TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 6B, pp. 2011-2020, December 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

PRIME SUBMODULES OF ARTINIAN MODULES

A. Azizi

Abstract. Prime submodules and weakly prime submodules of Artinian modules are characterized. Furthermore, some previous results on prime modules are generalized.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we consider R to be a ring and M a unitary R-module.

A proper submodule N of M is a *prime* submodule of M, if for each $r \in R$ and $a \in M$, the condition $ra \in N$ implies that $a \in N$ or $rM \subseteq N$. In this case, $P = (N : M) = \{t \in R|\ tM \subseteq N\}$ is a prime ideal of R, and we say N is a P-*prime* submodule of M. (See [1-3], [5-8, 10, 11, 13, 14, 16, 17]).

Recall that an R-module M is said to be a *multiplication* module if for any submodule L of M, $L = (L : M)M$. (See [4, 6, 9])

Let N be a proper submodule of M. If for any element x of M and elements a, b of R, $abx \in N$ implies that $ax \in N$ or $bx \in N$, then N is called a *weakly prime* submodule of M. (See [2, 7, 10]).

Prime and weakly prime submodules are generalizations of prime ideals in commutative rings. Obviously any prime submodule is a weakly prime submodule, but the converse is not always correct.

Example 1. Let R be a ring of positive Krull dimension and $P \subset Q$ a chain of prime ideals of R. Then one can see that for the free R-module $M = R \oplus R$, the submodule $P \oplus Q$ is a weakly prime submodule, which is not a prime submodule.

Received October 4, 2007, accepted February 23, 2008.
Communicated by Wen-Fong Ke.
2000 *Mathematics Subject Classification*: 13C99, 13C13, 13E05, 13F05, 13F15.
Key words and phrases: Catenary modules, Dimension of modules, Multiplication modules, Prime submodules, Reduced dimension of modules, Weakly prime submodules.

We denote the set of prime submodules [resp. weakly prime submodules] of M by $Spec(M)$ [resp. $WSpec(M)$].

Recall that the *height* of a prime submodule N of an R-module M, denoted by $ht\ N$, is n, if there exists a chain of prime submodules $N_0 \subset N_1 \subset N_2 \cdots \subset N_n = N$ of M and there does not exist such a chain of greater length.

Also the *dimension* of an R-module M denoted by $\dim\ M$ is defined by

$$\sup\{ht\ N : N \text{ is a prime submodule of M }\},$$

if $Spec(M) \neq \emptyset$, otherwise it is defined to be -1 (see [1, 8]).

Definition. The reduced dimension of an R-module M denoted by $r.\dim\ M$ or $r.\dim_R\ M$ is defined by

$$\sup\{n \mid \exists N_0 \subset N_1 \subset \cdots \subset N_n \ni N_i \in Spec(M), (N_i : M) \neq (N_j : M) \text{ for } i \neq j\},$$

if $Spec(M) \neq \emptyset$, otherwise it is defined to be -1.

In this paper we will characterize prime submodules of Artinian modules. It is proved that N is a prime submodule of an Artinian module M, if and only if $(N : M)$ is a maximal ideal of R (Corollary 2.4). Moreover, the dimension and reduced dimension of Artinian modules are studied (see Proposition 2.5, Corollary 2.6, and Proposition 2.8). We will prove that in a module with DCC on cyclic submodules, a submodule is a prime submodule if and only if it is a weakly prime submodule (Theorem 2.7). Furthermore, we will generalize Theorem 2.3, Proposition 3.1, and Proposition 3.2 of [17] (see Proposition 2.1, Corollaries 2.2, 2.3, Proposition 2.10, and Corollary 2.12).

2. Artinian Modules

A module M of which the 0 submodule is a prime submodule is called a *prime module*. It is easy to prove that M is a prime module if and only if $Ann\ N = Ann\ M$, for all non-zero submodules N of M. In [17, Theorem 2.3], it is proved that an Artinian faithful multiplication R-module is a prime module if and only if R is a Dedekind domain. The following two results are generalizations of this theorem.

Proposition 2.1. *An Artinian R-module M is a prime module if and only if $\frac{R}{Ann\ M}$ is a field.*

Proof. Let $T = \{N \mid N$ is a non-trivial submodule of $M\}$. Suppose that N_0 is a minimal element of T. Obviously N_0 is a non-zero simple module. Hence there exists an element $0 \neq a \in M$ such that $N_0 = Ra \cong \frac{R}{Ann\ a}$, and $Ann\ a$ is a maximal ideal of R. Since M is a prime module, $Ann\ a = Ann\ M$. Consequently, $Ann\ M$ is a maximal ideal of R.

For the converse note that in a vector space every proper submodule (subspace) is a prime submodule. Now since 0 is a prime submodule of M as an $\frac{R}{Ann\ M}$-module, obviously it is a prime submodule of M as an R-module. ∎

Corollary 2.2. *An Artinian faithful R-module is a prime module if and only if R is a field.*

Proof. The proof is clear by Proposition 2.1. ∎

Recall that an R-module M is said to be a π*-module* if for every non-zero submodule N of M, $\sum_{\phi \in Hom_R(N,M)} \phi(N) = M$.

Now we are ready to give a simple proof for [17, Theorem 1.3].

Corollary 2.3. *Every Artinian prime module is a π-module.*

Proof. By Proposition 2.1, M is a vector space over the field $\frac{R}{Ann\ M}$. Let N be a non-zero submodule (subspace) of M, $\mathbb{B}$ a basis for N, and $\mathbb{C}$ a basis for M. Consider $b_0 \in \mathbb{B}$. Evidently for any $c \in \mathbb{C}$, there exist a linear transformation $\phi_c : N \longrightarrow M$ such that $\phi_c(b_0) = c$, and clearly $\phi_c \in Hom_R(N, M)$. So $M = \sum_{c \in \mathbb{C}} \phi_c(N) \subseteq \sum_{\phi \in Hom_R(N,M)} \phi(N) \subseteq M$.

Recall that an R-module M is said to be a *torsion-free* module if $T(M) = \{m \in M |\ \exists r \in R,\ rm = 0\} = 0$.

It is easy to see that a submodule N of an R-module M is a prime submodule if and only if $(N : M)$ is a prime ideal of R and $\frac{M}{N}$ is a torsion-free $\frac{R}{(N:M)}$-module.

Corollary 2.4. *Let N be a submodule of an Artinian R-module M. Then N is a prime submodule of M if and only if $(N : M)$ is a maximal ideal of R.*

Proof. Suppose that N is a prime submodule of M. Then $\frac{M}{N}$ is an Artinian prime R-module, consequently by Proposition 2.1, $\frac{R}{(N:M)} = \frac{R}{Ann\ \frac{M}{N}}$ is a field.

Conversely if $(N : M)$ is a maximal ideal of R, then $\frac{M}{N}$ is a vector space over the field $\frac{R}{(N:M)}$. Thus it is torsion-free. Hence N is a prime submodule of M. ∎

Recall that if R is an integral domain with the quotient field K, the *rank* of an R-module M which is written as $rank_R\ M$, is the dimension (rank) of the vector space KM over the field K; i.e., $rank_R\ M = rank_K\ KM$ (see, [15, p. 84]).

A module M is called a *catenary* module if for any prime submodules N and N' of M with $N \subset N'$, all the saturated chains of prime submodules of M starting from N and ending at N' have the same length. (See [16]).

Proposition 2.5. *Let M be an Artinian R-module.*

(i) *M is catenary on prime submodules.*

(ii) *If N is a P-prime submodule of M, then* $\dim\ \frac{M}{N} = rank_{\frac{R}{P}}\ \frac{M}{N} - 1.$

(iii) $\dim M = \sup\{rank_{\frac{R}{m}}\ \frac{M}{mM} \mid m$ *is a maximal ideal containing* $AnnM\} - 1.$

(iv) $r.\dim\ M \leq 0.$

Proof.

(i) Let $N \subset N'$ be a chain of prime submodules of M, where $P = (N : M)$. Let T be a submodule of M between N and N'. Then $P = (N : M) \subseteq (T : M)$. By Corollary 2.4, P is a maximal ideal of R, then $(T : M) = P$. Now since $(T : M)$ is a maximal ideal of R, again by Corollary 2.4, T is a prime submodule of M.

One checks easily that T is a (prime) submodule of M between N and N', if and only if $\frac{T}{N}$ is a $\frac{P}{P}$-prime submodule of the $\frac{R}{P}$-module (vector space) $\frac{M}{N}$ contained in $\frac{N'}{N}$. Hence, $N \subset T_1 \subset T_2 \subset T_3 \subset \cdots \subset N'$ is a saturated chain of prime submodules of M if and only if $\frac{N}{N} \subset \frac{T_1}{N} \subset \frac{T_2}{N} \subset \frac{T_3}{N} \subset \cdots \subset \frac{N'}{N}$ is a saturated chain of subspaces of $\frac{M}{N}$ over the filed $\frac{R}{P}$. Consequently for any saturated chain $\mathbb{C}$ of prime submodules of M starting from N and ending at N', we have $\ell(\mathbb{C}) = rank_{\frac{R}{P}}\ \frac{N'}{N}$.

(ii) Suppose that $\mathbb{C}' : N \subset N_1 \subset N_2 \subset \cdots$ is a saturated chain of prime submodules of M. By Corollary 2.4, $(N : M)$ is a maximal ideal of R, and so $\forall i,\ (N : M) = (N_i : M)$. Let $K = \frac{R}{(N:M)}$. Hence $\frac{N}{N} \subset \frac{N_1}{N} \subset \frac{N_2}{N} \subset \frac{N_3}{N} \subset \cdots$ is a saturated chain of proper subspaces of the vector space $\frac{M}{N}$ over the field K, and since for each i, $\frac{N_i}{N} \subset \frac{M}{N}$, $\ell(\mathbb{C}') \leq rank_K\ \frac{M}{N} - 1$, and so $\dim\ \frac{M}{N} \leq rank_K\ \frac{M}{N} - 1$.

Conversely if $\frac{N}{N} \subset \frac{L_1}{N} \subset \frac{L_2}{N} \subset \frac{L_3}{N} \subset \cdots$ is a saturated chain of proper subspaces of the vector space $\frac{M}{N}$ over the field K, then clearly $N \subset L_1 \subset L_2 \subset \cdots$ is a saturated chain of prime submodules of M. So $rank_K\ \frac{M}{N} - 1 \leq \dim\ \frac{M}{N}$.

(iii) Let m be a maximal ideal of R containing $Ann\ M$. If $mM = M$, then $rank_{\frac{R}{m}}\ \frac{M}{mM} - 1 = 0 - 1 \leq \dim\ M$. Otherwise, since $(mM : M) = m$ is a maximal ideal of R, mM is a prime submodule of M, and so by part (ii),

$rank_{\frac{R}{m}} \frac{M}{mM} - 1 = \dim \frac{M}{mM} \leq \dim M$. Hence,
$\sup\{rank_{\frac{R}{m}} \frac{M}{mM} \mid m$ is a maximal ideal containing $Ann\ M\} - 1 \leq \dim M$.

Now assume that N is an arbitrary prime submodule of M and the chain $N_0 \subset N_1 \subset N_2 \subset \cdots \subset N$ is the longest saturated chain of prime submodules of M ending at N. Let $(N_0 : M) = m'$. Corollary 2.4 shows that m' is a maximal ideal of R. So $m'M \subseteq N_0 \subset N_1 \subset N_2 \subset \cdots \subset N$ is a saturated chain of prime submodules of M.

Clearly $ht\ N \leq \dim \frac{M}{m'M}$, and by part (ii), $\dim \frac{M}{m'M} = rank_{\frac{R}{m}} \frac{M}{mM} - 1$, that is, $ht\ N \leq rank_{\frac{R}{m}} \frac{M}{mM} - 1$. Consequently,
$\dim M \leq \sup\{rank_{\frac{R}{m}} \frac{M}{mM} | m$ is a maximal ideal containing $AnnM\} - 1$.

(iv) The proof is clear by Corollary 2.4. ■

Recall that a module M is said to be a *weak multiplication* module if for every prime submodule N of M, $N = (N : M)M$ (see [6]).

Corollary 2.6. *Let M be an Artinian weak multiplication R-module.*

(*i*) *If M is a prime module, then M is a simple module.*

(*ii*) $\dim M \leq 0$.

(*iii*) $Spec\ M = \{mM \mid m$ *is a maximal ideal of* R *and* $mM \neq M\}$.

Proof.

(i) By Proposition 2.1, M is a vector space over the field $\frac{R}{Ann\ M}$. Thus every proper submodule (subspace) of M as an $\frac{R}{Ann\ M}$-module is a prime submodule of M. Evidently M is a weak multiplication $\frac{R}{Ann\ M}$-module. Hence if N is an R-submodule of M, then $N = \frac{I}{Ann\ M}M$, where I is an ideal of R containing $Ann\ M$. Note that $I = Ann\ M$ or $I = R$, which implies that $N = 0$ or $N = M$.

(ii) If $Spec\ M = \emptyset$, then by the definition $\dim M = -1$. Now assume that N is a prime submodule of M. Obviously $\frac{M}{N}$ is an Artinian weak multiplication prime R-module, then by part (i), $\frac{M}{N}$ is a simple module. Hence N is a maximal submodule of M. So in this case $\dim M = 0$.

(iii) The proof is clear by Corollary 2.4. ■

As it was mentioned in Example 1 of introduction, a weakly prime submodule of a module is not necessary a prime submodule. So we need some conditions on modules, which one of them is given in the following.

Theorem 2.7. *In a module with DCC on cyclic submodules, a submodule is a prime submodule if and only if it is a weakly prime submodule.*

Proof. Let M be an R-module with DCC on cyclic submodules, and W a weakly prime submodule of M. Suppose that $ra \in W$, where $a \in M$ and $r \in R \setminus (W : M)$. Assume $rb \notin W$ for some $b \in M$. Consider the following chain of submodules

$$\cdots \subseteq Rr^3(a+b) \subseteq Rr^2(a+b) \subseteq Rr(a+b)$$

For some positive number n, we have $\mathrm{R}r^{n+1}(a+b) = Rr^n(a+b)$, that is, $r^n(rt-1)(a+b) = 0$, for some $t \in R$. Now $r^n(rt-1)(a+b) = 0 \in W$. If $r^n(a+b) \in W$, then evidently $r(a+b) \in W$, and since $ra \in W$, we will have $rb \in W$, which is impossible. Hence $rta - a + (rt-1)b = (rt-1)(a+b) \in W$. Note that $rta \in W$, then,

$$-a + (rt-1)b \in W. \qquad (*)$$

We get that $-ra + r(rt-1)b = r(-a + (rt-1)b) \in W$ and then $r(rt-1)b \in W$. Since W is weakly prime and $rb \notin W$, it follows that $(rt-1)b \in W$, and by $(*)$, we get $a \in W$. ■

Recall that an R-module M is said to be a *torsion* module if $T(M) = M$.

An R-module M is said to be a *semi-non-torsion* module if M is not torsion as an $\frac{R}{Ann\ M}$-module, that is $T_{\frac{R}{Ann\ M}}(M) \neq M$, (see [4]). It is easy to see that M is a semi-non-torsion module if and only if for some $0 \neq a \in M$, $Ann\ a = Ann\ M$. Therefore every prime module is a semi-non-torsion module, that is the concept semi-non-torsion is a generalization of the concept prime for modules. In general a semi-non-torsion module is not necessarily a prime module.

Example 2. Let I be a proper ideal of a ring R and consider $M = \frac{R}{I}$ as an R-module. Note that $Ann(1+I) = I = Ann\ M$, then M is a semi-non-torsion R-module. Particularly let $R = \mathbb{Z}$, the set of integer numbers and put $I = 4\mathbb{Z}$. Then $M = \frac{\mathbb{Z}}{4\mathbb{Z}}$ is a semi-non-torsion $\mathbb{Z}$-module. But $2(2+4\mathbb{Z}) = 0$, $2 \notin 4\mathbb{Z} = (0 : M)$ and $0 \neq 2 + 4\mathbb{Z}$, which implies that M is not a prime $\mathbb{Z}$-module.

Proposition 2.8. *Let M be a non-zero R-module. The following are equivalent.*

(*i*) *M is a semi-non-torsion Artinian weak multiplication module.*

(*ii*) *M is a cyclic module and $\frac{R}{Ann\ M}$ is an Artinian ring.*

Proof. $(i) \Longrightarrow (ii)$ Let M be a semi-non-torsion Artinian weak multiplication R-module. Then there exist an element $0 \neq a \in M$ such that $Ann\ a = Ann\ M$. Since Ra is a finitely generated Artinian R-module, $\frac{R}{Ann\ a} = \frac{R}{Ann\ M}$ is an Artinian ring (see [12, p. 388, Lemma 4.3]). Now M is a weak multiplication $\frac{R}{Ann\ M}$-module, where $\frac{R}{Ann\ M}$ is an Artinian ring, so by [6, Proposition 2.11], M is a cyclic $\frac{R}{Ann\ M}$-module and obviously a cyclic R-module.

$(ii) \Longrightarrow (i)$ Let M be a cyclic module. Then obviously it is multiplication and particularly weak multiplication. Also since M is finitely generated and $\frac{R}{Ann\ M}$ is an Artinian ring, then M is an Artinian module. Suppose that $M = Ra$, evidently $Ann\ a = Ann\ M$, thus M is semi-non-torsion.

In [17, Proposition 3.1 and Proposition 3.2], the authors proved that:
Let M be a finitely generated faithful multiplication R-module. Then

(1) If N is a minimal prime submodule of M, then $(N:M)$ is a minimal prime ideal of R.

(2) If P is a minimal prime ideal of R, then PM is a minimal prime submodule of M.

For the rest of this paper, we will simply generalize these results, in Proposition 2.10 and Corollary 2.12. First we need the following lemma.

Lemma 2.9. *Let M be a finitely generated R-module. Then the following are equivalent.*

(i) *M is a multiplication module.*

(ii) *For each prime ideal P of R containing $Ann\ M$, PM is the only P-prime submodule of M.*

(iii) *For each maximal ideal P of R containing $Ann\ M$, PM is the only P-prime submodule of M.*

Proof. See [1, Theorem 2.16]. ■

Proposition 2.10. *Let M be a finitely generated multiplication R-module, and B and C two submodules of M.*

(i) *There is a one-to-one correspondence between prime submodules of M between B and C and prime ideals of R between $(B:M)$ and $(C:M)$.*

(ii) *If N is a prime submodule of M, then $ht\ N = ht_{\frac{R}{Ann\ M}} \frac{(N:M)}{Ann\ M}$, and $\dim \frac{M}{N} = \dim \frac{R}{(N:M)}$. In particular if N is a minimal prime submodule of M, then $(N:M)$ is a prime ideal of R, minimal over $Ann\ M$.*

(*iii*) *If P is a prime ideal of R containing $Ann\ M$, then PM is a prime submodule of M, $ht\ PM = ht_{\frac{R}{Ann\ M}} \frac{P}{Ann\ M}$ and $\dim\ \frac{M}{PM} = \dim\ \frac{R}{P}$. Particularly if P is a prime ideal of R, minimal over $Ann\ M$, then PM is a minimal prime submodule of M.*

(*iv*) $\dim\ M = cl.\dim\ M$.

(*v*) *M is a catenary module if and only if $\frac{R}{Ann\ M}$ is a catenary ring.*

Proof.

(i) Put

$$A = \{N |\ N \text{ is a prime submodule of } M \text{ and } B \subseteq N \subseteq C\},$$

and

$$B = \{P |\ P \text{ is a prime ideal of } R \text{ and } (B : M) \subseteq P \subseteq (C : M)\},$$

and the function $\phi : A \longrightarrow B$, $\phi(N) = (N : M)$.

We show that ϕ is a bijective function.

If $N_1, N_2 \in A$ with $(N_1 : M) = (N_2 : M)$, then since M is multiplication, $N_1 = (N_1 : M)M = (N_2 : M)M = N_2$.

Now suppose that P is a prime ideal of R with $(B : M) \subseteq P \subseteq (C : M)$. Evidently $Ann\ M = (0 : M) \subseteq (B : M) \subseteq P$. Lemma 2.9, shows that PM is a P-prime submodule of M. Note that $B = (B : M)M \subseteq PM \subseteq (C : M)M = C$. Hence $PM \in A$ and $\phi(PM) = (PM : M) = P$.

(ii) Put $B = 0$, and $C = N$. Then clearly by part (i), $ht\ N = ht_{\frac{R}{Ann\ M}} \frac{(N:M)}{Ann\ M}$. Now if we put $B = N$ and $C = M$, then again by part (i), we get $\dim\ \frac{M}{N} = \dim\ \frac{R}{(N:M)}$.

(iii) The proof is given by Lemma 2.9, and part (ii).

The proofs of parts (iv) and (v) are clear according to part (i). ■

Lemma 2.11. *Let M be a finitely generated R-module and B a submodule of M. If $(B : M) \subseteq P$, where P is a prime ideal of R, then there exists a P-prime submodule N of M containing B.*

Proof. See [1, Lemma 4], or [14, Theorem 3.3]. ■

Corollary 2.12. *Let M be a finitely generated R-module. Then the following are equivalent.*

(i) *M is a multiplication module.*

(ii) *For every two submodules B and C of M, there is a one-to-one correspondence between prime submodules of M between B and C, and prime ideals of R between $(B : M)$ and $(C : M)$.*

(iii) *If B is a submodule of M, and P a prime ideal of R, minimal over $(B : M)$, then PM is a prime submodule of M, minimal over B.*

(iv) *M is a weak multiplication module.*

Proof. (i) $\Longrightarrow$ (ii) By Proposition 2.10(i).

(ii)$\Longrightarrow$ (i) Let P be a maximal ideal of R containing $Ann\ M$. By Lemma 2.11, there exists a prime submodule N of M with $(N : M) = P$. Since $PM \subseteq N$, $P \subseteq (PM : M) \subseteq (N : M) = P$, and so $(PM : M) = P$. Now $(PM : M) = P$ is a maximal ideal of R, then PM is a P-prime submodule of M.

Put $B = PM$ and $C = M$. Since P is the only prime ideal of R between $(B : M) = P$ and $(C : M) = R$, then there is exactly one prime submodule of M (between $B = PM$ and $C = M$), which is PM. Now by Lemma 2.9(iii), M is a multiplication module.

(i) $\Longrightarrow$ (iii) By Lemma 2.9(ii), PM is a P-prime submodule of M. Put $C = PM$. Note that $(B : M) \subseteq P$, so $B = (B : M)M \subseteq PM = C$. Since P is the only prime ideal of R, between $(B : M)$ and $P = (C : M)$, by Proposition 2.10(i), there is exactly one prime submodule of M between B and $C = PM$, which is PM.

(iii) $\Longrightarrow$ (iv) Let N be a P-prime submodule of M. Since $(N : M) = P$, then by assumption PM is a prime submodule minimal over N, and since N is a prime submodule, $N = PM$. Hence M is a weak multiplication module.

(iv) $\Longrightarrow$ (i) Let P be a maximal ideal of R containing $Ann\ M$. By Lemma 2.11, there exists a prime submodule N of M with $(N : M) = P$. Since M is weak multiplication, $N = (N : M)M = PM$. So PM is the only P-prime submodule of M, and by Lemma 2.9(iii), M is a multiplication module.

ACKNOWLEDGMENT

The author would like to thank the referee for his comments and suggestions.

REFERENCES

1. A. Azizi, Intersectin of prime submodules and dimension of modules, *Acta Math. Scientia*, **25B(3)** (2005), 385-394.
2. A. Azizi, On prime and weakly prime submodules, *Vietnam J. Math.*, **36(3)** (2008), 315-325.

3. A. Azizi, Prime submodules and flat modules, *Acta Math. Sinica, English Series*, **23(1)** (2007), 147-152.

4. A. Azizi, Principal ideal multiplication modules, *Algebra Colloquium.*, **15(4)** (2008), 637-648.

5. A. Azizi, Radical formula and prime submodules, *Journal of Algebra*, **307** (2007), 454-460.

6. A. Azizi, Weak multiplication modules, *Czech Mathematical Journal*, **53(128)** (2003), 529-534.

7. A. Azizi, Weakly prime submodules and prime submodules, *Glasgow Mathematical Journal*, **48(2)** (2006), 343-346.

8. A. Azizi and H. Sharif, On prime submodules, *Honam Mathematical Journal*, **21(1)** (1999), 1-12.

9. A. Barnard, Multiplication modules, *Journal of Algebra*, **71** (1981), 174-178.

10. M. Behboodi and H. Koohi, Weakly prime submodules, *Vietnam Journal of Math.*, **32(2)** (2004), 185-195.

11. A. M. George, R. L. McCasland and P. F. Smith, A principal ideal theorem analogue for modules over commutative rings, *Comm. Algebra*, **22** (1994), 2083-2099.

12. T. W. Hungerford, *Algebra*, Springer-Verlog, New York Inc., 1989.

13. C. P. Lu, Spectra of modules, *Comm. Algebra*, **23(10)** (1995), 3741-3752.

14. R. L. McCasland and M. E. Moore, Prime submodules, *Comm. Algebra*, **20(6)** (1992), 1803-1817.

15. H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1992.

16. S. Namazi and Y. Sharifi, Catenary modules, *Acta Math. Hungarica*, **85(3)** (1999), 211-218.

17. Y. Tiras and M. Alkan, Prime modules and submodules, *Comm. Algebra*, **31(11)** (2003), 5253-5261.

A. Azizi
Department of Mathematics,
College of Sciences,
Shiraz University,
Shiraz 71457-44776,
Iran
E-mail: a_azizi@yahoo.com
aazizi@shirazu.ac.ir