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STABILITY OF HOMOMORPHISMS AND DERIVATIONS IN C^*-TERNARY RINGS

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Abstract. In this paper, we prove the Hyers-Ulam-Rassias stability of homomorphisms in C^*-ternary rings and of derivations on C^*-ternary rings for the following generalized Cauchy-Jensen additive mapping:

$$2f\left(\frac{\sum_{j=1}^{p} x_j + \sum_{j=1}^{q} y_j}{2} + \sum_{j=1}^{d} z_j\right) = \sum_{j=1}^{p} f(x_j) + \sum_{j=1}^{q} f(y_j) + 2\sum_{j=1}^{d} f(z_j).$$

This is applied to investigate isomorphisms in C^*-ternary rings.

1. INTRODUCTION AND PRELIMINARIES

A C^*-ternary ring is a complex Banach space A, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x, x, x]\| = \|x\|^3$ (see [21]). Every left Hilbert C^*-module is a C^*-ternary ring via the ternary product $[x, y, z] := \langle x, y\rangle z$.

If a C^*-ternary ring $(A, [\cdot, \cdot, \cdot])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x, e, y]$ and $x^* := [e, x, e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary ring (see [9]).

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a C^**-ternary ring homomorphism* if

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$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$. If, in addition, the mapping H is bijective, then the mapping $H : A \to B$ is called a *C^*-ternary ring isomorphism*. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$ (see [9]).

A classical question in the theory of functional equations is the following: "When is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If the problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [20] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [6]. It gave rise the stability theory for functional equations. Aoki [1] generalized Hyers' theorem for approximately additive mappings. Th.M. Rassias [17] extended Hyers' Theorem by obtaining a unique linear mapping under certain continuity assumption when the Cauchy difference is allowed to be unbounded. Subsequently, various approach to the problem have been introduced by several authors such as Gajda [4] and Gavruta [5]. For the history and various aspects of this theory we refer the reader to [3, 8, 18, 19]. Recently the stability problem in ternary structures has extensively been investigated (see [2, 9-16])

Throughout this paper, assume that p, q, d are nonnegative integers with $p+q+d \geq 3$.

In Section 2, we prove the Hyers–Ulam–Rassias stability of homomorphisms in C^*-ternary rings for the generalized Cauchy–Jensen additive mapping.

In Section 3, we investigate isomorphisms in unital C^*-ternary rings associated with the generalized Cauchy–Jensen additive mapping.

In Section 4, we prove the Hyers–Ulam–Rassias stability of derivations on C^*-ternary rings for the generalized Cauchy–Jensen additive mapping.

2. Stability of Homomorphisms in C^*-Ternary Rings

Throughout this section, assume that A is a C^*-ternary ring with norm $\|\cdot\|_A$ and that B is a C^*-ternary ring with norm $\|\cdot\|_B$.

For a given mapping $f : A \to B$, we define

$$\begin{aligned} C_\mu f(x_1, ..., x_p, y_1, ..., y_q, z_1, ..., z_d) := 2f\left(\frac{\sum_{j=1}^{p} \mu x_j + \sum_{j=1}^{q} \mu y_j}{2} + \sum_{j=1}^{d} \mu z_j\right) \\ - \sum_{j=1}^{p} \mu f(x_j) - \sum_{j=1}^{q} \mu f(y_j) - 2\sum_{j=1}^{d} \mu f(z_j) \end{aligned}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_d \in A$.

It is easy to show that a mapping $f : A \to B$ satisfies $C_1 f(x_1, \cdots, x_p, y_1, \cdots, y_d, z_1, \cdots, z_d) = 0$ if and only if f is Cauchy additive, and $f(0) = 0$.

We prove the Hyers–Ulam–Rassias stability of homomorphisms in C^*-ternary rings for the functional equation $C_\mu f(x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_d) = 0$.

Theorem 2.1. *Let $r > 3$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\begin{aligned} &\|C_\mu f(x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_d)\|_B \\ &\leq \theta \left(\sum_{j=1}^{p} \|x_j\|_A^r + \sum_{j=1}^{q} \|y_j\|_A^r + \sum_{j=1}^{d} \|z_j\|_A^r \right), \end{aligned} \tag{2.1}$$

$$\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_d \in A$. Then there exists a unique C^*-ternary ring homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2^r(p+q+d)}{2(p+q+2d)^r - (p+q+2d)2^r}\theta\|x\|_A^r \tag{2.3}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x_1 = \cdots = x_p = y_1 = \cdots = y_q = z_1 = \cdots z_d = x$ in (2.1), we get

$$\left\|2f\left(\frac{p+q+2d}{2}x\right) - (p+q+2d)f(x)\right\|_B \leq (p+q+d)\theta\|x\|_A^r \tag{2.4}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{(p+q+2d)}{2} f\left(\frac{2}{p+q+2d}x\right)\right\|_B \leq \frac{2^r(p+q+d)}{2(p+q+2d)^r}\theta\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned} &\left\|\frac{(p+q+2d)^l}{2^l} f\left(\frac{2^l}{(p+q+2d)^l}x\right) - \frac{(p+q+2d)^m}{2^m} f\left(\frac{2^m}{(p+q+2d)^m}x\right)\right\|_B \\ &\leq \sum_{j=l}^{m-1} \left\| \frac{(p+q+2d)^j}{2^j} f\left(\frac{2^l}{(p+q+2d)^j}x\right) \right. \\ &\quad \left. - \frac{(p+q+2d)^{j+1}}{2^{j+1}} f\left(\frac{2^{j+1}}{(p+q+2d)^{j+1}}x\right) \right\|_B \\ &\leq \frac{2^r(p+q+d)}{2(p+q+2d)^r} \sum_{j=l}^{m-1} \frac{2^{rj}(p+q+2d)^j}{2^j(p+q+2d)^{rj}}\theta\|x\|_A^r \end{aligned} \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.5) that the sequence $\{\frac{(p+q+2d)^n}{2^n} f(\frac{2^n}{(p+q+2d)^n} x)\}$ is Cauchy for all $x \in A$. Since B is complete, the sequence $\{\frac{(p+q+2d)^n}{2^n} f(\frac{2^n}{(p+q+2d)^n} x)\}$ converges. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n}{(p+q+2d)^n} x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.5), we get (2.3).

It follows from (2.1) that

$$\begin{aligned}
&\left\| 2H\left(\frac{\sum_{j=1}^p x_j + \sum_{j=1}^q y_j}{2} + \sum_{j=1}^d z_j\right) - \sum_{j=1}^p H(x_j) - \sum_{j=1}^q H(y_j) - 2\sum_{j=1}^d H(z_j)\right\|_B \\
&= \lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} \left\| 2f\left(\frac{2^n}{(p+q+2d)^n} \frac{\sum_{j=1}^p x_j + \sum_{j=1}^q y_j}{2} + \frac{2^n}{(p+q+2d)^n} \sum_{j=1}^d z_j\right)\right. \\
&\quad \left. - \sum_{j=1}^p f\left(\frac{2^n}{(p+q+2d)^n} x_j\right) - \sum_{j=1}^q f\left(\frac{2^n}{(p+q+2d)^n} y_j\right) - 2\sum_{j=1}^d f\left(\frac{2^n}{(p+q+2d)^n} z_j\right)\right\|_B \\
&\le \lim_{n\to\infty} \frac{2^{nr}(p+q+2d)^n}{2^n(p+q+2d)^{nr}} \theta \left(\sum_{j=1}^p \|x_j\|_A^r + \sum_{j=1}^q \|y_j\|_A^r + \sum_{j=1}^d \|z_j\|_A^r\right) = 0
\end{aligned}$$

for all $x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_d \in A$. Hence

$$2H\left(\frac{\sum_{j=1}^p x_j + \sum_{j=1}^q y_j}{2} + \sum_{j=1}^d z_j\right) = \sum_{j=1}^p H(x_j) + \sum_{j=1}^q H(y_j) + 2\sum_{j=1}^d H(z_j)$$

for all $x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_d \in A$. So the mapping $H : A \to B$ is Cauchy additive.

By the same reasoning as in the proof of Theorem 2.1 of [13], the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.2) that

$$\begin{aligned}
&\|H([x,y,z]) - [H(x), H(y), H(z)]\|_B \\
&= \lim_{n\to\infty} \frac{(p+q+2d)^{3n}}{8^n} \left\| f\left(\frac{8^n[x,y,z]}{(p+q+2d)^{3n}}\right)\right. \\
&\quad \left. - \left[f\left(\frac{2^n x}{(p+q+2d)^n}\right), f\left(\frac{2^n y}{(p+q+2d)^n}\right), f\left(\frac{2^n z}{(p+q+2d)^n}\right)\right]\right\|_B \\
&\le \lim_{n\to\infty} \frac{2^{nr}(p+q+2d)^{3n}}{8^n(p+q+2d)^{nr}} \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. Thus

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$.

Now, let $T : A \to B$ be another generalized Cauchy-Jensen additive mapping satisfying (2.3). Then we have

$$\begin{aligned}
&\|H(x) - T(x)\|_B \\
= \ & \frac{(p+q+2d)^n}{2^n} \|H\left(\frac{2^n x}{(p+q+2d)^n}\right) - T\left(\frac{2^n x}{(p+q+2d)^n}\right)\|_B \\
\le \ & \frac{(p+q+2d)^n}{2^n} \left(\|H(\frac{2^n x}{(p+q+2d)^n}\right) - f\left(\frac{2^n x}{(p+q+2d)^n}\right)\|_B \\
& + \|T\left(\frac{2^n x}{(p+q+2d)^n}\right) - f\left(\frac{2^n x}{(p+q+2d)^n})\|_B\right) \\
\le \ & \frac{2^r(p+q+d)}{2(p+q+2d)^r - (p+q+2d)2^r} \cdot \frac{2^{nr+1}(p+q+2d)^n}{2^n(p+q+2d)^{nr}} \theta \|x\|_A^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of H. Thus the mapping $H : A \to B$ is a unique C^*-ternary ring homomorphism satisfying (2.3). ■

In the following theorem we have an alternative result of Theorem 2.1

Theorem 2.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.1) *and* (2.2). *Then there exists a unique C^*-ternary ring homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{2^r(p+q+d)}{2^r(p+q+2d) - 2(p+q+2d)^r} \theta \|x\|_A^r \tag{2.6}$$

for all $x \in A$.

Proof. It follows from (2.4) that

$$\left\| f(x) - \frac{2}{p+q+2d} f\left(\frac{p+q+2d}{2} x\right) \right\|_B \le \frac{p+q+d}{p+q+2d} \theta \|x\|_A^r$$

for all $x \in A$. So

$$
\begin{aligned}
&\left\|\frac{2^l}{(p+q+2d)^l}f\left(\frac{(p+q+2d)^l}{2^l}x\right)-\frac{2^m}{(p+q+2d)^m}f\left(\frac{(p+q+2d)^m}{2^m}x\right)\right\|_B\\
&\le \sum_{j=l}^{m-1}\left\|\frac{2^j}{(p+q+2d)^j}f\left(\frac{(p+q+2d)^j}{2^j}x\right)\right.\\
&\qquad\left.-\frac{2^{j+1}}{(p+q+2d)^{j+1}}f\left(\frac{(p+q+2d)^{j+1}}{2^{j+1}}x\right)\right\|_B \qquad (2.7)\\
&\le \frac{p+q+d}{p+q+2d}\sum_{j=l}^{m-1}\frac{2^j(p+q+2d)^{jr}}{2^{jr}(p+q+2d)^j}\theta\|x\|_A^r
\end{aligned}
$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.7) that the sequence $\{\frac{2^n}{(p+q+2d)^n}f(\frac{(p+q+2d)^n}{2^n}x)\}$ is Cauchy for all $x \in A$. Since B is complete, the sequence $\{\frac{2^n}{(p+q+2d)^n}f(\frac{(p+q+2d)^n}{2^n}x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{2^n}{(p+q+2d)^n}f\left(\frac{(p+q+2d)^n}{2^n}x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.6). ■

The rest of the proof is similar to the proof of Theorem 2.1.

Theorem 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$
\begin{aligned}
&\|C_\mu f(x_1,\cdots,x_p,y_1,\cdots,y_q,z_1,\cdots,z_d)\|_B\\
&\le \theta\prod_{j=1}^p\|x_j\|_A^r\cdot\prod_{j=1}^q\|y_j\|_A^r\cdot\prod_{j=1}^d\|z_j\|_A^r, \qquad (2.8)
\end{aligned}
$$

$$\|f([x,y,z])-[f(x),f(y),f(z)]\|_B \le \theta\cdot\|x\|_A^r\cdot\|y\|_A^r\cdot\|z\|_A^r \qquad (2.9)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z, x_1,\cdots,x_p, y_1,\cdots,y_q, z_1,\cdots,z_d \in A$. Then there exists a unique C^*-ternary ring homomorphism $H : A \to B$ such that

$$
\begin{aligned}
&\|f(x)-H(x)\|_B\\
&\le \frac{2^{(p+q+d)r}}{2(p+q+2d)^{(p+q+d)r}-2^{(p+q+d)r}(p+q+2d)}\theta\|x\|_A^{(p+q+d)r} \qquad (2.10)
\end{aligned}
$$

for all $x\in A$.

Proof. Letting $\mu = 1$ and $x_1 = \cdots = x_p = y_1 = \cdots = y_q = z_1, \cdots z_d = x$ in (2.8), we get

$$\left\| 2f\left(\frac{p+q+2d}{2}x\right) - (p+q+2d)f(x)\right\|_B \le \theta\|x\|_A^{(p+q+d)r} \tag{2.11}$$

for all $x \in A$. So

$$\left\| f(x) - \frac{p+q+2d}{2} f\left(\frac{2}{p+q+2d}x\right)\right\|_B \le \frac{2^{(p+q+d)r}}{2(p+q+2d)^{(p+q+d)r}}\theta\|x\|_A^{(p+q+d)r}$$

for all $x \in A$. Hence

$$\begin{aligned}
&\left\|\frac{(p+q+2d)^l}{2^l} f\left(\frac{2^l}{(p+q+2d)^l}x\right) - \frac{(p+q+2d)^m}{2^m} f\left(\frac{2^m}{(p+q+2d)^m}x\right)\right\|_B \\
&\le \sum_{j=l}^{m-1}\left\|\frac{(p+q+2d)^j}{2^j} f\left(\frac{2^j}{(p+q+2d)^j}x\right)\right. \\
&\quad\left. - \frac{(p+q+2d)^{j+1}}{2^{j+1}} f\left(\frac{2^{j+1}}{(p+q+2d)^{j+1}}x\right)\right\|_B \\
&\le \frac{2^{(p+q+d)r}}{2(p+2d)^{(p+q+d)r}} \sum_{j=l}^{m-1} \frac{2^{(p+q+d)rj}(p+q+2d)^j}{2^j(p+q+2d)^{(p+q+d)rj}}\theta\|x\|_A^{(p+q+d)r}
\end{aligned} \tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.12) that the sequence $\{\frac{(p+q+2d)^n}{2^n} f(\frac{2^n}{(p+q+2d)^n}x)\}$ is Cauchy for all $x \in A$. Since B is complete, the sequence $\{\frac{(p+q+2d)^n}{2^n} f(\frac{2^n}{(p+q+2d)^n}x)\}$ converges. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n}{(p+q+2d)^n}x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.10). ■

The rest of the proof is similar to the proof of Theorem 2.1.

Theorem 2.4. *Let $r < \frac{1}{p+q+d}$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.8) *and* (2.9). *Then there exists a unique C^*-ternary ring homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{2^{(p+q+d)r}}{2^{(p+q+d)r}(p+q+2d) - 2(p+q+2d)^{(p+q+d)r}}\theta\|x\|_A^{(p+q+d)r} \tag{2.13}$$

for all $x \in A$.

Proof. It follows from (2.11) that

$$\left\| f(x) - \frac{2}{p+q+2d} f\left(\frac{p+q+2d}{2}x\right)\right\|_B \le \frac{\theta}{p+q+2d}\|x\|_A^{(p+q+d)r}$$

for all $x \in A$. So

$$\begin{aligned} &\left\| \frac{2^l}{(p+q+2d)^l} f\left(\frac{(p+q+2d)^l}{2^l}x\right) - \frac{2^m}{(p+q+2d)^m} f\left(\frac{(p+q+2d)^m}{2^m}x\right)\right\|_B \\ &\le \sum_{j=l}^{m-1} \left\| \frac{2^j}{(p+q+2d)^j} f\left(\frac{(p+q+2d)^j}{2^j}x\right) \right. \\ &\quad \left. - \frac{2^{j+1}}{(p+q+2d)^{j+1}} f\left(\frac{(p+q+2d)^{j+1}}{2^{j+1}}x\right)\right\|_B \\ &\le \frac{\theta}{p+q+2d} \sum_{j=l}^{m-1} \frac{2^j (p+q+2d)^{j(p+q+d)r}}{2^{j(p+q+d)r}(p+q+2d)^j} \|x\|_A^{(p+q+d)r} \end{aligned} \tag{2.14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.14) that the sequence $\{\frac{2^n}{(p+q+2d)^n} f(\frac{(p+q+2d)^n}{2^n}x)\}$ is Cauchy for all $x \in A$. Since B is complete, the sequence $\{\frac{2^n}{(p+q+2d)^n} f(\frac{(p+q+2d)^n}{2^n}x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n}{2^n}x\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.14), we get (2.13). ■

The rest of the proof is similar to the proof of Theorem 2.1.

3. Isomorphisms in C^*-Ternary Rings

Throughout this section, assume that A is a unital C^*-ternary ring with norm $\|\cdot\|_A$ and unit e, and that B is a unital C^*-ternary ring with norm $\|\cdot\|_B$ and unit e'.

We investigate isomorphisms in C^*-ternary rings associated with the functional equation $C_\mu f(x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_d) = 0$.

Theorem 3.1. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.1) *such that*

$$f([x,y,z]) = [f(x), f(y), f(z)] \tag{3.1}$$

for all $x, y, z \in A$. If $\lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n e}{(p+q+2d)^n}\right) = e'$, then the mapping $f : A \to B$ is a C^*-ternary ring isomorphism.

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{2^r(p+q+d)}{2(p+q+2d)^r - (p+q+2d)2^r}\theta\|x\|_A^r$$

for all $x \in A$. The mapping $H : A \to B$ is defined by

$$H(x) := \lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n}{(p+q+2d)^n}x\right)$$

for all $x \in A$.

Since $f([x,y,z]) = [f(x), f(y), f(z)]$ for all $x, y, z \in A$,

$$\begin{aligned}
&H([x,y,z])\\
&= \lim_{n\to\infty} \frac{(p+q+2d)^{3n}}{8^n} f\left(\left[\frac{2^n x}{(p+q+2d)^n}, \frac{2^n y}{(p+q+2d)^n}, \frac{2^n z}{(p+q+2d)^n}\right]\right)\\
&= \lim_{n\to\infty} \left[\frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n x}{(p+q+2d)^n}\right), \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n y}{(p+q+2d)^n}\right),\right.\\
&\quad \left.\frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n z}{(p+q+2d)^n}\right)\right] = [H(x), H(y), H(z)]
\end{aligned}$$

for all $x, y, z \in A$. So the mapping $H : A \to B$ is a C^*-ternary ring homomorphism.

It follows from (3.1) that

$$\begin{aligned}
H(x) &= H([e,e,x]) = \lim_{n\to\infty} \frac{(p+q+2d)^{2n}}{4^n} f\left(\frac{4^n}{(p+q+2d)^{2n}}[e,e,x]\right)\\
&= \lim_{n\to\infty} \frac{(p+q+2d)^{2n}}{4^n} f\left(\left[\frac{2^n e}{(p+q+2d)^n}, \frac{2^n e}{(p+q+2d)^n}, x\right]\right)\\
&= \lim_{n\to\infty} \left[\frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n e}{(p+q+2d)^n}\right), \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n e}{(p+q+2d)^n}\right), f(x)\right]\\
&= [e', e', f(x)] = f(x)
\end{aligned}$$

for all $x \in A$. Hence the bijective mapping $f : A \to B$ is a C^*-ternary ring isomorphism.

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.1) *and* (3.1). *If* $\lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f(\frac{(p+q+2d)^n}{2^n}e) = e'$, *then the mapping $f : A \to B$ is a C^*-ternary ring isomorphism.*

Proof. By the same reasoning as in the proofs of Theorems 2.1 and 2.2, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{2^r(p+q+d)}{2^r(p+q+2d) - 2(p+q+2d)^r}\theta\|x\|_A^r$$

for all $x \in A$. The mapping $H : A \to B$ is defined by

$$H(x) := \lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n}{2^n}x\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1.

Theorem 3.3. *Let $r > \frac{1}{p+q+d}$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.8) *and* (3.1). *If* $\lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n e}{(p+q+2d)^n}\right) = e'$*, then the mapping $f : A \to B$ is a C^*-ternary ring isomorphism.*

Proof. By the same reasoning as in the proofs of Theorems 2.1 and 2.3, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{2^{(p+q+d)r}}{2(p+q+2d)^{(p+q+d)r} - 2^{(p+q+d)r}(p+q+2d)}\theta\|x\|_A^{(p+q+d)r}$$

for all $x \in A$. The mapping $H : A \to B$ is defined by

$$H(x) := \lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n}{(p+q+2d)^n}x\right)$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 2.3 and 3.1.

Theorem 3.4. *Let $r < \frac{1}{p+q+d}$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.8) *and* (3.1). *If* $\lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n}{2^n}e\right) = e'$*, then the mapping $f : A \to B$ is a C^*-ternary ring isomorphism.*

Proof. By the same reasoning as in the proofs of Theorems 2.1 and 2.4, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{2^{(p+q+d)r}}{2^{(p+q+d)r}(p+q+2d) - 2(p+q+2d)^{(p+q+d)r}}\theta\|x\|_A^{(p+q+d)r}$$

for all $x \in A$. The mapping $H : A \to B$ is defined by

$$H(x) := \lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n}{2^n}x\right)$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 2.4 and 3.1.

4. Stability of Derivations on C^*-Ternary Rings

Throughout this section, assume that A is a C^*-ternary ring with norm $\|\cdot\|_A$.

We prove the Hyers–Ulam–Rassias stability of derivations on C^*-ternary rings for the functional equation $C_\mu f(x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_d) = 0$.

Theorem 4.1. *Let $r > 3$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (2.1) *such that*

$$\begin{aligned} &\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\|_A \\ &\leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) \end{aligned} \tag{4.1}$$

for all $x, y, z \in A$. Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{2^r(p+q+d)}{2(p+q+2d)^r - (p+q+2d)2^r}\theta\|x\|_A^r \tag{4.2}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (4.2). The mapping $\delta : A \to A$ is defined by

$$\delta(x) := \lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n}{(p+q+2d)^n}x\right)$$

for all $x \in A$.

It follows from (4.1) that

$$\begin{aligned} &\|\delta([x,y,z]) - [\delta(x),y,z] - [x,\delta(y),z] - [x,y,\delta(z)]\|_A \\ &= \lim_{n\to\infty} \frac{(p+q+2d)^{3n}}{8^n} \left\| f\left(\frac{8^n}{(p+q+2d)^{3n}}[x,y,z]\right) \right. \\ &\quad - \left[f\left(\frac{2^n x}{(p+q+2d)^n}\right), \frac{2^n y}{(p+q+2d)^n}, \frac{2^n z}{(p+q+2d)^n}\right] \end{aligned}$$

$$- \left[\frac{2^n x}{(p+q+2d)^n}, f\left(\frac{2^n y}{(p+q+2d)^n} \right), \frac{2^n z}{(p+q+2d)^n} \right]$$
$$- \left[\frac{2^n x}{(p+q+2d)^n}, \frac{2^n y}{(p+q+2d)^n}, f\left(\frac{2^n z}{(p+q+2d)^n} \right) \right] \Big\|_A$$
$$\leq \lim_{n\to\infty} \frac{2^{nr}(p+q+2d)^{3n}}{8^n (p+q+2d)^{nr}} \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0$$

for all $x, y, z \in A$. Hence

$$\delta([x,y,z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$. Thus the mapping $\delta : A \to A$ is a unique C^*-ternary derivation satisfying (4.2), as desired.

Theorem 4.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (2.1) *and* (4.1). *Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \leq \frac{2^r(p+q+d)}{2^r(p+q+2d) - 2(p+q+2d)^r} \theta \|x\|_A^r \tag{4.3}$$

for all $x \in A$.

Proof. By the same reasoning as in the proofs of Theorems 2.1 and 2.2, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (4.3). The mapping $\delta : A \to A$ is defined by

$$\delta(x) := \lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n}{2^n} x \right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.1.

Theorem 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (2.8) *such that*

$$\|f([x,y,z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)]\|_A$$
$$\leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \cdot \|z\|_A^r \tag{4.4}$$

for all $x, y, z \in A$. Then there exists a unique C^*-ternary derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{2^{(p+q+d)r}}{2(p+q+2d)^{(p+q+d)r} - 2^{(p+q+d)r}(p+q+2d)} \theta \|x\|_A^{(p+q+d)r} \tag{4.5}$$

for all $x \in A$.

Proof. By the same reasoning as in the proofs of Theorems 2.1 and 2.3, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (4.5). The mapping $\delta : A \to A$ is defined by

$$\delta(x) := \lim_{n\to\infty} \frac{(p+q+2d)^n}{2^n} f\left(\frac{2^n}{(p+q+2d)^n}x\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.1.

Theorem 4.4. *Let* $r < \frac{1}{p+q+d}$ *and* θ *be nonnegative real numbers, and let* $f : A \to A$ *be a mapping satisfying* (2.8) *and* (4.4)*. Then there exists a unique* C^**-ternary derivation* $\delta : A \to A$ *such that*

$$\|f(x)-\delta(x)\|_A \leq \frac{2^{(p+q+d)r}}{2^{(p+q+d)r}(p+q+2d) - 2(p+q+2d)^{(p+q+d)r}}\theta\|x\|_A^{(p+q+d)r} \tag{4.6}$$

for all $x \in A$.

Proof. By the same reasoning as in the proofs of Theorems 2.1 and 2.4, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (4.6). The mapping $\delta : A \to A$ is defined by

$$\delta(x) := \lim_{n\to\infty} \frac{2^n}{(p+q+2d)^n} f\left(\frac{(p+q+2d)^n}{2^n}x\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 4.1.

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