TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 6B, pp. 2001-2009, December 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

SOME INCLUSION PROPERTIES OF CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Abstract. We use a property of the Bernardi operator in the theory of the Briot–Bouquet differential subordinations to prove several theorems for some classes of analytic functions defined by using the Dziok-Srivastava operator. Some of these results we obtain applying the convolution property due to Rusheweyh. We take advantage of the Miller–Mocanu differential subordinations.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in $\mathcal{U} = \mathcal{U}(1)$, where $\mathcal{U}(r) = \{z : z \in \mathbf{C} \text{ and } |z| < r\}$.

For analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \mathcal{U}),$$

by $f * g$ we denote the Hadamard product or convolution of f and g, defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Received October 8, 2007, accepted February 21, 2008.
Communicated by H. M. Srivastava.
2000 *Mathematics Subject Classification*: 30C45, 26A33.
Key words and phrases: Dziok–Srivastava operator, Bernardi operator, Hadamard product (or convolution), Subordination, Starlike functions, Convex functions.

Moreover, we say that a function f is subordinate to a function g, and write $f(z) \prec g(z)$, if and only if there exists a function ω, analytic in $\mathcal{U}$ such that

$$\omega(0) = 0,\ |\omega(z)| < 1 \quad (z \in \mathcal{U}),$$

and

$$f(z) = g(\omega(z)) \quad (z \in \mathcal{U}).$$

In particular, if g is univalent in $\mathcal{U}$, we have the following equivalence

$$f(z) \prec g(z) \iff f(0) = g(0) \ \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}). \tag{2}$$

Let $\mathcal{K}$ denote the class of convex function defined by

$$\mathcal{K} := \left\{ f \in \mathcal{A} :\ \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathcal{U} \right\}.$$

Moreover we recall the class of function introduced by Janowski [6]

$$\mathcal{S}^*\left[\frac{1+az}{1+bz}\right] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+az}{1+bz},\ z \in \mathcal{U} \right\} \ (-1 \le b < a \le 1). \tag{3}$$

In particular we have the class of starlike functions $\mathcal{S}^* := \mathcal{S}^*\left[\frac{1+z}{1-z}\right]$. In this paper we take advantage of $\mathcal{S}^*\left[\frac{1+az}{1+bz}\right]$ to define other class of functions.

Let $q, s \in \mathbf{N} = \{1, 2, ...\},\ q \le s+1$. For complex parameters $a_1, \ldots, a_q$ and $b_1, \ldots, b_s$, $(b_j \neq 0, -1, -2, \ldots;\ j = 1, \ldots, s)$, the generalized hypergeometric function ${}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z)$ is defined by

$${}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdot \ldots \cdot (a_q)_n}{(b_1)_n \cdot \ldots \cdot (b_s)_n} \frac{z^n}{n!} \quad (z \in \mathcal{U}),$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1) \cdot \ldots \cdot (\lambda+n-1) & (n \in \mathbf{N}). \end{cases}$$

Let us consider the Dziok–Srivastava operator [4] (see also [3] and [5])

$$\mathcal{H} : \mathcal{A} \to \mathcal{A}$$

such that

$$\mathcal{H}f(z) = \mathcal{H}(a_1, \ldots, a_q; b_1, \ldots, b_s)f(z) = \{z \cdot {}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z)\} * f(z).$$

We observe that for a function f of the form (1), we have

$$\mathcal{H}(a_1,\ldots,a_q;b_1,\ldots,b_s)f(z) = z+\sum_{n=2}^{\infty}A_n a_n z^n, \tag{4}$$

where

$$A_n = \frac{(a_1)_{n-1}\cdot\ldots\cdot(a_q)_{n-1}}{(b_1)_{n-1}\cdot\ldots\cdot(b_s)_{n-1}\cdot(n-1)!}. \tag{5}$$

The Dziok-Srivastava operator $\mathcal{H}(a_1,\ldots,a_q;b_1,\ldots,b_s)$ includes various other linear operators which were considered in earlier works (see [11], [12] and [13]). In particular we recall the Bernardi integral operator [1]

$$\mathcal{J}_\nu : \mathcal{A}\to\mathcal{A},$$

defined by

$$\mathcal{J}_\nu[f(z)] = \frac{\nu+1}{z^\nu}\int_0^z t^{\nu-1}f(t)dt \quad (\nu\in\mathbf{C}). \tag{6}$$

For $f\in\mathcal{A}$ of the form (1) we have

$$\mathcal{J}_\nu[f(z)] = z+\sum_{n=2}^{\infty}\frac{\nu+1}{\nu+n}a_n z^n. \tag{7}$$

The Bernardi operator and the Dziok–Srivastava operator are connected in the following way

$$\mathcal{J}_\nu[f(z)] = \mathcal{H}(1+\nu,1;\nu+2)f(z).$$

Let suppose

$$-1\le B\le 0 \ \text{ and } \ |A|<1 \quad (A\in\mathbf{C}). \tag{8}$$

We denote by $V(q,s;A,B)$ the class of functions f of the form (1) which satisfy the following condition:

$$\frac{z\,[\mathcal{H}f(z)]'}{\mathcal{H}f(z)} \prec \frac{1+Az}{1+Bz} \quad (z\in\mathcal{U}). \tag{9}$$

By (8) we have $\mathrm{Re}\left(\frac{1+Az}{1+Bz}\right)>0$ for $z\in\mathcal{U}$, Thus

$$f\in V(q,s;A,B)\Rightarrow \mathcal{H}f(z)\in S^*. \tag{10}$$

Moreover for $-1\le B<A\le 1$ this means that $\mathcal{H}f(z)$ belongs to the class $\mathcal{S}^*\left[\frac{1+Az}{1+Bz}\right]$ defined by (3). After some calculations we obtain

$$a_i\mathcal{H}(a_i+1)f(z) = z\mathcal{H}'f(z)+(a_i-1)\mathcal{H}f(z), \quad i=1,...,q, \tag{11}$$

where, for convenience,

$$\mathcal{H}(a_i+m)f(z) = \mathcal{H}(a_1,\ldots,a_i+m,\ldots,a_q;b_1,\ldots,b_s)f(z), \quad i=1,...,q.$$

By (11) the condition (9) is for each a_i, $i=1,...,q$ equivalent the following subordination

$$a_i\frac{\mathcal{H}(a_i+1)f(z)}{\mathcal{H}f(z)}+1-a_i \prec \frac{1+Az}{1+Bz} \quad (z\in\mathcal{U})\,. \tag{12}$$

Therefore we use following alternatively notation

$$V(q,s;A,B)=V(a_i;A,B).$$

Dziok and Srivastava [4] making use of the generalized hypergeometric function, have introduced a class of analytic functions with negative coefficients. They considered the class $V(q,s;A,B)$ defined by condition (12) where parameters $a_1,\ldots,a_q$, $b_1,\ldots,b_s$ are positive real and -1$\leq A<B\leq 1$. Some inclusion for this class was given in [2].

The main object of this paper is to investigate a inclusion properties of the classes $V(q,s;A,B)$.

2. Main Results

We begin with a lemma, which will be useful later on.

Lemma 1. [8]. *Let $\nu, A\in \mathbf{C}$ and $B\in[-1;0]$ satisfy either*

$$\operatorname{Re}\left[1+AB+\nu(1+B^2)\right]\geq\mid A+B+B(\nu+\bar{\nu})\mid \;\; for\ B\in(-1;0], \tag{13}$$

or

$$1+A>0\ and\ \operatorname{Re}[1-A+2\nu]\geq 0\ for\ B=-1 \tag{14}$$

If $f\in\mathcal{A}$ and $F(z)=\mathcal{J}_\nu[f(z)]$ is given by (6), then $F\in\mathcal{A}$ and

$$\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz}\Rightarrow\frac{zF'(z)}{F(z)}\prec\frac{1+Az}{1+Bz}.$$

Lemma 1 in the more general case is in [8], p. 111.

Lemma 2. *If the function f is of the form (1), then*

$$\mathcal{H}f(z)=\mathcal{J}_{a_i-1}\left[\mathcal{H}(a_i+1)f(z)\right] \quad (i=1,2,...,q)\,, \tag{15}$$

where $\mathcal{J}_{a_i-1}$ is the Bernardi operator (6).

Proof. From (4) and from (5) we have

$$\begin{aligned}\mathcal{H}f(z) &= z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\cdot\ldots\cdot(a_q)_{n-1}}{(b_1)_{n-1}\cdot\ldots\cdot(b_s)_{n-1}\cdot(n-1)!}a_nz^n\\ &= z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}\cdot\ldots\cdot\frac{a_i}{a_i+n-1}\cdot(a_i+1)_{n-1}\cdot\ldots\cdot(a_q)_{n-1}}{(b_1)_{n-1}\cdot\ldots\cdot(b_s)_{n-1}\cdot(n-1)!}a_nz^n\\ &= \left[\sum_{n=1}^{\infty}\frac{a_i}{a_i+n-1}z^n\right]*[\mathcal{H}(a_i+1)f(z)]\\ &= \left[\sum_{n=1}^{\infty}\frac{(a_i-1)+1}{(a_i-1)+n}z^n\right]*[\mathcal{H}(a_i+1)f(z)]\,.\end{aligned}$$

Thus by (7) with $\nu=a_i+1$ we obtain (15).

Theorem 1. *If* $m\in\mathbf{N}$ *and* $i\in\{1,...,q\}$*, then*

$$V(a_i+m;A,B)\subseteq V(a_i;A,B), \tag{16}$$

whenever $A,\ B$ *satisfy either* (13) *or* (14) *with* $\nu=a_i-1$.

Proof. It is clear that it is sufficient to prove (16) only for $m=1$. Let $f\in V(a_i+1;A,B)$, then from (9) we have

$$\frac{z[\mathcal{H}(a_i+1)f(z)]'}{\mathcal{H}(a_i+1)f(z)}\prec\frac{1+Az}{1+Bz}\quad(z\in\mathcal{U})\,.$$

Applying Lemma 1 and Lemma 2, by (9) we obtain that $f\in V(a_i;A,B)$.

It is natural to ask about the inclusion relation (16) when m is not positive integer. Using a different method we will give a partial answer to this question. We will need the following lemma.

Lemma 3. [10]. *If* $f\in\mathcal{K}$, $g\in\mathcal{S}^*$*, then for each analytic function* h *in* $\mathcal{U}$*,*

$$\frac{(f*hg)(\mathcal{U})}{(f*g)(\mathcal{U})}\subseteq\overline{co}h(\mathcal{U}),$$

where $\overline{co}h(\mathcal{U})$ *denotes the closed convex hull of* $h(\mathcal{U})$.

Theorem 2. *If* $G(z)=\sum\limits_{n=0}^{\infty}\frac{(a_i)_n}{(\tilde{a}_i)_n}z^{n+1}\in\mathcal{K}$*, then* $V(\tilde{a}_i;A,B)\subset V(a_i;A,B)$.

Proof. Let $f \in V(\tilde{a}_i; A, B)$. *By the definition of the subordination we have*

$$z\frac{[\mathcal{H}(\tilde{a}_i)f(z)]'}{\mathcal{H}(\tilde{a}_i)f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} := \phi[\omega(z)] \ \ (z \in \mathcal{U}), \tag{17}$$

where ϕ *is convex univalent mapping of* $\mathcal{U}$ *and* $|\omega(z)| < 1$ *in* $\mathcal{U}$ *with* $\omega(0) = 0 = \phi(0) - 1$. *Moreover,* $\mathrm{Re}[\phi(z)] > 0, \ z \in \mathcal{U}$. Applying (17) and the properties of convolution we get

$$\begin{aligned} \frac{z[\mathcal{H}(a_i)f(z)]'}{\mathcal{H}(a_i)f(z)} &= \frac{z\left[\sum\limits_{n=0}^{\infty}\frac{(a_i)_n}{(\tilde{a}_i)_n}z^{n+1} * \mathcal{H}(\tilde{a}_i)f(z)\right]'}{\sum\limits_{n=0}^{\infty}\frac{(a_i)_n}{(\tilde{a}_i)_n}z^{n+1} * \mathcal{H}(\tilde{a}_i)f(z)} \\ &= \frac{G(z) * zH'(z)}{G(z) * H(z)} = \frac{G(z) * \phi[\omega(z)]H(z)}{G(z) * H(z)} =: g(z). \end{aligned} \tag{18}$$

Because $H(z) \in \mathcal{S}^*$, $G(z) \in \mathcal{K}$ and ϕ is convex univalent, then by Lemma 3 we obtain that for $z \in \mathcal{U}$ the quantity (18) lies in $\overline{co}\phi[\omega(\mathcal{U})]$. By (2) and from the above-mentioned properties of ϕ we conclude that g defined by(18) is subordinated to ϕ. Thus, by (9) we have that $\mathcal{H}(a_i)f(z) \in \mathcal{S}^*\left[\frac{1+Az}{1+Bz}\right] \subseteq \mathcal{S}^*$ and finally $f \in V(a_i; A, B)$.

Lemma 4. [9]. *If either* $0 < a \leq c$ *and* $c \geq 2$ *when* a, c *are real number, or* $\mathrm{Re}[a + c] \geq 3$, $\mathrm{Re}[a] \leq \mathrm{Re}[c]$ *and* $\mathrm{Im}[a] = \mathrm{Im}[c]$ *when* a, c *are complex, then the function*

$$f(z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1} \ \ (z \in \mathcal{U})$$

belongs to the class $\mathcal{K}$ *of convex functions.*

Lemma 4 is a special case of Theorem 2.12 or Theorem 2.13 contained in [9].

Theorem 3. *Let* $i \in \{1, 2, ..., q\}$. *If* $a_i, \tilde{a}_i$ *are real number such that*

$$0 < a_i \leq \tilde{a}_i \ \ and \ \ \tilde{a}_i \geq 2$$

or $a_i, \tilde{a}_i$ *are complex number such that*

$$\mathrm{Re}[a_i + \tilde{a}_i] \geq 3, \ \mathrm{Re}[a_i] \leq \mathrm{Re}[\tilde{a}_i] \ \ and \ \ \mathrm{Im}[a_i] = \mathrm{Im}[\tilde{a}_i],$$

then

$$V(\tilde{a}_i; A, B) \subseteq V(a_i; A, B).$$

Proof. Since $\mathcal{H}(\tilde{a}_i)f(z) \in \mathcal{S}^*$, by Lemma 4 the function

$$G(z) = \sum_{n=0}^{\infty} \frac{(a_i)_n}{(\tilde{a}_i)_n} z^{n+1} \ (z \in \mathcal{U})$$

belongs to the class of convex functions $\mathcal{K}$. Using Theorem 1 we obtain that $f \in V(a_i; A, B)$.

Lemma 5. ([8], p.240). *If a, b, c are real and satisfy $-2 \leq a < 0$, $b \neq 0$, $-1 \leq b$ and $c > M(a, b)$, where*

$$M(a, b) = \max\{2 + |a + b|, 1 - ab\},$$

then the Gaussian hypergeometric function

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

is convex in $\mathcal{U}$.

Lemma 6. *Let $-1 \leq a_i < 1$, $i \in \{1, ..., q\}$. If $\tilde{a}_i > 3 + |a_i|$, then*

$$\sum_{n=0}^{\infty} \frac{(a_i)_n}{(\tilde{a}_i)_n} z^{n+1} \in \mathcal{K}.$$

Proof. Let we chose $b = 1$, $a = a_i - 1$, $c = \tilde{a}_i - 1$ in Lemma 5. Then we obtain that the function

$$F(z) = \sum_{n=0}^{\infty} \frac{(a_i - 1)_n}{(\tilde{a}_i - 1)_n} z^n$$

is convex for $-2 \leq a_i - 1 < 0$ and $\tilde{a}_i - 1 > M(a, b) = 2 + |a_i|$. It is clear that $G(z) = \frac{\tilde{a}_i - 1}{a_i - 1} [F(z) - 1] \in \mathcal{K}$. After some calculations we obtain that

$$G(z) = \sum_{n=0}^{\infty} \frac{(a_i)_n}{(\tilde{a}_i)_n} z^{n+1}$$

and this ends the proof.

Theorem 4. *Let $-1 \leq a_i < 1$, $i \in \{1, ..., q\}$. If $\tilde{a}_i > 3 + |a_i|$, then*

$$V(\tilde{a}_i; A, B) \subseteq V(a_i; A, B).$$

Proof. The proof runs as the proof of Theorem 3 by using Lemma 6.

Theorem 5. *Let* $m \in \mathbf{N}$, $i \in \{1, ..., q\}$. *If* $\operatorname{Re} a_i > 1$, *then*

$$V(a_i + m; A, B) \subseteq V(a_i; A, B). \tag{19}$$

Proof. It is clear that it is sufficient to prove (19) only for $m = 1$. If $f \in V(a_i+1; A, B)$, then by (10) we have $H(z) := \mathcal{H}(a_i+1)f(z) \in \mathcal{S}^* \left[\frac{1+Az}{1+Bz}\right] \subseteq \mathcal{S}^*$. Let us denote

$$\frac{zH'(z)}{H(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} := \phi[\omega(z)] \ \ (z \in \mathcal{U}),$$

where ϕ is convex univalent and $|\omega(z)| < 1$ in $\mathcal{U}$ with $\omega(0) = 0 = \phi(0) - 1$ Moreover $\operatorname{Re}[\phi(z)] > 0$. If $\operatorname{Re} a_i > 1$, then the function

$$G(z) = \sum_{n=1}^{\infty} \frac{(a_i - 1) + 1}{(a_i - 1) + n} z^n \ \ (z \in \mathcal{U})$$

belongs to the class of convex functions $\mathcal{K}$, (Ruscheweych,[9]). Recall that

$$f(z) * G(z) = \mathcal{J}_{1,a_i-1}\left[f(z)\right],$$

where $\mathcal{J}_{a_i-1}$ is the Bernardi operator defined by (6). From the proof of Lemma 2 we have

$$\mathcal{H}(a_i)f(z) = G(z) * \mathcal{H}(a_i + 1)f(z).$$

Thus

$$\begin{aligned} \frac{z[\mathcal{H}(a_i)f(z)]'}{\mathcal{H}(a_i)f(z)} &= \frac{[G(z) * zH(z)]'}{G(z) * H(z)} = \frac{G(z) * zH'(z)}{G(z) * H(z)} \\ &= \frac{G(z) * \phi[\omega(z)]H(z)}{G(z) * H(z)} \in \overline{co}\phi(\mathcal{U}). \end{aligned}$$

For the same reasons as in the proof of Theorem 2 we obtain that $f \in V(a_i; A, B)$.

References

1. S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, **135** (1969), 429-446.

2. J. Dziok, On some applications of the Briot–Bouquet differential subordinations, *J. Math. Anal. Appl.*, (2007), 295-301.

3. J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transform. Spec. Funct.*, **14** (2003), 7-18.

4. J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.*, **103** (1999), 1-13.

5. J. Dziok and H. M. Srivastava, Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function, *Advanced Studies in Contemporary Mathematics*, **5(2)** (2002), 115-125.

6. W. Janowski, Extremal problems for a family of functions with positive real part and for some related families, *Ann. Polon. Math.*, **23** (1970), 159-177.

7. Y. C. Kim and H. M. Srivastava, Fractional integral and other linear operators associated with the Gaussian hypergeometric function, *Complex Variables Theory Appl.*, **34** (1997), 293-312.

8. S. S. Miller and P. T. Mocanu, *Differential subordinations: theory and applications*, Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York · Basel, 2000.

9. St. Ruscheweyh, *Convolutions in Geometric Function Theory*, Sem. Math. Sup., 83, Presses Univ. Montreal, 1982.

10. St. Ruscheweyh and T. Sheil-Small, Hadamard product of schlicht functions and the Poyla-Schoenberg conjecture, *Comm. Math. Helv.*, **48** (1973), 119-135.

11. H. M. Srivastava and M. K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II, *J. Math. Anal. Appl.*, **171** (1992), 1-13; *ibid*. **192** (1995), 673-688.

12. H. M. Srivastava and S. Owa, Some characterization and distortion theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions, *Nagoya Math. J.*, **106** (1987), 1-28.

13. H. M. Srivastava and S. Owa (eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.

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