mu'(t)|$.
Thus, by integrating (3.12) over $(0,t)$ and using (3.13), (3.14), (3.11) and (3.10),

we obtain

$$\left\|u_n''(t)\right\|_2^2+\alpha\left\|\Delta u_n'(t)\right\|_2^2+\mu(t)\left\|\nabla u_n'(t)\right\|_2^2$$
$$\leq c_2+\int_0^t\left(2+\frac{|\mu'(s)|}{\mu(s)}\right)\left(\left\|u_n''(s)\right\|_2^2+\mu(s)\left\|\nabla u_n'(s)\right\|_2^2\right)ds,$$

where $c_2=\mu(0)\left\|\nabla u_{1n}\right\|_2^2+L_2^2+\alpha\left\|\Delta u_{1n}\right\|_2^2+TL_1^2M_1^2+\int_0^T\left\|f_1'\right\|_2^2dt$.
Then, by Gronwall's Lemma and using $(3.5)-(3.6)$, we have

(3.15) $$\left\|u_n''(t)\right\|_2^2+\alpha\left\|\Delta u_n'(t)\right\|_2^2+\mu(t)\left\|\nabla u_n'(t)\right\|_2^2\leq L_{3,}$$

for all $t\in[0,T]$ and L_3 is a positive constant independent of $n\in N$.
Therefore, from (3.10) and (3.15), we see that

(3.16) $$u_i\rightarrow u\ \text{ weak-* in }\ L^\infty\left(0,T;H_0^2(\Omega)\right),$$

(3.17) $$u_i'\rightarrow u'\ \text{ weak-* in }\ L^\infty\left(0,T;H_0^1(\Omega)\cap H^2(\Omega)\right),$$

(3.18) $$u_i'\rightarrow u'\ \text{ weak-* in }\ L^\infty\left(0,T;L^2(\Omega)\right),$$

(3.19) $$u_i''\rightarrow u''\ \text{ weak-* in } L^\infty\left(0,T;L^2(\Omega)\right).$$

Thus, by passing the limit in (3.4) and using $(3.16)-(3.19)$, we obtain

$$\int_0^T\int_\Omega\left(u_{tt}+\alpha\Delta^2u-\mu(t)\Delta u\right)v\theta dxdt=\int_0^T\int_\Omega f_1(x,t)v\theta dxdt,$$

for all $\theta\in D\left(0,T\right)$ and for all $v\in U$. From above identity, we have

(3.20) $$u_{tt}+\alpha\Delta^2u-\mu(t)\Delta u=f_1(x,t)\text{ in }D'\left(\Omega\times(0,T)\right).$$

On the other hand, since u'', $\mu\Delta u$ and $f_1\in L^\infty\left(0,T;L^2(\Omega)\right)$ and by (3.20), we deduce that $\Delta^2u\in L^\infty\left(0,T;L^2(\Omega)\right)$, so $u\in L^\infty\left(0,T;U\right)$.
In addition

$$u_{tt}+\alpha\Delta^2u-\mu(t)\Delta u=f_1(x,t)\text{ in }L^\infty\left(0,T;L^2(\Omega)\right).$$

Next, we want to show the uniqueness of $(3.1)-(3.3)$. Let $u^{(1)},u^{(2)}$ be two solutions of $(3.1)-(3.3)$. Then $z=u^{(1)}-u^{(2)}$ satisfies

(3.21) $$\int_\Omega z''(t)wdx+\alpha\int_\Omega\Delta z\Delta wdx+\mu(t)\int_\Omega\nabla z(t)\cdot\nabla wdx=0\text{ for }w\in U,$$

$$z(x,0)=0, z'(x,0)=0, \quad x\in\Omega,$$
$$z(x,t)=\frac{\partial}{\partial\nu}z(x,t)=0,\ x\in\partial\Omega,\ t\geq 0.$$

Setting $w=2z'(t)$ in (3.21), then as in deriving (3.10), we see that

$$\left\|z'(t)\right\|_2^2+\mu(t)\left\|\nabla z(t)\right\|_2^2+\alpha\left\|\Delta z(t)\right\|_2^2$$
$$\leq \int_0^t\left[1+\frac{|\mu'(s)|}{\mu(s)}\right]\left[\left\|z'(s)\right\|_2^2+\mu(s)\left\|\nabla z(s)\right\|_2^2\right]ds.$$

Thus, by employing Gronwall's Lemma, we conclude that

$$\left\|z'(t)\right\|_2=\left\|\nabla z(t)\right\|_2=\left\|\Delta z(t)\right\|_2=0 \text{ for all } t\in[0,T].$$

Therefore, we have the uniqueness.

Now, we are ready to to show the local existence of the problem $(1.1)-(1.3)$.

Theorem 3.2. (Regular Solution). *Suppose that (A1) holds, and that* $u_0\in U$, $u_1\in H_0^1(\Omega)\cap H^2(\Omega)$, *then there exists a unique solution* u *of* $(1.1)-(1.3)$ *satisfying*

$$u\in L^\infty(0,T;U), u_t\in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))$$

and

$$u_{tt}\in L^\infty(0,T;L^2(\Omega)).$$

Proof. Define the following two-parameter space :

$$X_{T,R_0}=\left\{\begin{array}{c} \upsilon\in L^\infty(0,T;H_0^2(\Omega)), \upsilon_t\in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)): \\ e(\upsilon(t))\leq R_0^2,\ t\in[0,T],\ \text{with } \upsilon(0)=u_0 \text{ and } \upsilon_t(0)=u_1. \end{array}\right\},$$

for $T>0$, $R_0>0$ and $e(\upsilon(t))\equiv\left\|\upsilon_t(t)\right\|_2^2+\left\|\Delta\upsilon(t)\right\|_2^2$. Then X_{T,R_0} is a complete metric space with the distance

$$d(y,z)=\sup_{0\leq t\leq T}\left[\left\|\Delta(y-z)\right\|_2^2+\left\|(y-z)_t\right\|_2^2\right]^{\frac{1}{2}}. \tag{3.22}$$

where $y,\ z\in X_{T,R_0}$.

Given $\upsilon\in X_{T,R_0}$, we consider the following problem

$$u_{tt}+\alpha\Delta^2u-M(\left\|\nabla\upsilon\right\|_2^2)\Delta u=f(\upsilon), \tag{3.23}$$

with initial conditions

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \tag{3.24}$$

and boundary condition

$$u(x,t) = \frac{\partial}{\partial \nu} u(x,t) = 0, x \in \partial\Omega, t \geq 0. \tag{3.25}$$

First of all, we observe that

$$\begin{aligned} \frac{d}{dt} M(\|\nabla v\|_2^2) &= 2M'(\|\nabla v\|_2^2) \int_\Omega \nabla v \cdot \nabla v_t dx \\ &\leq 2M_2 \|\Delta v\|_2 \|v_t\|_2 \\ &\leq M_2 R_0^2, \end{aligned} \tag{3.26}$$

where $M_2 = \sup\{|M'(s)|; 0 \leq s \leq B_1^2 R_0^2\}$. And by (A1), we note that $f \in W^{1,2}(0,T;L^2(\Omega))$. Thus, by Lemma 3.1, there exists a unique solution u of $(3.23)-(3.25)$. We define the nonlinear mapping $Sv = u$, and then, we shall show that there exist $T > 0$ and $R_0 > 0$ such that

(i) $S : X_{T,R_0} \to X_{T,R_0}$,

(ii) S is a contraction mapping in X_{T,R_0} with respect to the metric $d(\cdot,\cdot)$ defined in (3.22).

Multiplying (3.23) by $2u_t$, and then integrating it over $\Omega \times (0,t)$, we obtain

$$\frac{d}{dt} e_1(u(t)) = I_1 + I_2, \tag{3.27}$$

where

$$\begin{aligned} e_1(u(t)) &= \|u_t\|_2^2 + \alpha \|\Delta u\|_2^2 + M(\|\nabla v\|_2^2) \|\nabla u\|_2^2, \\ I_1 &= \left(\frac{d}{dt} M(\|\nabla v\|_2^2) \right) \|\nabla u\|_2^2, \end{aligned} \tag{3.28}$$

and

$$I_2 = 2 \int_\Omega f(v) u_t dx.$$

By (3.26) and (3.28), we have

$$|I_1| \leq M_2 R_0^2 e_1(u(t)), \tag{3.29}$$

and by (A1), Hölder inequality and Lemma 2.1, we get

$$\begin{aligned} |I_2| &\leq 2k_1 \int_\Omega |v|^{p-1} |u_t| dx \\ &\leq 2k_1 B_1^{p-1} \|\Delta v\|_2^{p-1} \|u_t\|_2 \\ &\leq 2k_1 B_1^{p-1} R_0^{p-1} e_1(u(t))^{\frac{1}{2}}. \end{aligned} \tag{3.30}$$

Then, by integrating (3.27) over $(0,t)$ and using $(3.29)-(3.30)$, we deduce

$$e_1(u(t)) \le e_1(u_0) + \int_0^t \left(2M_2R_0^2 e_1(u(s)) + 2k_1 B_1^{p-1} R_0^{p-1} e_1(u(s))^{\frac{1}{2}}\right) ds.$$

Thus, by Gronwall's Lemma, we have

(3.31) $$e_1(u(t)) \le \chi(u_0,u_1,R_0,T)^2 \mathrm{e}^{2M_2R_0^2T},$$

where

$$\chi(u_0,u_1,R_0,T) = \sqrt{e_1(u_0)} + k_1 B_1^{p-1} R_0^{p-1} T.$$

Hence, from (3.31) and (3.28), we obtain

$$e(u(t)) \le k_2 \chi(u_0,u_1,R_0,T)^2 \mathrm{e}^{2M_2R_0^2T},$$

where $k_2 = \frac{1}{\min\{1,\alpha\}}$.
Therefore if the parameters T and R_0 satisfy

(3.32) $$k_2 \chi(u_0,u_1,R_0,T)^2 \mathrm{e}^{2M_2R_0^2T} \le R_0^2,$$

then S maps X_{T,R_0} into itself.

Next, we will show that S is a contraction mapping with respect to the metric $d(\cdot,\cdot)$. Let $\upsilon_i \in X_{T,R_0}$ and $u^{(i)} \in X_{T,R_0}$, $i=1,2$ be the corresponding solution to $(3.23)-(3.25)$. Let $w(t) = \left(u^{(1)} - u^{(2)}\right)(t)$, then w satisfy the following system :

(3.33) $$\begin{aligned} &w_{tt} + \alpha\Delta^2 w - M\left(\|\nabla \upsilon_1\|_2^2\right)\Delta w \\ &= f(\upsilon_1) - f(\upsilon_2) + \left[M\left(\|\nabla \upsilon_1\|_2^2\right) - M\left(\|\nabla \upsilon_2\|_2^2\right)\right]\Delta u^{(2)}, \end{aligned}$$

with initial conditions

(3.34) $$w(0) = 0,\ w_t(0) = 0,$$

and boundary condition

(3.35) $$w(x,t) = \frac{\partial}{\partial\nu} w(x,t) = 0,\ x \in \partial\Omega \text{ and } t \ge 0.$$

Multiplying (3.33) by $2w_t$, and integrating it over Ω, we have

(3.36) $$\frac{d}{dt}\left[\|w_t\|_2^2 + M(\|\nabla \upsilon_1\|_2^2)\,\|\nabla w(t)\|_2^2 + \alpha\,\|\Delta w\|_2^2\right] = I_3 + I_4 + I_5,$$

where

$$I_3 = 2\left[M\left(\|\nabla \upsilon_1\|_2^2\right) - M\left(\|\nabla \upsilon_2\|_2^2\right)\right] \int_\Omega \Delta u^{(2)} w_t dx,$$

$$I_4 = 2\int_\Omega \left(f(\upsilon_1) - f(\upsilon_2)\right) w_t dx,$$

and

$$I_5 = \left(\frac{d}{dt}M(\|\nabla \upsilon_1\|_2^2)\right) \|\nabla w(t)\|_2^2 .$$

To proceed the estimates of $I_i, i = 3, 4, 5$, we observe that

$$\begin{aligned} |I_3| &\leq 2L\left(\|\nabla \upsilon_1\|_2 + \|\nabla \upsilon_2\|_2\right) \|\nabla \upsilon_1 - \nabla \upsilon_2\|_2 \left\|\Delta u^{(2)}\right\|_2 \|w_t\|_2 \\ &\leq 4LB_1^2 R_0^2 e(\upsilon_1 - \upsilon_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{3.37}$$

$$|I_4| \leq 4k_1 B_1^p R_0^{p-2} e(\upsilon_1 - \upsilon_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \tag{3.38}$$

and

$$|I_5| \leq M_2 R_0^2 e(w(t)), \tag{3.39}$$

where $L = L(R_0)$ is the Lipschitz constant of $M(r)$ in $[0, R_0]$.
Thus, by using $(3.37) - (3.39)$ in (3.36), we get

$$\begin{aligned} &\frac{d}{dt}\left[\|w_t\|_2^2 + M(\|\nabla \upsilon_1\|_2^2) \|\nabla w(t)\|_2^2 + \alpha \|\Delta w\|_2^2\right] \\ &\leq 2M_2 R_0^2 e(w(t)) + c_3 e(\upsilon_1 - \upsilon_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{3.40}$$

where $c_3 = 4\left(LB_1^2 R_0^2 + k_1 B_1^p R_0^{p-2}\right)$.
Then, integrating (3.40) over $(0, t)$ and using $(3.34) - (3.35)$, we deduce

$$e(w(t)) \leq \int_0^t [2M_2 R_0^2 e(w(s)) + c_3 e(\upsilon_1 - \upsilon_2)^{\frac{1}{2}} e(w(s))^{\frac{1}{2}}]ds. \tag{3.41}$$

Thus, by Gronwall's Lemma, we obtain

$$e(w(t)) \leq c_3^2 T^2 \mathrm{e}^{2M_2 R_0^2 T} \sup_{0\leq t\leq T} e(\upsilon_1 - \upsilon_2).$$

By (3.22), we have

$$d(u^1, u^2) \leq C(T, R_0)^{\frac{1}{2}} d(\upsilon_1, \upsilon_2), \tag{3.42}$$

where

$$C(T, R_0) = c_3^2 T^2 \mathrm{e}^{2M_2 B_1^2 R_0^2 T}.$$

Hence, under inequality (3.32), S is a contraction mapping if $C(T, R_0) < 1$. Indeed, we choose R_0 sufficiently large and T sufficiently small so that (3.32) and (3.42) are satisfied at the same time. By applying Banach fixed point theorem, we obtain the local existence result.

Next, we are in condition to show the existence of weak solution for the problem $(1.1)-(1.3)$.

Theorem 3.3. (Weak Solution). *Supposed that (A1) holds and that* $u_0 \in H_0^2(\Omega)$ *and* $u_1 \in L^2(\Omega)$. *Then the problem* $(1.1)-(1.3)$ *possesses a unique solution* u *such that*

$$u \in C\left([0,T]; H_0^2(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right).$$

Proof. Since $U \times \left(H_0^1(\Omega) \cap H^2(\Omega)\right)$ is dense in $H_0^2(\Omega) \times L^2(\Omega)$, there exists $\{u_0^m, u_1^m\} \subset U \times H_0^1(\Omega) \cap H^2(\Omega)$ such that $\{u_0^m, u_1^m\} \to \{u_0, u_1\}$ in $H_0^2(\Omega) \times L^2(\Omega)$ as $m \to \infty$.

By Theorem 3.2, for each $m \in N$, there exists a unique solution u_m such that $u_m \in L^\infty(0,T;U)$, $u_m' \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega))$ and $u_m'' \in L^\infty(0,T;L^2(\Omega))$ satisfies

$$u_m'' + \alpha \Delta^2 u_m - M(\|\nabla u_m\|_2^2)\Delta u_m = f(u_m), \tag{3.43}$$

$$u_m(x,0) = u_0^m(x), u_m'(x,0) = u_1^m(x), x \in \Omega, \tag{3.44}$$

$$u_m(x,t) = \frac{\partial}{\partial \nu} u_m(x,t) = 0, x \in \partial\Omega, t \geq 0. \tag{3.45}$$

By using similar arguments as in the Step 1 of Lemma 3.1, we deduce

$$\left\|u_m'(t)\right\|_2^2 + \alpha \left\|\Delta u_m(t)\right\|_2^2 + \widehat{M}\left(\left\|\nabla u_n(t)\right\|_2^2\right) \leq L \tag{3.46}$$

for all $t \in [0,T]$ and L is a positive constant independent of $m \in N$, where $\widehat{M}(s) = \int_0^s M(r)dr$.

Let $m_2 \geq m_1$ be two natural numbers and consider $z_m = u_{m_2} - u_{m_1}$. Repeating similar discussions used in $(3.33)-(3.40)$ and observing that $\{u_0^m\}, \{u_1^m\}$ are Cauchy sequence in U and $H_0^1(\Omega) \cap H^2(\Omega)$, respectively, we, then, have

$$\left\|z_m'(t)\right\|_2^2 + M\left(\left\|\nabla z_{m_2}\right\|_2^2\right)\left\|\nabla z_m\right\|_2^2 + \alpha\left\|\Delta z_m\right\|_2^2 \to 0, \tag{3.47}$$

as $m \to \infty$, for all $t \in [0,T]$.

Therefore, from (3.46) and (3.47), we see that

$$u_m \to u \text{ in } C\left([0,T]; H_0^2(\Omega)\right),$$

$$u_m' \to u' \text{ in } C\left([0,T]; L^2(\Omega)\right),$$

$$u_m \to u \text{ weak-* in } L^\infty\left(0,T; H_0^2(\Omega)\right),$$

$$u_m' \to u' \text{ weak-* in } L^\infty\left(0,T; L^2(\Omega)\right).$$

By the above convergence results, it is sufficient to to pass the limit in (3.43), we obtain

$$u_{tt}+\alpha\Delta^2u-M\left(\|\nabla u\|_2^2\right)\Delta u=f(u)\ \text{ in }\ L^\infty(0,T;H^{-2}(\Omega)).$$

The uniqueness of weak solutions can be obtained by using the similar discussions as in [4]. We omit the details.

4. Global Existence

In this section, we consider the global existence of solutions for a kind of the problem $(1.1)-(1.3)$:

$$u_{tt}+\alpha\Delta^2u-M(\|\nabla u\|_2^2)\Delta u=|u|^{p-2}\,u,\ p>2, \tag{4.1}$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega, \tag{4.2}$$

$$u(x,t)=\frac{\partial}{\partial\nu}u(x,t)=0,x\in\partial\Omega,t\geq 0. \tag{4.3}$$

Let

$$I_1(t)\equiv I_1(u(t))=\alpha\,\|\Delta u\|_2^2+m_0\,\|\nabla u\|_2^2-\|u\|_p^p, \tag{4.4}$$

$$I_2(t)\equiv I_2(u(t))=\alpha\,\|\Delta u\|_2^2+M(\|\nabla u\|_2^2)\,\|\nabla u\|_2^2-\|u\|_p^p, \tag{4.5}$$

and

$$J(t)\equiv J(u(t))=\frac{\alpha}{2}\,\|\Delta u\|_2^2+\frac{1}{2}\widehat{M}\left(\|\nabla u\|_2^2\right)-\frac{1}{p}\|u\|_p^p, \tag{4.6}$$

for $u(t)\in H_0^2(\Omega)$, $t\geq 0$ and $\widehat{M}(s)=\int_0^s M(r)dr$.
We define the energy of the solution u of $(4.1)-(4.3)$ by

$$E(t)=\frac{1}{2}\,\|u_t\|_2^2+J(t). \tag{4.7}$$

Lemma 4.1. *$E(t)$ is a constant function on $[0,T]$.*

Proof. Multiplying (4.1) by u_t, integrating by parts over $\Omega\times(0,t)$, and using the boundary conditions (4.3), we obtain

$$E(t)=E(0),\ \text{for } t\in[0,T].$$

Remark. By (4.6), (4.7), the assumption of M and Lemma 2.1, we have

$$\begin{aligned} E\,(t)\,&=\,\frac{1}{2}\,\|u_t\|_2^2+\frac{\alpha}{2}\,\|\Delta u\|_2^2+\frac{1}{2}\widehat{M}\left(\|\nabla u\|_2^2\right)-\frac{1}{p}\|u\|_p^p\\ &\geq\,\frac{1}{2}l\|\nabla u\|_2^2-\frac{1}{p}\|u\|_p^p,\ t\geq 0,\end{aligned} \tag{4.8}$$

where $l = \alpha B_1^{-2} + m_0$ and B_1 is the Sobolev's constant given in Lemma 2.1.
By Poincaré inequality, we get

(4.9) $$E(t) \geq G(\|\nabla u(t)\|_2),\ t \geq 0,$$

where

(4.10) $$G(\lambda) = \frac{1}{2} l\lambda^2 - \frac{B_1^p}{p}\lambda^p.$$

Note that $G(\lambda)$ has the maximum at $\lambda_1 = \left(\frac{l}{B_1^p}\right)^{\frac{1}{p-2}}$ and the maximum value E_1 is

(4.11) $$E_1 = G(\lambda_1) = l^{\frac{p}{p-2}} \left(\frac{1}{2} - \frac{1}{p}\right) B_1^{\frac{-2p}{p-2}}.$$

Adapting the idea of Vitillaro [25], we have the following Lemma:

Lemma 4.2. *Assume that* $E(0) < E_1$. *Then*

(i) *if* $\|\nabla u_0\|_2 < \lambda_1$, *then* $\|\nabla u(t)\|_2 < \lambda_1$ *for* $t \geq 0$.

(ii) *If* $\|\nabla u_0\|_2 > \lambda_1$, *then there exists* $\lambda_2 > \lambda_1$ *such that* $\|\nabla u(t)\|_2 \geq \lambda_2$ *for* $t \geq 0$.

Lemma 4.3. *Let* u *be a solution of* $(4.1)-(4.3)$. *Assume that* $0 < \|\nabla u_0\|_2 < \lambda_1$ *and*

(4.12) $$\beta = \frac{B_1^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} < 1,$$

then $I_2(t) > 0$, *for all* $t \in [0, T)$, *where* l *is given in (4.8).*

Proof. We note that $\|\nabla u_0\|_2 < \lambda_1$ implies $I_1(u_0) > 0$, hence by the continuity of $u(t)$, we have

(4.13) $$I_1(t) > 0,$$

for some interval near $t = 0$. Let $t_{\max} > 0$ be a maximal time (possibly $t_{\max} = T$), when (4.13) holds on $[0, t_{\max})$.
From (4.6) and (4.4), we have

(4.14) $$\begin{aligned} J(t) &\geq \frac{\alpha}{2}\|\Delta u\|_2^2 + \frac{m_0}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p}\|u(t)\|_p^p \\ &= \frac{p-2}{2p}\left[\alpha\|\Delta u\|_2^2 + m_0\|\nabla u(t)\|_2^2\right] + \frac{1}{p}I_1(t). \end{aligned}$$

From (4.14) and using Poincaré inequality and Lemma 4.1, we get

(4.15) $$l\|\nabla u\|_2^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) = \frac{2p}{p-2}E(0).$$

Then, from Poincaré inequality, (4.15) and (4.12), we obtain

$$\|u\|_p^p \le B_1^p \|\nabla u\|_2^p \le \frac{B_1^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} l\,\|\nabla u\|_2^2 = \beta l\,\|\nabla u\|_2^2 < l\,\|\nabla u\|_2^2 \text{ on } [0,t_{\max}). \tag{4.16}$$

Thus

$$I_1(t) \ge l\,\|\nabla u\|_2^2 - \|u\|_p^p > 0 \text{ on } [0,t_{\max}). \tag{4.17}$$

This implies that we can take $t_{\max} = T$. But, from (4.4) and (4.5), we see that

$$I_2(t) \ge I_1(t), t \in [0,T].$$

Therefore, we have $I_2(t) > 0$, for $t \in [0,T]$.

Remark. (4.12) holds if and only if $0 < E(0) < E_1$.

Next, we want to show that $T = \infty$, by using the similar arguments as that of [11].

Theorem 4.4. (Global existence). *Assume that* $u_0 \in H_0^2(\Omega)$ *and* $u_1 \in L^2(\Omega)$ *with the conditions that* $0 < \|\nabla u_0\|_2 < \lambda_1$ *and* $0 < E(0) < E_1$. *Then the problem* $(4.1)-(4.3)$ *has a unique weak global solution satisfying*

$$u \in C\left(0,\infty;\ H_0^2(\Omega)\right) \cap C^1\left(0,\infty;\ L^2(\Omega)\right).$$

Proof. We define

$$E_2(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{\alpha}{2}\|\Delta u\|_2^2 + \frac{1}{2}\widehat{M}\left(\|\nabla u(t)\|_2^2\right) + \frac{1}{p}\|u(t)\|_p^p.$$

Then, from Lemma 4.1, we obtain

$$E_2'(t) = 2\int_\Omega |u|^{p-2} u u_t dx. \tag{4.19}$$

Note that by using (4.11), (4.7) and Lemma 4.1, we have

$$\alpha\,\|\Delta u\|_2^2 \le \frac{2p}{p-2}J(t) \le \frac{2p}{p-2}E(t) = \frac{2p}{p-2}E(0). \tag{4.20}$$

On the other hand, by Hölder inequality, Poincaré inequality and (4.20), we get

$$\begin{aligned}\left|\int_\Omega |u|^{p-2} u u_t dx\right| &\le \|u_t\|_2\,\|u\|_{2(p-1)}^{p-1}, \\ &\le \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u\|_{2(p-1)}^{2(p-1)} \\ &\le \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}B_1^{2(p-1)}\|\Delta u\|_2^{2(p-1)} \\ &\le \frac{1}{2}\|u_t\|_2^2 + \frac{B_1^{2(p-1)}}{\alpha}\left(\frac{2p}{\alpha\,(p-2)}E(0)\right)^{p-2}\frac{\alpha}{2}\|\Delta u\|_2^2.\end{aligned}$$

Then integrating (4.19) over $(0,t)$ and using above inequality, we obtain

(4.21) $$E_2'(t) \le c_4 E_2(t),$$

where $c_4 = \max\left\{1, \frac{B_1^{2(p-1)}}{\alpha}\left(\frac{2p}{\alpha(p-2)}E(0)\right)^{p-2}\right\}$.
Thus, we deduce

$$E_2(t) \le E_2(0)\exp(c_4 t),$$

for any $t \ge 0$. Therefore by the standard continuation principle, we have $T = \infty$.

5. Blow-up Property

In this section, we shall discuss the blow up phenomena for a kind of the problem $(1.1) - (1.3)$:

(5.1) $$u_{tt} + \alpha\Delta^2 u - M(\|\nabla u\|_2^2)\Delta u = |u|^{p-2}u, \ p > 2.$$

In order to state our results, we make further assumptions on M :

(A2) There exists a positive constant $0 < \delta \le \frac{p-2}{4}$ such that

$$(2\delta+1)\widehat{M}(s) - M(s)s \ge 2\delta m_0 s, \text{ for all } s \ge 0.$$

Definition. A solution u of $(5.1), (1.2)$ and (1.3) is called blow-up if there exists a finite time T^* such that

(5.2) $$\lim_{t \to T^{*-}} \left(\int_\Omega u^2 dx\right)^{-1} = 0.$$

Now, let u be a solution of (5.1) and define

(5.3) $$a(t) = \int_\Omega u^2 dx, \ t \ge 0.$$

Lemma 5.1. *Assume that (A2) holds, then we have*

(5.4) $$a''(t) - 4(\delta+1)\|u_t\|_2^2 \ge Q_1(t), \text{ for } t \ge 0,$$

where

(5.5) $$Q_1(t) = -4(1+2\delta)E(0) + 4\delta l\|\nabla u\|_2^2.$$

Proof. Form (5.3), we have

$$a'(t) = 2\int_{\Omega} u u_t dx. \tag{5.6}$$

By (5.1) and Divergence theorem, we get

$$a''(t) = 2\|u_t\|_2^2 - 2\alpha\|\Delta u\|_2^2 - 2M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 + 2\|u\|_p^p. \tag{5.7}$$

Then, by (4.7), we arrive at

$$\begin{aligned} &a''(t) - 4(\delta+1)\|u_t\|_2^2 \\ &= (-4-8\delta)E(0) + 4\delta\alpha\|\Delta u\|_2^2 + 2\left(1 - \frac{2+4\delta}{p}\right)\|u\|_p^p \\ &+ \left[(2+4\delta)\widehat{M}\left(\|\nabla u(t)\|_2^2\right) - 2M\left(\|\nabla u(t)\|_2^2\right)\|\nabla u(t)\|_2^2\right]. \end{aligned}$$

Therefore by $(\mathrm{A2})$ and Poincaré inequality, we obtain (5.4).

Now, we consider four different cases on the initial energy $E(0)$.
(1) If $E(0) < 0$, then from (5.4), we have

$$a'(t) \geq a'(0) - 4(1+2\delta)E(0)t, \ t \geq 0.$$

Thus we get $a'(t) > 0$ for $t > t_1^*$, where

$$t_1^* = \max\left\{\frac{a'(0)}{4(1+2\delta)E(0)}, \ 0\right\}. \tag{5.8}$$

(2) If $E(0) = 0$, then $a''(t) \geq 0$ for $t \geq 0$.
Furthermore, if $a'(0) > 0$, then $a'(t) > 0, \ t \geq 0$

(3) If $0 < E(0) < E_1$ and $\|\nabla u_0\|_2 > \lambda_1$.
From (5.5) and Lemma 4.2, we see that

$$\begin{aligned} Q_1(t) &= (-4-8\delta)E(0) + 4\delta l\|\nabla u\|_2^2 \\ &> (-4-8\delta)E(0) + 4\delta l^{\frac{p}{p-2}} B_1^{-\frac{2p}{p-2}} \\ &= (4+8\delta)\left[-E(0) + \frac{4\delta}{4+8\delta}\frac{2p}{p-2}E_1\right]. \end{aligned} \tag{5.9}$$

Then, choosing $\delta = \frac{p-2}{4}$ and from (5.4) and (5.9), we obtain

$$a''(t) \geq Q_1(t) > k_3 > 0, \tag{5.10}$$

where $k_3 = 2p\left(E_1 - E(0)\right).$
Thus we get $a'(t) > 0$ for $t > t_2^*$, where

$$t_2^* = \max\left\{\frac{-a'(0)}{k_3},\ 0\right\}. \tag{5.11}$$

(4) For the case that $E(0) \geq E_1$, we first note that, by using Hölder inequality and Young's inequality, we have from (5.6)

$$a'(t) \leq a(t) + \|u_t\|_2^2. \tag{5.12}$$

Hence by (5.4) and (5.12), we deduce

$$a''(t) - 4(\delta+1)a'(t) + 4(\delta+1)a(t) + K_1 \geq 0,$$

where

$$K_1 = (4 + 8\delta)E(0).$$

Let

$$b(t) = a(t) + \frac{K_1}{4(1+\delta)},\ \ t > 0.$$

Then $b(t)$ satisfies (2.1). By (2.2), we see that if

$$a'(0) > r_2\left[a(0) + \frac{K_1}{4(1+\delta)}\right], \tag{5.13}$$

then $a'(t) > 0$, $t > 0$, here $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$.

Consequently, we have

Lemma 5.2. *Assume that (A2) holds and that either one of the following statements is satisfied:*

(i) $E(0) < 0$,
(ii) $E(0) = 0$ *and* $a'(0) > 0$,
(iii) $0 < E(0) < E_1$ *and* $\|\nabla u_0\|_2 > \lambda_1$,
(iv) $E_1 \leq E(0)$ *and* (5.13) *holds,*

then $a'(t) > 0$ *for* $t > t_0$, *where* $t_0 = t_1^*$ *is given by* (5.8) *in case* (i), $t_0 = t_2^*$ *is given by* (5.11) *in case* (iii) *and* $t_0 = 0$ *in cases* (ii) *and* (iv).

Now, we will find the estimate for the life span of $a(t)$.
Let

$$J(t) = a(t)^{-\delta},\ \text{ for } t \geq 0. \tag{5.14}$$

Then we have

$$J'(t) = -\delta J(t)^{1+\frac{1}{\delta}} a'(t)$$

and

$$J''(t) = -\delta J(t)^{1+\frac{2}{\delta}} V(t), \tag{5.15}$$

where

$$V(t) = a''(t) a(t) - (1+\delta) a'(t)^2. \tag{5.16}$$

By using Hölder inequality in (5.6), we get

$$a'(t) \leq 2 \|u\|_2 \|u_t\|_2. \tag{5.17}$$

Thus, by (5.4) and (5.17), we obtain from (5.16)

$$\begin{aligned} V(t) &\geq \left[Q_1(t) + 4(1+\delta)\|u_t\|_2^2\right] a(t) - 4(1+\delta) a(t) \|u_t\|_2^2 \\ &= Q_1(t) J(t)^{-\frac{1}{\delta}}, \ t \geq t_0. \end{aligned}$$

Therefore, by (5.15), we have

$$J''(t) \leq -\delta Q_1(t) J(t)^{1+\frac{1}{\delta}}, \ t \geq t_0. \tag{5.18}$$

Theorem 5.3. (Nonexistence of global solutions). *Assume that (A2) holds and that either one of the following statements is satisfied:*

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $a'(0) > 0$,

(iii) $0 < E(0) < E_1$ *and* $\|\nabla u_0\|_2 > \lambda_1$

(iv) $E_1 \leq E(0) < \frac{a'(t_0)^2}{8a(t_0)}$ *and* (5.13) *holds,*

then the solution u *blows up at finite time* T^* *in the sense of* (5.2).
Moreover, the upper bound of T^* *is estimated as follows:*
In case (i),

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\left\{1, \sqrt{\frac{\alpha_1}{-\beta_1}}\right\}$, *then we have*

$$T^* \leq t_0 + \frac{1}{\sqrt{-\beta_1}} \ln \frac{\sqrt{\frac{\alpha_1}{-\beta_1}}}{\sqrt{\frac{\alpha_1}{-\beta_1}} - J(t_0)}.$$

In case (ii),

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}$$

or

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{\alpha_1}}.$$

In case (iii),

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\left\{1, \sqrt{\frac{\alpha_2}{-\beta_2}}\right\}$, *we have*

$$T^* \leq t_0 + \frac{1}{\sqrt{-\beta_2}} \ln \frac{\sqrt{\frac{\alpha_2}{-\beta_2}}}{\sqrt{\frac{\alpha_2}{-\beta_2}} - J(t_0)}.$$

In case (iv) ,

$$T^* \leq \frac{J(t_0)}{\sqrt{\alpha_1}}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha_1}} \left\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\right\},$$

where $c = \left(\frac{\beta_1}{\alpha_1}\right)^{\frac{\delta}{2+\delta}}$, *here* α_1 *and* β_1 *are in* (5.21) *and* (5.22) *and* α_2 *and* β_2 *are in* (5.23) *and* (5.24)*respectively.*
Note that in case (i), $t_0 = t_1^*$ *is given by* (5.8), $t_0 = t_2^*$ *is given by* (5.11) *in case* (iii) *and* $t_0 = 0$ *in cases* (ii) *and* (iv).

Proof. (1) For $E(0) \leq 0$, from (5.18) and (5.5), we have

$$J''(t) \leq \delta(4 + 8\delta) E(0) J(t)^{1+\frac{1}{\delta}}. \tag{5.19}$$

Note that by Lemma 5.2, $J'(t) < 0$ for $t > t_0$. Multiplying (5.19) by $J'(t)$ and integrating it from t_0 to t, we have

$$J'(t)^2 \geq \alpha_1 + \beta_1 J(t)^{2+\frac{1}{\delta}} \text{ for } t \geq t_0,$$

where

$$\begin{aligned} \alpha_1 &= \delta^2 J(t_0)^{2+\frac{2}{\delta}} \left[a'(t_0)^2 - 8E(0) J(t_0)^{\frac{-1}{\delta}}\right] \\ &> 0. \end{aligned} \tag{5.21}$$

and

$$\beta_1 = 8\delta^2 E(0). \tag{5.22}$$

Then by Lemma 2.3, there exists a finite time T^* such that $\lim_{t\to T^{*-}} J(t) = 0$ and this will imply that $\lim_{t\to T^{*-}} \left(\int_\Omega u^2 dx\right)^{-1} = 0$.

(2) For the case of $0 < E(0) < E_1$, from (5.18) and (5.10), we get

$$J''(t) \le -\delta k_3 J(t)^{1+\frac{1}{\delta}} \text{ for } t \ge t_0.$$

Then as the same arguments in (1), we have

$$J'(t)^2 \ge \alpha_2 + \beta_2 J(t)^{2+\frac{1}{\delta}} \text{ for } t \ge t_0,$$

where

$$\begin{aligned} \alpha_2 &= \delta^2 J(t_0)^{2+\frac{2}{\delta}} \left[a'(t_0)^2 + \frac{2k_3}{1+2\delta} J(t_0)^{\frac{-1}{\delta}} \right] \\ &> 0. \end{aligned} \tag{5.23}$$

and

$$\beta_2 = -\frac{2k_3\delta^2}{1+2\delta}. \tag{5.24}$$

Thus, by Lemma 2.3, there exists a finite time T^* such that $\lim_{t\to T^{*-}} \left(\int_\Omega u^2 dx\right)^{-1} = 0$.

(3) For the case of $E_1 \le E(0)$

Applying the same arguments as in part (1), we also have (5.21) and (5.22). We observe that

$$\alpha_1 > 0 \text{ iff } E(0) < \frac{a'(t_0)^2}{8a(t_0)}.$$

Hence, by Lemma 2.3, there exists a finite time T^* such that $\lim_{t\to T^{*-}} \left(\int_\Omega u^2 dx\right)^{-1} = 0$.

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