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ON GLOBAL PERIODICITY OF DIFFERENCE EQUATIONS

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Abstract. We study the global periodicity of difference equations of the form $x_{n+1} = f(x_n)$, $x_{n+1} = f_n(x_n)$ and $x_{n+2} = f(x_n, x_{n+1})$. We characterize the n-cycles in the case of first order equations and give some partial results for the second order equation. In particular, we find some examples of 3-cycles which are different from the equation $x_{n+2} = \frac{c}{x_n x_{n+1}}$, solving a question of [2] and [4].

1. Introduction

Given a continuous function, $f : (0,\infty)^l \to (0,\infty)$, $l \geq 1$, an *(autonomous) difference equation of order* l is a expression of the form

$$x_{n+l} = f(x_{n+l-1}, x_{n+l-2}, \ldots, x_n), \quad n \in \mathbb{N} \cup \{0\}. \tag{1}$$

If we fix l real numbers, $\{x_0, x_1, \ldots x_{l-1}\}$, we obtain a unique sequence (x_n) satisfying (1), the l real numbers are said to be the *initial conditions* of the *solution* (x_n). The sequence (x_n) is said to be *periodic* if there is $p \in \mathbb{N}$ so that $x_{n+p} = x_n$ for any $n \in \mathbb{N} \cup \{0\}$; if moreover p is the smallest integer satisfying the previous condition then we say that (x_n) is *periodic of period* p or p*-periodic*. We will say that (1), or the map f, is a k-cycle if any solution is periodic and moreover k is the least common multiple of all the periods. Remark that if (1) is a k-cycle then, by force, $k \geq l$.

If now we take a sequence of continuous functions (f_n), $f_n : (0,\infty)^l \to (0,\infty)$, then

$$x_{n+l} = f_n(x_{n+l-1}, x_{n+l-2}, \ldots, x_n), \quad n \in \mathbb{N} \cup \{0\}, \tag{2}$$

is said to be a *non autonomous difference equation* for which the concepts of initial conditions, solution, periodicity and k-cycle can be adapted in a natural way.

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Both autonomous and non autonomous difference equations have a ubiquitous presence in applications from Economics, Biology, ..., so the main task when dealing them is to know the asymptotical behaviour of their solutions. In this paper we focus our attention on the topic of the periodicity, and in particular, on k-cycles. The problem of characterizing maps, $f : (0,\infty)^l \to (0,\infty)$, so that (1) is a k-cycle is too much general, extremely difficult even for $l = 2$ and it may depend too much on the value of k. Thus, we begin by considering the (autonomous) difference equation $x_{n+1} = f(x_n)$ (see [7, Open Problem 3.4.1]) and we obtain, in Theorem A, that if it is a k-cycle then $k \in \{1, 2\}$. Moreover we give a full description of the maps f generating k-cycles: when $k = 1$ f has to be the identity and if $k = 2$, then we show that all the cycles are topologically conjugate to the map $g(x) = \frac{1}{x}$, see Theorem B. Finally, we study the same question but adding the condition that f is differentiable and we characterize all the differentiable 2-cycles.

In a next step we study the first order non-autonomous difference equation

$$x_{n+1} = f_n(x_n), \tag{3}$$

where each $f_n : (0,\infty) \to (0,\infty)$ is a continuous map, $n = 0, 1, \ldots$ We prove that if (3) is a k-cycle, $k \geq 1$, then the sequence of maps (f_n) must be also periodic. Even, in such a case if the sequence of maps (f_n) is periodic of period p, f_n is bijective for any n and either $f_{p-1} \circ \cdots \circ f_0 = \mathrm{Id}|_{(0,\infty)}$ or $f_{2p-1} \circ \cdots \circ f_0 = \mathrm{Id}|_{(0,\infty)}$, where $\mathrm{Id}|_A$ denotes the *identity map* on A, see Theorem C. In addition, we provide examples of non autonomous difference equations of first order such that all their solutions are periodic, but their set of periods is infinite.

Finally we study some topics about the second order autonomous difference equation:

$$x_{n+2} = f(x_{n+1}, x_n), \tag{4}$$

where $f : (0,\infty)^2 \to (0,\infty)$ is a continuous map. Our source of inspiration is [7, Open Problem 3.4.2] where it is proposed to characterize the k-cycles with $k \geq 3$. We investigate this question when $k = 3$ and hence any solution of the difference equation has period 1 or 3. In particular we study this problem for maps of the form $f(x, y) = \xi(xy)$ and we prove that then $f(x, y) = \dfrac{c}{xy}$, $c > 0$, see Theorem D. Remark that we obtain the same maps as in [2] and then it is natural to wonder if these maps are the only for which (4) is a 3-cycle. We answer negatively this question by finding several families of 3-cycles different from $\dfrac{c}{xy}$, solving a question stated in [2] and [4]. Moreover, since the method for constructing these families provides symmetric maps (see Theorem E), we also construct in the last subsection of the paper an example of a non-symmetric map which is also a 3-cycle (Theorem F).

The paper is organized as follows. Section 2 is devoted to study both one dimensional cases: autonomous and non-autonomous. In Section 3 we study global periodic aspects of equation (4).

2. On Cycles in Difference Equations of Order One

In this section we study difference equations of order one of two types: autonomous and non autonomous ones. In the autonomous case, we prove that there only exist 1-cycles and 2-cycles, and we show that all 2-cycles are topologically conjugate to the map $\phi(x) = \frac{1}{x}$. Concerning the non autonomous case, given a positive integer $k,$ we can choose the sequence (f_n) such that the corresponding difference equation is a k-cycle, and then the sequence (f_n) is periodic itself. Moreover, we are able to construct non autonomous difference equations of order one holding that all their solutions are periodic and having an infinite set of periods.

2.1. On the equation $x_{n+1} = f(x_n)$

In this subsection we characterize k-cycles for first order (autonomous) difference equations for any $k \in \mathbb{N}$. We begin by proving the following result.

Theorem A. *Consider the difference equation* $x_{n+1} = f(x_n)$*, where* $f : (0,\infty) \to (0,\infty)$ *is continuous. If* f *is a* k*-cycle, then* $k \in \{1,2\}$*. Moreover, if* f *is not the identity map on* $(0,\infty)$*, then*

$$f(x) = \begin{cases} f_0(x) \text{ if } x \in (0, x_0), \\ x_0 \text{ if } x = x_0, \\ f_0^{-1}(x) \text{ if } x \in (x_0, \infty), \end{cases}$$

where $x_0 > 0$ *and* $f_0 : (0, x_0) \to (x_0, \infty)$ *is a continuous strictly decreasing map such that* $\lim_{x\to x_0} f_0(x) = x_0$.

Proof. Assume that f is a k-cycle. First, we are going to prove that f is bijective. To this end, notice that if $z \notin f(0,\infty)$, then the sequence $(z, f(z), ...)$ cannot be periodic and hence f has to be surjective. On the other hand, if f would not be injective and $f(z_1) = f(z_2)$ for two different points $z_1, z_2 \in (0,\infty)$, then the sequences $(z_1, f(z_1), ...)$ and $(z_2, f(z_2), ...)$ are equal except for the first element and hence both of them cannot be periodic. This proves that f is injective.

Now suppose that f is strictly increasing. If $x < f(x)$ for some $x \in (0,\infty)$ then $x < f(x) < f^2(x) <$ .... and consequently $(x, f(x), f^2(x), \ldots)$ cannot be periodic, hence $x \geq f(x)$. If $x > f(x)$ then $x > f(x) > f^2(x) >$ .... and $(x, f(x), f^2(x), \ldots)$ cannot be periodic, therefore $x = f(x)$ for all $x \in (0,\infty)$.

Finally, suppose that f is strictly decreasing. Then f^2 is strictly increasing and $x_{n+1} = f^2(x_n)$ is either a k-cycle (if k odd) or a $k/2$-cycle (if k even). We apply the argument of the above paragraph to f^2 and we obtain $f^2(x) = x$ for all $x \in (0, +\infty)$. Then by [6, Lemma 15.2, page 290] we deduce the result. ■

Remark 1. We must point out that different versions of Theorem can be found in [1] and [5]. However, our proof is slightly different from those and we decide to include it for the sake of completeness. In addition, Theorem solves Open Problem 3.4.1 from [7].

Two maps, $f, g \in C[(0,\infty), (0,\infty)]$ are said to be *topologically conjugate* if there exists a homeomorphism $\varphi : (0,\infty) \to (0,\infty)$ such that $\varphi \circ f = g \circ \varphi$. We write $f \sim g$ if and only if they are topologically conjugate. Clearly $\sim$ is an equivalence relationship. In Theorem A we see that there exists a unique map, the identity, which is a 1-cycle. The following result shows that there is only an equivalence class on the set of 2-cycles.

Theorem B. *Any* 2*-cycle* $f \in C[(0,\infty), (0,\infty)]$ *is topologically conjugate to the map* $\phi(x) = 1/x$.

Proof. We are going to define a homeomorphism $\varphi : (0,\infty) \to (0,\infty)$ such that

$$\varphi \circ f = \phi \circ \varphi. \tag{5}$$

Notice that equality (5) can be rewritten for any $x \in (0,\infty)$ as

$$\varphi(f(x)) = 1/\varphi(x). \tag{6}$$

Let $x_0 \in (0,\infty)$ be the unique point such that $f(x_0) = x_0$. Then (6) gives us $\varphi(x_0) = 1/\varphi(x_0)$, and hence $\varphi(x_0) = 1$. Now, define $\varphi : (0, x_0] \to (0, 1]$ to be strictly increasing, continuous and such that $\lim_{x\to 0} \varphi(x) = 0$. We extend this map to (x_0,∞) as follows. Let $x \in (x_0,\infty)$ and let $y = f(x) \in (0, x_0)$. Define $\varphi(x) := 1/\varphi(y) = 1/\varphi(f(x))$. Notice that since $\varphi(x) \neq 0$ for all $x \in (0, x_0]$ and

$$\lim_{\substack{x\to x_0 \\ x>x_0}} \varphi(x) = \lim_{\substack{x\to x_0 \\ x<x_0}} 1/\varphi(x) = 1,$$

the map φ is continuous. Now, we prove that $\varphi : (0,\infty) \to (0,\infty)$ is a homeomorphism. Since

$$\lim_{x\to\infty} \varphi(x) = \lim_{x\to 0} 1/\varphi(x) = \infty,$$

we conclude that φ is surjective. Now, we finish by proving that φ is strictly increasing. To this end, note that $\varphi(x) < \varphi(y)$ for any $x \in (0, x_0]$, $y \in (x_0,\infty)$ by definition. If $x > y > x_0$, then since f is strictly decreasing $f(x) < f(y)$. Then $\varphi(f(x)) < \varphi(f(y))$ and therefore $\varphi(x) = 1/\varphi(f(x)) > 1/\varphi(f(y)) = \varphi(y)$, which finishes the proof. ■

When the map f of Theorem A is of class C^1 the following additional condition must be satisfied.

Proposition 2. *Let f be as in Theorem A, $f \neq \mathrm{Id}|_{(0,\infty)}$. If f is a C^1-map then $f'(x_0) = -1$, where x_0 is the fixed point of f.*

Proof. Since f is of C^1-class and $f^2(x) = x$, we have that

$$f'(f(x))f'(x) = 1,$$

for any $x \in (0,\infty)$. In particular, for the fixed point x_0 it is held $f'(x_0)^2 = 1$. Since f is decreasing, the result follows. ■

It is a simple task to check that the following C^1 maps are 2-cycles:

$$f_1(x) = \frac{c}{x},\ c > 0,$$

$$f_2(x) = \begin{cases} -\dfrac{1}{e}\log x \text{ if } x \in \left(0, \dfrac{1}{e}\right], \\ \left(\dfrac{1}{e}\right)^{ex} \quad \text{if } x \in \left(\dfrac{1}{e}, \infty\right). \end{cases}$$

2.2. On the equation $x_{n+1} = f_n(x_n)$

Now, we will investigate Equation (3) and study what properties have to satisfy the sequence of continuous maps (f_n) in order to be a k-cycle. We adopt the following notation

$$f_i^j := f_{i+j-1} \circ \cdots \circ f_{i+1} \circ f_i$$

for any $i \in \mathbb{N} \cup \{0\}$ and $j \in \mathbb{N}$. Hence, we assume that $f_0 : (0,\infty) \to f_0(0,\infty)$ and for $n \geq 1$, $f_n : f_0^{n-1}(0,\infty) \to f_0^n(0,\infty)$.

Theorem C. *Assume that Equation (3) is a k-cycle. Then:*

(*i*) *f_i is bijective for any $i \in \mathbb{N} \cup \{0\}$.*

(*ii*) *(f_n) is periodic.*

(*iii*) *If p is the period of (f_n) then either $f_{p-1} \circ \ldots \circ f_0 = \mathrm{Id}|_{(0,\infty)}$ or $f_{2p-1} \circ \ldots \circ f_0 = \mathrm{Id}|_{(0,\infty)}$.*

Proof. Since any solution $(x_0, x_1, x_2, \ldots)$ from (3) is periodic of period less than or equal to k and its period has to divide k then

$$x_i = x_{i+kj} \text{ for any } i \in \mathbb{N} \cup \{0\} \text{ and any } j \in \mathbb{N}. \tag{7}$$

Therefore $f_{hk}^k = \mathrm{Id}|_{(0,\infty)}$ for any $h \in \mathbb{N} \cup \{0\}$.

We begin by proving **(i).** From $f_0^k = f_{k-1} \circ \cdots \circ f_1 \circ f_0 = \mathrm{Id}|_{(0,\infty)}$ we deduce that f_0 is injective and, consequently, strictly monotone. Since f_0 is also surjective, it is bijective. Now $f_{k-1} \circ \ldots f_2 \circ f_1 = f_0^{-1}$ and we can repeat the previous argument to prove that f_1 is bijective. Reasoning in this way we have $f_{k-1} \circ \cdots \circ f_{i+1} \circ f_i = (f_{i-1} \circ \cdots \circ f_0)^{-1}$ and each f_i is bijective, $i \in \{1, 2, \ldots, k-1\}$. Using the relations $f_{hk}^k = \mathrm{Id}|_{(0,\infty)}$, $h \in \mathbb{N}$, in the same way it is easy to prove that f_i is bijective for any $i \in \mathbb{N}$.

Next we prove **(ii)**. Let $x \in (0, \infty)$ and $i \in \mathbb{N} \cup \{0\}$ then there is a solution $(x_0, x_1, \ldots)$ of (3) so that $x_i = x$: it suffices to take $x_0 = x$ if $i = 0$ and $x_0 = f_0^{-1} \circ f_1^{-1} \circ \cdots \circ f_{i-1}^{-1}(x)$ if $i > 0$. From (7) we have

$$f_{i+lj}(x) = f_{i+lj}(x_i) = f_{i+lj}(x_{i+lj}) = x_{i+lj+1} = x_{i+1} = f_i(x_i) = f_i(x)$$

for any $j \in \mathbb{N}$, thus (f_n) is periodic.

Finally we prove **(iii)**. Let p be the period of (f_n) and note that any solution of

$$x_{n+1} = f_0^p(x_n) \tag{8}$$

has the form (x_{pn}) where (x_n) is a solution of (3). Thus (8) is a h-cycle for some $h \leq k$ and by Theorem A, $h \in \{1, 2\}$. If $h = 1$ then $f_0^p = \mathrm{Id}|_{(0,\infty)}$, otherwise $(f_0^p)^2 = f_0^{2p} = \mathrm{Id}|_{(0,\infty)}$. ∎

Example 3. Consider the sequence (f_n), where $f_{2n+1}(x) = x$, and $f_{2n}(x) = 1/x$, $n \geq 0$. Then (3) is a 4-cycle and (f_n) has period 2. If for instance we consider $\alpha > 1$, $f_{2n+1}(x) = \alpha x$ and $f_{2n}(x) = \frac{1}{\alpha} x$, we obtain that all the solutions of (3) have period 2 and (3) is a 2-cycle, moreover (f_n) has period 2.

Example 4. Let $k, r \in \mathbb{N}$, $I_k := [k, k+1]$ and $p_{I_k}^r, q_{I_k}^r : I_k \to I_k$ defined by $p_{I_k}^r(x) = k + (x-k)^r$ and $q_{I_k}^r(x) = k + (x-k)^{1/r}$. Clearly $(p_{I_k}^r)^{-1} = q_{I_k}^r$, $p_{I_k}^r(k) = q_{I_k}^r(k) = k$ and $p_{I_k}^r(k+1) = q_{I_k}^r(k+1) = k+1$.

Now we are going to define a sequence (f_n) of continuous maps so that any solution of (3) is periodic and the set of periods of (3) is $\mathbb{N}$. We define any f_n by pieces as follows:

- $f_n|_{(0,1]} = \mathrm{Id}|_{(0,1]}$;
- If $t = 2s$ for some $s \in \mathbb{N}$ then we take $(f_n|_{I_t})$ to be periodic of period $t+1$, $f_{2h}|_{I_t} = p_{I_t}^{h+2}$ if $0 \leq h < s$, $f_{2h+1}|_{I_t} = q_{I_t}^{h+2}$ if $0 \leq h < s$ and $f_{2s}|_{I_t} = \mathrm{Id}|_{I_t}$.

- If $t = 2s+1$ for some $s \in \mathbb{N} \cup \{0\}$, then we take $(f_n|_{I_t})$ to be periodic of period $t+1$, $f_{2h}|_{I_t} = p_{I_t}^{h+2}$ if $0 \leq h \leq s$, $f_{2h+1}|_{I_t} = q_{I_t}^{h+2}$ if $0 \leq h \leq s$.

In order to clarify the previous definition of (f_n) we introduce the below table where we write the definition of the maps $f_0, f_1, \ldots f_9$ when restricted to $(0, 7]$.

	I_0	I_1	I_2	I_3	I_4	I_5	I_6
f_0	$\mathrm{Id}\|_{I_0}$	$p_{I_1}^2$	$p_{I_2}^2$	$p_{I_3}^2$	$p_{I_4}^2$	$p_{I_5}^2$	$p_{I_6}^2$
f_1	$\mathrm{Id}\|_{I_0}$	$q_{I_1}^2$	$q_{I_2}^2$	$q_{I_3}^2$	$q_{I_4}^2$	$q_{I_5}^2$	$q_{I_6}^2$
f_2	$\mathrm{Id}\|_{I_0}$	$p_{I_1}^2$	$\mathrm{Id}\|_{I_2}$	$p_{I_3}^3$	$p_{I_4}^3$	$p_{I_5}^3$	$p_{I_6}^3$
f_3	$\mathrm{Id}\|_{I_0}$	$q_{I_1}^2$	$p_{I_2}^2$	$q_{I_3}^3$	$q_{I_4}^3$	$q_{I_5}^3$	$q_{I_6}^3$
f_4	$\mathrm{Id}\|_{I_0}$	$p_{I_1}^2$	$q_{I_2}^2$	$p_{I_3}^2$	$\mathrm{Id}\|_{I_4}$	$p_{I_5}^4$	$p_{I_6}^4$
f_5	$\mathrm{Id}\|_{I_0}$	$q_{I_1}^2$	$\mathrm{Id}\|_{I_2}$	$q_{I_3}^2$	$p_{I_4}^2$	$q_{I_5}^4$	$q_{I_6}^4$
f_6	$\mathrm{Id}\|_{I_0}$	$p_{I_1}^2$	$p_{I_2}^2$	$p_{I_3}^3$	$q_{I_4}^2$	$p_{I_5}^2$	$\mathrm{Id}\|_{I_6}$
f_7	$\mathrm{Id}\|_{I_0}$	$q_{I_1}^2$	$q_{I_2}^2$	$q_{I_3}^3$	$p_{I_4}^3$	$q_{I_5}^2$	$p_{I_6}^2$
f_8	$\mathrm{Id}\|_{I_0}$	$p_{I_1}^2$	$\mathrm{Id}\|_{I_2}$	$p_{I_3}^2$	$q_{I_4}^3$	$p_{I_5}^3$	$q_{I_6}^2$
f_9	$\mathrm{Id}\|_{I_0}$	$q_{I_1}^2$	$p_{I_2}^2$	$q_{I_3}^2$	$\mathrm{Id}\|_{I_4}$	$q_{I_5}^3$	$p_{I_6}^3$

Let (x_n) be a solution of (3), then it is easy to check: (i) if $x_0 \in (0, 1] \cup \mathbb{N}$ then (x_n) is 1-periodic; (ii) if $t \in \mathbb{N}$ and $x_0 \in (t, t+1)$ then (x_n) is $(t+1)$-periodic.

It is an open question to know whether autonomous difference equations such that all their solutions are periodic and having an infinite set of periods can exist.

Before finishing this section we assume that the sequence (f_n) converges to a continuous map f. Then if $x_{n+1} = f(x)_n$ is a k-cycle (with $k \in \{1, 2\}$ by Theorem A) it is interesting to know if the solutions of (3) are necessarily asymptotically periodic. In the following two remarks we answer negatively this question for both the pointwise and the uniform convergence.

Remark 5. [Pointwise convergence]. Let $f_n(x) = a_n x$, where a_n is a sequence converging to 1. Then clearly f_n converges to $f(x) = x$ and:

- If we take $a_n < 1$ for any n, then $f_0^n(x) = \prod_{i=0}^{n-1} a_i x$ converges to 0 because $\prod_{i=0}^{n-1} a_i$ converges to 0. Then the solutions of (3) are asymptotically periodic.

- However if we choose $a_n > 1$ for any n, then $f_0^n(x) = \prod_{i=0}^{n-1} a_i x$ converges to ∞ because $\prod_{i=0}^{n-1} a_i$ converges to ∞ and then the solutions of (3) are not asymptotically periodic.

- If we choose $a_{2n} \cdot a_{2n+1} = 1$ for any n, then $f_0^n(x) = x$ if n is odd and $f_0^n(x) = a_n x$ if n is even. In both cases $f_0^n(x)$ converges to x and the solutions of (3) are asymptotically periodic.

Remark 6. [Uniform convergence] Let (a_n) be a sequence of real numbers converging to 0 and so that $a_n > -1$ for any n. Let $f_n(x) = x + a_n$ if $x \in [1, \infty)$ and $f_n(x) = x(1 + a_n)$ when $x \in (0, 1)$. Then it is easy to check that f_n uniformly converges to $f(x) = x$. Moreover:

- if $a_n = -\frac{1}{n+1}$ then any solution of (3) converges to 0 and then is asymptotically periodic;
- if $a_n = \frac{1}{n+1}$ then any solution of (3) converges to ∞ and then it is not asymptotically periodic.

3. On the Equation $x_{n+2} = f(x_{n+1}, x_n)$

Let $f : (0, \infty)^2 \to (0, \infty)$ be a continuous map *separating variables*, that is, $f(x, y) = f_1(x)f_2(y)$ for some continuous maps $f_1, f_2 : (0, \infty) \to (0, \infty)$. Then it is known, see [2], that the unique 3-cycles of the form

$$x_{n+2} = f_1(x_{n+1})f_2(x_n), \tag{9}$$

are given by

$$x_{n+2} = \frac{c}{x_n x_{n+1}}, \; n = 0, 1, 2, ..., \; c > 0. \tag{10}$$

However it is an open problem, cf. [4], to know whether the above cycles are or not the unique 3-cycles of the form (9) for general continuous maps $f : (0, \infty)^2 \to (0, \infty)$.

The aim of this section is answering this question. We will build difference equations which are 3-cycle different to (10). In fact we will construct 3-cycles for which the corresponding maps $f(x, y)$ are *symmetric* (i.e. $f(x, y) = f(y, x)$) and *non-symmetric* (i.e. $f(x, y) \neq f(y, x)$). Nevertheless, we will see that if $f(x, y) = \xi(xy)$, where $\xi : (0, \infty) \to (0, \infty)$ is a C^1 map, then necessarily $\xi(z) = \frac{c}{z}$, where c is an arbitrary positive real constant.

3.1. The unique 3-cycle of the form $x_{n+2} = \xi(x_{n+1}x_n)$

We begin by recalling some results from [2] when f separates the variables.

Proposition 7. *Let (9) be a 3-cycle and let $f(x, y) = f_1(x)f_2(y)$. Then:*

(i) $\varphi_u = \psi_u$ *for any $u > 0$, where φ_u and ψ_u are the fiber maps defined by $\varphi_u(x) = f(u, x)$ and $\psi_u(x) = f(x, u)$ for any $x > 0$.*

(ii) $\lim_{x\to\infty}\varphi_u(x) = 0$ *for all $u > 0$.*

(iii) *The one-dimensional map $h(x) = f(x, x)$ is a decreasing homeomorphism.*

(*iv*) *There is a unique* $\lambda \in (0, \infty)$ *such that* $f(\lambda, \lambda) = \lambda$.
(*v*) $x = f(f(x, y), y)$ *and* $y = f(x, f(x, y))$, *for all* $x, y > 0$.

Doing the derivatives in the second equality from Proposition 7**(v)** we obtain:

Lemma 8. *If* $f : (0, \infty)^2 \to (0, \infty)$ *is a symmetric differentiable map and* (9) *is a 3-cycle, then*

$$\begin{aligned} 1 &= \frac{\partial f}{\partial y}(a, f(a, b)) \frac{\partial f}{\partial y}(a, b), \\ 0 &= \frac{\partial f}{\partial x}(a, f(a, b)) + \frac{\partial f}{\partial y}(a, f(a, b)) \frac{\partial f}{\partial x}(a, b). \end{aligned}$$

As a direct consequence,

$$\frac{\dfrac{\partial f}{\partial x}(a, b)}{\dfrac{\partial f}{\partial y}(a, b)} = -\frac{\partial f}{\partial x}(a, f(a, b)).$$

Now we are in position to prove:

Theorem D. *Assume that* $f : (0, \infty)^2 \to (0, \infty)$ *is a differentiable map having the form* $f(x, y) = \xi(xy)$. *Then (9) is a* 3*-cycle if and only if*

$$f(x, y) = \frac{c}{xy},$$

for some positive constant $c > 0$.

Proof. Let $\alpha > 0$ and define $K_\alpha = f(\alpha, \alpha)$. Then it is easily seen that

$$f(x, \frac{\alpha^2}{x}) = \xi(\alpha^2) = K_\alpha \text{ for any } x > 0.$$

Hence,

$$0 = \frac{\partial f}{\partial x}(x, \frac{\alpha^2}{x}) - \frac{\alpha^2}{x^2} \frac{\partial f}{\partial y}(x, \frac{\alpha^2}{x}) \quad \text{and} \quad \frac{\alpha^2}{x^2} = \frac{\dfrac{\partial f}{\partial x}(x, \dfrac{\alpha^2}{x})}{\dfrac{\partial f}{\partial y}(x, \dfrac{\alpha^2}{x})}.$$

According to Lemma 8, we obtain

$$\frac{\alpha^2}{x^2} = \frac{\dfrac{\partial f}{\partial x}(x, \dfrac{\alpha^2}{x})}{\dfrac{\partial f}{\partial y}(x, \dfrac{\alpha^2}{x})} = -\frac{\partial f}{\partial x}(x, f(x, \frac{\alpha^2}{x})) = -\frac{\partial f}{\partial x}(x, K_\alpha) = -\varphi'_{K_\alpha}(x).$$

As a consequence,

$$\varphi_{K_\alpha}(x) = \frac{\alpha^2}{x} + c, \quad c \in \mathbb{R}.$$

Since $\lim_{x\to\infty} \varphi_{K_\alpha}(x) = 0$ (see Proposition 7), we find $c = 0$ and so

$$\varphi_{K_\alpha}(x) = \frac{\alpha^2}{x},$$

or equivalently $\varphi_{f(\alpha,\alpha)}(x) = \frac{\alpha^2}{x}$, $x > 0$. Using the notation $h(z) := f(z, z)$, the above equality is rewritten as

$$\varphi_{h(\alpha)}(x) = \frac{\alpha^2}{x},\ x > 0.$$

Taking into account Proposition 7**(v)**, for any $u > 0$ we write

$$\varphi_u(x) = \frac{[h^{-1}(u)]^2}{x},\ x > 0.$$

At the same time, the symmetry of f yields

$$\frac{[h^{-1}(u)]^2}{x} = \varphi_u(x) = \varphi_x(u) = \frac{[h^{-1}(x)]^2}{u},$$

so

$$\frac{[h^{-1}(u)]^2}{[h^{-1}(x)]^2} = \frac{x}{u},\ \text{for any } x, u > 0.$$

By Proposition 7**(iv)**, let $\lambda \in (0, \infty)$ be the unique equilibrium point of the equation. Then if $u = \lambda$, $h(\lambda) = \lambda = h^{-1}(\lambda)$, we obtain

$$\frac{\lambda^2}{[h^{-1}(x)]^2} = \frac{x}{\lambda}, \quad \text{that is, } h^{-1}(x) = \sqrt{\frac{\lambda^3}{x}},\ x > 0.$$

Finally,

$$f(x, y) = \varphi_y(x) = \frac{[h^{-1}(y)]^2}{x} = \frac{\frac{\lambda^3}{y}}{x} = \frac{\lambda^3}{xy},$$

which ends the proof. ■

3.2. New symmetric 3-cycles

In this subsection we construct new 3-cycles, different to $x_{n+2} = \frac{c}{x_{n+1}x_n}$, by making use of a typical strategy appearing in the setting of discrete dynamical systems; namely we deal with topological conjugations of the map $F_c : (0,\infty)^2 \to (0,\infty)^2$ given by $F_c(x, y) = (y, \frac{c}{xy})$. We precise this idea in Theorem E.

First we clarify the relationship between difference equations and (discrete) dynamical systems. We call *associated dynamical system* to (1) to the map

$$\begin{aligned} F &: (0,\infty)^l \to (0,\infty)^l \\ F(x_1, ..., x_{l-1}, x_l) &= (x_2, ..., x_{l-1}, f(x_l, x_{l-1}, ..., x_1)). \end{aligned} \tag{11}$$

Conversely (1) is said to be the *associated difference equation to the dynamical system* (11). Now it is a simple task to prove the following result (we leave the details to the reader).

Lemma 9. *If (1) is a k-cycle and F is its associated dynamical system, then $F^k = \mathrm{Id}|_{(0,\infty)^l}$. Conversely, if F is given by (11) and k is the smallest positive integer such that $F^k = \mathrm{Id}|_{(0,\infty)^l}$ then its associated difference equation is a k-cycle.*

Theorem E. *Assume that (1) is a k-cycle $(k \geq l)$ and let $\alpha : (0,\infty) \to (0,\infty)$ be a homeomorphism. Then*

$$x_{n+l} = \alpha\left(f[\alpha^{-1}(x_{n+l-1}), \alpha^{-1}(x_{n+l-2}), ..., \alpha^{-1}(x_{n+1}), \alpha^{-1}(x_n)]\right)$$

is also a k-cycle.

Proof. Let $F : (0,\infty)^l \to (0,\infty)^l$ be the associated dynamical system to (1). Then $F^k = \mathrm{Id}|_{(0,\infty)^l}$ by Lemma 9. Now take $H : (0,\infty)^l \to (0,\infty)^l$ defined by $H(x_1, x_2, ..., x_l) = (\alpha(x_1), \alpha(x_2), ..., \alpha(x_l))$ and consider $G = H \circ F \circ H^{-1}$. Using that $H^{-1}(x_1, x_2, ..., x_l) = (\alpha^{-1}(x_1), \alpha^{-1}(x_2), ..., \alpha^{-1}(x_l))$, it is a simple matter to check that

$$G(x_1, x_2, ..., x_l) = \left(x_2, ..., x_l, \alpha(f[\alpha^{-1}(x_l), \alpha^{-1}(x_{l-1}), ..., \alpha^{-1}(x_2), \alpha^{-1}(x_1)])\right).$$

Moreover

$$G^k = H \circ F^k \circ H^{-1} = \mathrm{Id}|_{(0,\infty)^l}. \tag{12}$$

Then, by Lemma 9,

$$x_{n+l} = \alpha(f\left[\alpha^{-1}(x_{n+l-1}), \alpha^{-1}(x_{n+l-2}), ..., \alpha^{-1}(x_{n+1}), \alpha^{-1}(x_n)\right])$$

is a new k-cycle. ■

As an easy consequence of the above result we obtain:

Example 10. The following difference equations are 3-cycles:

- $x_{n+2} = \exp\left(\dfrac{c}{\log(x_n + 1)\log(x_{n+1} + 1)}\right) - 1,\ c \in (0,\infty);$

- $x_{n+2} = \frac{\left(-1+\sqrt{1+\frac{4}{x_{n+1}}}\right)^2\left(-1+\sqrt{1+\frac{4}{x_n}}\right)^2}{4c\left[4c+\left(-1+\sqrt{1+\frac{4}{x_{n+1}}}\right)\left(-1+\sqrt{1+\frac{4}{x_n}}\right)\right]}$, $c \in (0,\infty)$.

Indeed, both equations are obtained from the 3-cycle $x_{n+2} = \frac{c}{x_n x_{n+1}}$, $c \in (0,\infty)$, by applying Theorem E. The first one is got by taking $\alpha(x) = e^x - 1$. For the second one we have chosen $\alpha(x) = \frac{1}{x(1+x)}$.

It is interesting to point out that the maps generating the above 3-cycles are symmetric and neither are of the form $\xi(xy)$ nor separate variables.

Example 11. The difference equation

$$x_{n+3} = x_n \left(\frac{x_{n+2}}{x_{n+1}}\right)^{\Phi},$$

where $\Phi \in \{\phi, -\phi^{-1}\}$ and $\phi = \frac{1+\sqrt{5}}{2}$ is the golden number, is a 5-cycle. In fact the only one which has the form $x_{n+3} = x_n\rho(x_{n+1}, x_{n+2})$, see [3]. Then if we take $\alpha(x) = e^x - 1$ we obtain the following 5-cycle

$$x_{n+3} = (1 + x_{n+2})^{\left(\frac{\log(1+x_{n+1})}{\log(1+x_n)}\right)^{\Phi}} - 1.$$

Remark 12. It is interesting to emphasize that Theorem E partially solves the open problem proposed in the last section of [4]. For instance, in that paper it is shown that for an odd positive integer l, the difference equations

(13) $$x_{n+l} = \frac{c}{x_n x_{n+2} \cdots x_{n+l-1}}, \quad \text{for some } c > 0$$

and

(14) $$x_{n+l} = \frac{\prod_{j=1}^{(l+1)/2} x_{n+2j-2}}{\prod_{j=1}^{(l-1)/2} x_{n+2j-1}}$$

are the only $(l+1)$-cycles with the form

$$x_{n+l} = f_{l-1}(x_{n+l-1}) \cdots f_1(x_{n+1}) f_0(x_n).$$

If we consider an arbitrary homeomorphism $\alpha : (0,\infty) \to (0,\infty)$, then according to Proposition E we can obtain a new family of $(l+1)$-cycles

$$x_{n+l} = \alpha\left[f_{l-1}(\alpha^{-1}(x_{n+l-1})) \cdot \cdots \cdot f_2(\alpha^{-1}(x_{n+2})) \cdot f_1(\alpha^{-1}(x_{n+1}))\right],$$

different to (13) and (14).

3.3. Non-symmetric 3-cycles

The aim of this subsection is to construct a continuous non-symmetric map $\widetilde{f}: (0,\infty)^2 \to (0,\infty)$ such that the associated difference equation $x_{n+2} = \widetilde{f}(x_{n+1}, x_n)$ is a 3-cycle. In particular we will prove the following theorem at the end of this subsection.

Theorem F. *There exists a continuous map $\widetilde{F} : (0,\infty)^2 \to (0,\infty)^2$, with the form $\widetilde{F}(x,y) = (y, \widetilde{f}(x,y))$, that holds $\widetilde{F}^3 = \mathrm{Id}|_{(0,\infty)^2}$. Moreover $\widetilde{f}$ is non-symmetric.*

Consequently, the difference equation associated to $\widetilde{F}$ is a non-symmetric 3-cycle.

We are going to introduce several technical results in order to prove Theorem . We begin by defining $F : (0,+\infty)^2 \to (0,+\infty)^2$ as

$$F(x,y) = (y, f(x,y)), \quad \text{where } f(x,y) = \frac{1}{xy}.$$

Notice that for all $x, y > 0$ we have

$$F^2(x,y) = (f(x,y), x) \text{ and } F^3(x,y) = (x,y). \tag{15}$$

The main idea of the construction is to modify F in a suitable way on an appropriate rectangle R and its iterates. Later we consider the new map $\widetilde{F}(x,y) = (y, \widetilde{f}(x,y))$ which agrees with F outside the rectangle R and its iterates.

As usual, if $A \subset \mathbb{R}^2$ then ∂A and $\operatorname{Int} A$ denote the boundary and the interior of A, respectively.

Lemma 13. *There exists a rectangle $R = [a,b] \times [c,d]$ such that $R \cap F(R) = R \cap F^2(R) = F(R) \cap F^2(R) = \emptyset$.*

Proof. Notice that

$$F(R) = \{(v,w) : c \le v \le d, \frac{1}{bv} \le w \le \frac{1}{av}\}, \tag{16}$$

$$F^2(R) = \{(v,w) : a \le w \le b, \frac{1}{dw} \le v \le \frac{1}{cw}\}. \tag{17}$$

So it suffices to take $R = \left[\frac{1}{500}, \frac{1}{100}\right] \times \left[\frac{1}{10}, \frac{1}{5}\right]$. ■

In the sequel R will denote the rectangle

$$R = [a,b] \times [c,d] := \left[\frac{1}{500}, \frac{1}{100}\right] \times \left[\frac{1}{10}, \frac{1}{5}\right].$$

Now we modify the difference equation $x_{n+2} = \frac{1}{x_n x_{n+1}}$, or equivalently the map F, in the sets R, $F(R)$ and $F^2(R)$.

We will define homeomorphisms

$$F_1 : R \to F(R), \qquad F_2 : F(R) \to F^2(R), \qquad F_3 : F^2(R) \to F^3(R) = R,$$

such that F_1, F_2 and F_3 coincide with F on the boundary of their respective domains and satisfying

$$F_i(x,y) = (y, f_i(x,y)),\ 1 \leq i \leq 3,$$

for suitable maps $f_i : C_i \to (0,+\infty)$, $C_i = F^{i-1}(R)$, $i = 1,2,3$.

Proposition 14. *The map $f_1 : R \to \mathbb{R}$ given by*

$$f_1(x,y) = \frac{1}{xy} + \left(\frac{1}{ay} - \frac{1}{xy}\right)(x-a)^2(b-x)^2(d-y)^2(y-c)^2, \tag{18}$$

satisfies:

- (i) $\frac{\partial f_1}{\partial x}(x,y) < 0$ *for all* $(x,y) \in R$.
- (ii) $\frac{\partial f_1}{\partial y}(x,y) < 0$ *for all* $(x,y) \in R$.

Proof. (i). It is straightforward to check that

$$\begin{aligned} &\frac{\partial f_1}{\partial x}(x,y) \\ &= \frac{1}{-x^2 y}\left(\frac{1}{2} - \frac{1}{2}(x-a)^2(b-x)^2(d-y)^2(y-c)^2\right) \\ &\quad + \frac{1}{y}\Big\{(d-y)^2(y-c)^2(x-a)(b-x)\times \\ &\quad \left[\frac{(x-a)(b-x)}{2x^2} + 2\left(\frac{1}{a} - \frac{1}{x}\right)(a+b-2x)\right] - \frac{1}{2}\frac{1}{x^2}\Big\} =: D + \frac{1}{y}E. \end{aligned}$$

Since $D < 0$ for all $(x,y) \in R$, we will prove that E is also negative. Notice that

$$(d-y)^2(y-c)^2(x-a)(b-x)\frac{(x-a)(b-x)}{2x^2} < \frac{1}{2}\frac{1}{2x^2}. \tag{19}$$

On the other hand, we claim that

$$\left|2(d-y)^2(y-c)^2(x-a)(b-x)\left(\frac{1}{a} - \frac{1}{x}\right)(a+b-2x)\right| < \frac{1}{2}\frac{1}{2x^2} \tag{20}$$

for all $(x, y) \in R$. It is a simple matter to check that $|(x-a)(b-x)| \leq \frac{(b-a)^2}{4}$. Hence

$$\left|2(d-y)^2(y-c)^2(x-a)(b-x)\left(\frac{1}{a}-\frac{1}{x}\right)(a+b-2x)\right|$$
$$\leq 2(0.1)^4\frac{(b-a)^2}{4}\left(\frac{1}{a}-\frac{1}{x}\right)0.008.$$

The task now is to deduce that

$$2(0.1)^4\frac{(0.008)^2}{4}\left(\frac{1}{a}-\frac{1}{x}\right)0.008 < \frac{1}{2}\frac{1}{2x^2},$$

or equivalently

$$\frac{1}{a} < \frac{1}{x} + \frac{10^{13}}{2^{10}}\frac{1}{x^2}, \text{ that is } 500x^2 - x - \frac{10^{13}}{2^{10}} < 0.$$

But it is easily seen that the last inequality holds for all $x \in R$ (the details are left to the reader). This completes the proof of (20).

Finally, from (19) and (20) we deduce that $E < 0$ for all $(x, y) \in R$, and consequently $\frac{\partial f_1}{\partial x} < 0$ in R.

(ii). Similarly to the first part, we compute

$$\begin{aligned}\frac{\partial f_1}{\partial y}(x,y) &= \frac{-1}{y^2x}\left[\frac{1}{2}-(x-a)^2(b-x)^2(d-y)^2(y-c)^2\right]\\ &-\frac{1}{2}\frac{1}{xy^2}-\frac{1}{ay^2}(x-a)^2(b-x)^2(d-y)^2(y-c)^2\\ &+2\left(\frac{1}{ay}-\frac{1}{xy}\right)(x-a)^2(b-x)^2(d-y)(y-c)(c+d-2y).\end{aligned}$$

If we prove that

$$\left|2\left(\frac{1}{ay}-\frac{1}{xy}\right)(x-a)^2(b-x)^2(d-y)(y-c)(c+d-2y)\right| < \frac{1}{2}\frac{1}{xy^2},$$

we will obtain that $\frac{\partial f_1}{\partial y}(x, y) < 0$ in R. To see it, notice that

$$\left|2\left(\frac{1}{ay}-\frac{1}{xy}\right)[(x-a)^2(b-x)^2]\{(d-y)(y-c)\}(c+d-2y)\right|$$
$$< \frac{2}{y}\left(\frac{1}{a}-\frac{1}{x}\right)\frac{(b-a)^4}{16}\frac{(d-c)^2}{4}(d-c) = \frac{2}{y}\left(\frac{1}{a}-\frac{1}{x}\right)\frac{(0.008)^4(0.1)^3}{64}.$$

So, it suffices to prove that

$$\frac{2}{y}\left(\frac{1}{a}-\frac{1}{x}\right)\frac{(0.008)^4\,(0.1)^3}{64}<\frac{1}{2}\frac{1}{xy^2},$$

or equivalently

$$\frac{1}{a}<\frac{1}{x}+\frac{10^{15}}{2^8}\frac{1}{xy},\quad \text{i.e.}\quad 500x \quad < \quad 1+\frac{10^{15}}{2^8}\frac{1}{y}.$$

This last inequality holds since $1+\frac{10^{15}}{2^8}\frac{1}{y}\geq 1+5\frac{10^{15}}{2^8}>5\geq 500x$, which completes the proof of the second part of the result. ■

Next, we define the map $F_1 : R \to F_1(R)$ by

$$F_1(x,y)=(y,f_1(x,y)). \tag{21}$$

Lemma 15. *It holds*

$$F_1(R)=F(R).$$

Proof. First notice that $F_1(R)\subset F(R)$. To see it, take a point $(p,q)\in F_1(R)$, $(p,q)=F_1(x,y)=(y,f_1(x,y))$ for some $(x,y)\in R$. Then $y=p\in[c,d]$ and also

$$\frac{1}{bp}\leq q\leq\frac{1}{ap}$$

since the inequalities

$$\frac{1}{by}\leq\frac{1}{xy}+\left(\frac{1}{ay}-\frac{1}{xy}\right)(x-a)^2(b-x)^2(d-y)^2(y-c)^2\leq\frac{1}{ay}$$

hold. By (16) we deduce that $(p,q)\in F(R)$.

To obtain the reverse inclusion consider a point $(v,w)\in F(R)$, so $(v,w)=F(\widetilde{x},\widetilde{y})=(\widetilde{y},f(\widetilde{x},\widetilde{y}))$ for some $(\widetilde{x},\widetilde{y})\in R$. We are going to show that $(v,w)\in F_1(R)$. Taking into account that $f_1(a,\widetilde{y})=f(a,\widetilde{y})$, $f_1(b,\widetilde{y})=f(b,\widetilde{y})$ and applying Proposition 14(i), we deduce the existence of a point $x'\in(a,b)$ such that $f_1(x',\widetilde{y})=f(\widetilde{x},\widetilde{y})$. Finally $F_1(x',\widetilde{y})=(\widetilde{y},f_1(x',\widetilde{y}))=(\widetilde{y},f(\widetilde{x},\widetilde{y}))=(v,w)$. Therefore $(v,w)\in F_1(R)$, as desired. ■

Proposition 16. *The map $F_1 : R\to F(R)$ is a homeomorphism satisfying*

$$(F_1)|_{\partial R}=F|_{\partial R}.$$

Proof. From the definition of F_1 and F we deduce that $(F_1)|_{\partial R} = F|_{\partial R}$.

Lemma 15 yields the surjectivity of F_1. Moreover, F_1 is injective, since $F_1(x,y) = F_1(x',y')$ implies $(y, f_1(x,y)) = (y', f_1(x',y'))$, so $y = y'$, $f_1(x,y) = f_1(x',y)$. Now Proposition 14 leads to $x = x'$. Hence $(x,y) = (x',y')$.

Since F_1 is continuous, bijective and it is defined on a compact set of the plane, we conclude that it is a homeomorphism. ∎

We now present the second modification of F on $F(R)$. We define

$$F_2 \ : \ F(R) \to F_2(F(R)),$$

$$F_2(x,y) = (y, f \circ F \circ F_1^{-1}(x,y)).$$

Remark 17. From the definitions, we have

$$(F_2)|_{\partial F(R)} = F|_{\partial F(R)} \quad \textit{and} \quad F_2(\partial F(R)) = \partial(F^2(R)).$$

Lemma 18. *F_2 is an injective map.*

Proof. Suppose that $F_2(x,y) = F_2(x',y')$ for some $(x,y),(x',y') \in F(R)$. Then

$$(y, f \circ F \circ F_1^{-1}(x,y)) = (y', f \circ F \circ F_1^{-1}(x',y')),$$

so $y = y'$ and $f \circ F \circ F_1^{-1}(x,y) = f \circ F \circ F_1^{-1}(x',y)$.

Let $F_1^{-1}(x,y) = (u,x)$ and $F_1^{-1}(x',y) = (v,x')$. Then:

$$f \circ F \circ F_1^{-1}(x,y) = f \circ F(u,x) = f(x, f(u,x)) = u,$$

$$f \circ F \circ F_1^{-1}(x',y) = f \circ F(v,x') = f(x', f(v,x')) = v.$$

Hence $u = v$, and from $(x,y) = F_1(u,x) = (x, f_1(u,x))$, $(x',y) = F_1(v,x') = (x', f_1(v,x'))$, we obtain $f_1(u,x) = f_1(v,x') = f_1(u,x')$. Finally by Proposition 14(ii) we deduce that $x = x'$. ∎

Proposition 19. *F_2 is a homeomorphism from $F(R)$ into $F_2(F(R))$ and*

$$F_2(F(R)) = F^2(R). \tag{22}$$

Proof. Since F_2 is continuous, bijective (see Lemma 18) and it is defined on a compact set, we deduce that F_2 is a homeomorphism from $F(R)$ onto $F_2(F(R))$. Next, we are going to prove (22). From Remark 17 and the fact that F_2 is a homeomorphism, we deduce $F_2\,(\partial F(R)) = F_2(F(\partial R)) = \partial(F^2(R)) = F^2(\partial R)$, that is

$$F_2(F(\partial R)) = F^2(\partial R). \tag{23}$$

Moreover, $\gamma := F^2(\partial R)$ is a Jordan's curve.

Since $F_2(\text{Int}(F(R)))$ is an open set in $\mathbb{R}^2$ by the theorem of invariance of domain (see [8, p. 475]), we find

$$\partial(F_2(F(R))) \cap F_2(\text{Int}(F(R))) = \emptyset, \quad \partial(F_2(F(R))) \subseteq F_2(\partial F(R)) = \partial F^2(R).$$

Being $F_2(F(R))$ a bounded set, we have $\partial(F_2(F(R))) = F_2(\partial F(R)) = \partial F^2(R)$. In that case, either $F_2(\text{Int}(F(R))) = A$ (here A is the bounded region inside γ) or $F_2(\text{Int}(F(R))) = B$ (where B is the unbounded region outside γ). By noticing that $F_2(F(R))$ is bounded, we obtain

$$F_2(\text{Int}(F(R))) = A. \tag{24}$$

From (23) and (24) we conclude that $F_2(F(R)) = F^2(R)$. ■

Lemma 20. *For any* $(x, y) \in R$, $F^2(x, y) = (f(x, y), x)$ *and* $F_2 \circ F_1(x, y) = (f_1(x, y), x)$.

Proof. Given a point $(x, y) \in R$ it is clear that $F^2(x, y) = (f(x, y), x)$ and

$$\begin{aligned} &F_2 \circ F_1(x, y) \\ &= F_2(y, f_1(x, y)) = (f_1(x, y), f \circ F \circ F_1^{-1}(y, f_1(x, y))) \\ &= (f_1(x, y), f \circ F(x, y)) = (f_1(x, y), f(y, f(x, y))) = (f_1(x, y), x). \end{aligned}$$ ■

Finally we make a third modification of F on $F^2(R)$. We define the map F_3 as:

$$\begin{aligned} F_3 :\ & F^2(R) \to F_3(F^2(R)), \\ F_3(x, y) =\ & \big(y, f \circ F^2 \circ (F_2 \circ F_1)^{-1}(x, y)\big). \end{aligned} \tag{25}$$

Lemma 21. F_3 *is injective.*

Proof. Consider two elements (v, w), $(\widetilde{v}, \widetilde{w})$ from $F^2(R)$ and suppose that $F_3(v, w) = F_3(\widetilde{v}, \widetilde{w})$. According to (25) $w = \widetilde{w}$. Proposition 19 and Lemma 15 yield $F^2(R) = F_2(F(R)) = F_2(F_1(R))$, so there are $(x, y), (\widetilde{x}, \widetilde{y}) \in R$ such that $(v, w) = F_2 \circ F_1(x, y)$ and $(\widetilde{v}, \widetilde{w}) = F_2 \circ F_1(\widetilde{x}, \widetilde{y})$. Then using Lemma 20:

$$f \circ F^2 \circ (F_2 \circ F_1)^{-1}[F_2 \circ F_1(x, y)] = f \circ F^2(x, y) = f((f(x, y), x)) = y,$$

$$f \circ F^2 \circ (F_2 \circ F_1)^{-1}[F_2 \circ F_1(\widetilde{x}, \widetilde{y})] = f \circ F^2(\widetilde{x}, \widetilde{y}) = f((f(\widetilde{x}, \widetilde{y}), \widetilde{x})) = \widetilde{y}.$$

Consequently,

$$\begin{aligned} F_3(v,w) &= F_3(F_2\circ F_1(x,y)) = (w, f\circ F^2\circ (F_2\circ F_1)^{-1}[F_2\circ F_1(x,y)]) \\ &= (w,y), \\ F_3(\widetilde{v},\widetilde{w}) &= F_3(F_2\circ F_1(\widetilde{x},\widetilde{y})) = (\widetilde{w}, f\circ F^2\circ (F_2\circ F_1)^{-1}[F_2\circ F_1(\widetilde{x},\widetilde{y})]) \\ &= (\widetilde{w},\widetilde{y}) = (w,\widetilde{y}). \end{aligned}$$

Since $F_3(v,w) = F_3(\widetilde{v},\widetilde{w})$ we have $y=\widetilde{y}$ and using Lemma 20:

$$F_2\circ F_1(x,y) = (f_1(x,y),x) = (v,w), \quad F_2\circ F_1(\widetilde{x},\widetilde{y}) = (f_1(\widetilde{x},\widetilde{y}),\widetilde{x}) = (\widetilde{v},\widetilde{w}).$$

Therefore $x=\widetilde{x}$ and finally $(v,w)=(\widetilde{v},\widetilde{w})$, which proves the injectivity of F_3. ∎

Proposition 22. *$R = F_3(F^2(R))$, F_3 is a homeomorphism and satisfies*

$$F^3(x,y) = (F_3\circ F_2\circ F_1)(x,y) = (x,y) \qquad \textit{for all } (x,y)\in R.$$

Proof. First we obtain $F^3(x,y) = (F_3\circ F_2\circ F_1)(x,y)$, $(x,y)\in R$. Indeed, a straightforward calculation gives (we use Lemma 20 and (15))

$$\begin{aligned} (F_3\circ F_2\circ F_1)(x,y) &= F_3(f_1(x,y),x) \\ &= (x,(f\circ F^2\circ (F_2\circ F_1)^{-1}[F_2\circ F_1(x,y)]) \\ &= (x,(f\circ F^2)(x,y)) = (x,f(f(x,y),x)) = (x,y) = F^3(x,y). \end{aligned}$$

From $F^3|_R = F_3\circ F_2\circ F_1$, Proposition 16 and (22) it follows $R = F_3(F^2(R))$.

F_3 is bijective (use Lemma 21), then it is a homeomorphism from $F^2(R)$ onto R since $F^2(R)$ is compact and F_3 is continuous. ∎

Proof of Theorem F. Define the map $\widetilde{F}:(0,\infty)^2\to(0,\infty)^2$ by

$$\widetilde{F}(x,y) = \begin{cases} F(x,y), & \text{if } (x,y)\notin R\cup F(R)\cup F^2(R), \\ F_1(x,y), & \text{if } (x,y)\in R, \\ F_2(x,y), & \text{if } (x,y)\in F(R), \\ F_3(x,y), & \text{if } (x,y)\in F^2(R). \end{cases}$$

Since $\widetilde{f}|_R = f_1$ it is clear that $\widetilde{f}$ is not symmetric. Moreover applying Propositions 16, 19, 22 and (15) we obtain $\widetilde{F}^3 = \mathrm{Id}|_{(0,\infty)^2}$. Finally by Lemma 9 the associated difference equation to $\widetilde{f}$ is a 3-cycle. ∎

4. Final Remarks

We have studied the topic of global periodicity of difference equations of first and second order. Given a k-cycle we have shown a general method based in the topological conjugation for constructing new k-cycles. Moreover, by translating the study of the dynamics of a difference equation to the study of the associated dynamical system, we were able to do a suitable modification on a symmetric 3-cycle in order to obtain new non-symmetric 3-cycles linked to it. It has been proved that any 2-cycle of first order is topologically conjugated to $x_{n+1} = \frac{1}{x_n}$. However, it is an open problem to know whether any 3-cycle of second order is or not topologically conjugate to $x_{n+2} = \frac{1}{x_n x_{n+1}}$.

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