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ON SANDWICH THEOREMS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH DZIOK-SRIVASTAVA OPERATOR

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Abstract. In the present paper, we give some applications of first order differential subordination and superordinations to obtain sufficient conditions for normalized analytic functions defined by certain linear operators to be subordinated and superordinated to a given univalent function.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the open unit disc $\Delta := \{z : |z| < 1\}$. Let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2 z^2 + \cdots$.

With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Then we say that the functions f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in Δ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

such that

$$f(z) = g(w(z)) \quad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).$$

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In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{1.1}$$

then p is a solution of the differential superordination (1.1). (If f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.1). An univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [9] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Using the results of Miller and Mocanu [9], Bulboaca [3] considered certain classes of first order differential superordinations as well as superordination preserving integral operators [2]. Ali et al. [1] has used the results of Bulboaca [2] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$.

For $\alpha_j \in \mathbb{C}(j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^- : \{0, -1, -2, \ldots\}, (j = 1, 2, \ldots, m)$, the generalized hypergeometric function ${}_lF_m\ (\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n = 0); \\ a, (a+1), (a+2), \ldots, (a+n-1), & (n \in \mathbb{N}). \end{cases}$$

Corresponding to the function

$$h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z\ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [5, 6, 7] (See also [10]) $H^{(l,m)}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;$ $z)$ is defined by the Hadamard product

$$\begin{aligned} &H^{(l,m)}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)f(z):\\ &= h(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)*f(z)\\ &= z+\sum_{n=2}^{\infty}\frac{(\alpha_1)_{n-1}\dots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_m)_{n-1}}\frac{a_n z^n}{(n-1)!}. \end{aligned} \tag{1.2}$$

It is well known, from the work of Srivastava [5], that

$$\begin{aligned} &\alpha_1 H^{(l,m)}(\alpha_1+1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)f(z)\\ &= z[H^{(l,m)}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)f(z)]'\\ &+(\alpha_1-1)H^{(l,m)}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)f(z). \end{aligned} \tag{1.3}$$

To make the notation simple, we write

$$H^{(l,m)}[\alpha_1]f(z) := H^{(l,m)}(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)f(z).$$

We note that $H^{2,1}(a,1;c)f(z)=L(a,c)f(z)$, the familiar Carlson-Shaffer operator and $H^{2,1}(\delta+1,1;1)f(z)=D^{\delta}f(z)$, the familiar Ruscheweyh derivative operator.

The multiplier transformation of Srivastava [10] on $\mathcal{A}$, is the operator $I(r,\lambda)$ on $\mathcal{A}$ defined by the following infinite series

$$I(r,\lambda)f(z) := z+\sum_{k=2}^{\infty}\left(\frac{k+\lambda}{1+\lambda}\right)^r a_k z^k. \tag{1.4}$$

A straight forward calculation shows that

$$(1+\lambda)I(r+1,\lambda)f(z) = z(I(r,\lambda)f(z))'+\lambda I(r,\lambda)f(z). \tag{1.5}$$

The operator $I(r,0)$ is the Salagean derivative operators. The operator $I_\lambda^r := I(r,\lambda)$ was studied recently by Cho and Kim [4]. The operator $I_r : I(r,1)$ was studied by Uralegaddi and Somanatha [11].

In the present investigation, we obtain sufficient condition for a normalized analytic function $f(z)$ to satisfy

$$q_1(z)\prec f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma}\prec q_2(z),\quad (0<\gamma<1)$$

where q_1 and q_2 are given univalent function in Δ. Also, we obtain results for function defined by Dziok-Srivastava operator and multiplier transformation.

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 2.1. [9, Definition 2, p. 817]. Denote by $\mathcal{Q}$, the set of all functions $f(z)$ that are analytic and injective on $\bar{\Delta} - E(f)$, where

$$E(f) := \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Theorem 2.1. [8, Theorem 3.4h, p. 132]. *Let the function q be univalent in the unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta\{q(z)\} + Q(z)$. Suppose that,*

(1) *Q is starlike univalent in Δ and*
(2) $\Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for $z \in \Delta$.*

If $p(z)$ is analytic in Δ with $p(\Delta) \subseteq D$, and

$$\theta\{p(z)\} + zp'(z)\phi(p(z)) \prec \theta\{q(z)\} + zq'(z)\phi(q(z)), \tag{2.6}$$

then $p \prec q$ and q is the best dominant.

Theorem 2.2. [3]. *Let q be univalent in Δ, v and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

(1) $\Re\left\{\frac{v'(q(z))}{\varphi(q(z))}\right\} > 0$ *for $z \in \Delta$, and*
(2) *$Q(z) = zq'(z)\varphi(q(z))$ is starlike univalent function in Δ.*

If $p \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$, with $p(\Delta) \subset D$, and $v\{p(z)\} + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$v\{q(z)\} + zq'(z)\varphi(q(z)) \prec v\{p(z)\} + zp'(z)\varphi(p(z)), \tag{2.7}$$

then $q \prec p$ and q is the best subordinant.

3. Subordination and Superordination for Analytic Functions

We begin by proving the following result.

Theorem 3.1. *Let* $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$. *Let* $0 \neq q(z)$ *be univalent in* Δ *and* $\frac{zq'(z)}{q(z)}$ *be starlike univalent in* Δ. *Further assume that*

$$\Re\left(1 + \frac{\alpha}{\beta}q(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0. \tag{3.1}$$

Let

$$\psi(\alpha, \beta; z) := \alpha\left[f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma}\right] + \beta\left[\frac{zf''(z)}{f'(z)} + (1+\gamma)\left(1 - \frac{zf'(z)}{f(z)}\right)\right]. \tag{3.2}$$

If $f \in \mathcal{A}$ *satisfies*

$$\psi(\alpha, \beta; z) \prec \alpha q(z) + \beta\frac{zq'(z)}{q(z)},$$

then

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma} \prec q(z) \tag{3.3}$$

and q *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) := f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma} \quad (0 < \gamma < 1),\ z \in \Delta. \tag{3.4}$$

Then the function $p(z)$ is analytic in Δ and $p(0) = 1$.
By a straightforward computation, we have

$$\begin{aligned} &\alpha p(z) + \beta\frac{zp'(z)}{p(z)} \\ &= \alpha\left[f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma}\right] + \beta\left[\frac{zf''(z)}{f'(z)} + (1+\gamma)\left(1 - \frac{zf'(z)}{f(z)}\right)\right] \\ &= \psi(\alpha, \beta; z). \end{aligned} \tag{3.5}$$

By setting

$$\theta(w) := \alpha w \quad \text{and} \quad \phi(w) := \frac{\beta}{w},$$

it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} - \{0\}$ and that $\phi(w) \neq 0$ $(w \in \mathbb{C} - \{0\})$.

Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \beta\frac{zq'(z)}{q(z)} \tag{3.6}$$

and

(3.7) $$h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta \frac{zq'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in Δ and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left\{1 + \frac{\alpha}{\beta}q(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right\} > 0;$$

$$(z \in \Delta; \alpha, \beta \in \mathbb{C}; \beta \neq 0).$$

The assertion (3.3) of Theorem 3.1 now follows by an application of Theorem 2.1.

For the choice $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3.1, we get the following result.

Corollary 3.2. *Assume that (3.1) hold. If $f \in \mathcal{A}$ and*

$$\psi(\alpha, \beta; z) \prec \alpha\left(\frac{1+Az}{1+Bz}\right) + \beta \frac{(A-B)z}{(1+Az)(1+Bz)},$$

where $\psi(\alpha, \beta; z)$ is as defined by (3.2), then

$$f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

In particular

$$\psi(\alpha, \beta; z) \prec \alpha\left(\frac{1+z}{1-z}\right) + \frac{2\beta z}{1-z^2}$$

implies

$$\Re\left[f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma}\right] > 0.$$

Theorem 3.3. *Let $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$. Let q be analytic and convex univalent in Δ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in Δ. Let $f \in \mathcal{A}$, $0 \neq f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma} \in \mathcal{H}[1,1]$ $\cap \mathcal{Q}$, with*

(3.8) $$\Re\left(\frac{\alpha}{\beta}\ \ q(z)\right) > 0.$$

If $\psi(\alpha, \beta; z)$ as defined by (3.2) is univalent in Δ and

(3.9) $$\alpha q(z) + \beta \frac{zq'(z)}{q(z)} \prec \psi(\alpha, \beta; z),$$

then

(3.10) $$q(z) \prec f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma}$$

and q is the best subordinant.

Proof. By setting

$$v(w) := \alpha w \quad \text{and} \quad \varphi(w) := \frac{\beta}{w}\ ,$$

it is easily observed that $v(w)$ is analytic in $\mathbb{C}$. Also, $\varphi(w)$ is analytic in $\mathbb{C} - \{0\}$ and that $\varphi(w) \neq 0$ $(w \in \mathbb{C} - \{0\})$. Since q is convex (univalent) function it follows that,

$$\Re\left[\frac{v'(q(z))}{\varphi(q(z))}\right] = \Re\left[\frac{\alpha}{\beta}\ \ q(z)\right] > 0, \quad (z \in \Delta; \alpha, \beta \in \mathbb{C}; \beta \neq 0).$$

The assertion (3.10) of Theorem 3.3 follows by an application of Theorem 2.2.

We remark here that Theorem 3.3 can easily be restated, for different choices of the function $q(z)$. Combining Theorem 3.1 and Theorem 3.3, we get the following sandwich theorem.

Theorem 3.4. *Let q_1 and q_2 be convex univalent in Δ and satisfies (3.8) and (3.1) respectively. Suppose $\frac{zq_i'(z)}{q_i(z)}$ be starlike univalent in Δ for $i = 1, 2$. If $f \in \mathcal{A}$, $0 \neq f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ and $\psi(\alpha, \beta; z)$ as defined by (3.2) is univalent in Δ and*

$$\alpha q_1(z) + \beta \frac{zq_1'(z)}{q_1(z)} \prec \psi(\alpha, \beta; z) \prec \alpha q_2(z) + \beta \frac{zq_2'(z)}{q_2(z)}$$

then

$$q_1(z) \prec f'(z)\left(\frac{z}{f(z)}\right)^{1+\gamma} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and best dominant.

4. Application to Dziok-srivastava Operator

Theorem 4.1. *Let $0 \neq q(z)$ be univalent in Δ and satisfies (3.1). Suppose $\frac{zq'(z)}{q(z)}$ be starlike univalent in Δ. Let*

$$\begin{aligned}\phi(\alpha, \beta, l, m; z) := \ &\alpha\left(\frac{z^\gamma H^{l,m}[\alpha_1 + 1]f(z)}{(H^{l,m}[\alpha_1]f(z))^{1+\gamma}}\right) \\ &+\beta\left[(\gamma\alpha_1 - 1) + \frac{(\alpha_1 + 1)H^{l,m}[\alpha_1 + 2]f(z)}{H^{l,m}[\alpha_1 + 1]f(z)}\right. \\ &\left.-\frac{(1+\gamma)\alpha_1 H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)}\right].\end{aligned} \tag{4.1}$$

Let $f \in \mathcal{A}$ and

$$\phi(\alpha, \beta, l, m; z) \prec \alpha q(z) + \beta \frac{zq'(z)}{q(z)},$$

then

$$\frac{z^{\gamma} H^{l,m}[\alpha_1 + 1]f(z)}{(H^{l,m}[\alpha_1]f(z))^{1+\gamma}} \prec q(z) \tag{4.2}$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := \frac{z^{\gamma} H^{l,m}[\alpha_1 + 1]f(z)}{(H^{l,m}[\alpha_1]f(z))^{1+\gamma}} . \tag{4.3}$$

By taking logarithmic derivative of $p(z)$ given by (4.3), we have

$$\frac{zp'(z)}{p(z)} = \gamma + \frac{z(H^{l,m}[\alpha_1 + 1]f(z))'}{H^{l,m}[\alpha_1 + 1]f(z)} - (1+\gamma)\frac{z(H^{l,m}[\alpha_1 + 1]f(z))'}{H^{l,m}[\alpha_1]f(z)} \tag{4.4}$$

By using the identity

$$z(H^{l,m}[\alpha_1]f(z))' = \alpha_1 H^{l,m}[\alpha_1 + 1]f(z) - (\alpha_1 - 1)H^{l,m}[\alpha_1]f(z)$$

and (4.3) in (4.4) we obtain

$$\frac{zp'(z)}{p(z)} = (\gamma\alpha_1 - 1) + (\alpha_1 + 1)\frac{H^{l,m}[\alpha_1 + 2]f(z)}{H^{l,m}[\alpha_1 + 1]f(z)} - (1+\gamma)\alpha_1\frac{H^{l,m}[\alpha_1 + 1]f(z)}{H^{l,m}[\alpha_1]f(z)}$$

The assertion (4.2) of Theorem 4.1 now follows from Theorem 2.1.

Taking $l = 2$, $m = 1$ and $\alpha_2 = 1$ in Theorem 4.1, we get:

Corollary 4.2. *Let $0 \neq q(z)$ be univalent in Δ with $q(0) = 1$. If $f \in \mathcal{A}$ and*

$$\begin{aligned} \xi(\alpha, \beta, l, m; z) := {} & \alpha \left(\frac{z^{\gamma} L(a+1, c)f(z)}{(L(a, c)f(z))^{1+\gamma}} \right) \\ & + \beta \left[(\gamma a - 1) + \frac{(a+1)L(a+2, c)f(z)}{L(a+1, c)f(z)} \right. \\ & \left. - \frac{(1+\gamma)aL(a+1, c)f(z)}{L(a, c)f(z)} \right] . \end{aligned} \tag{4.5}$$

If

$$\xi(\alpha, \beta, l, m; z) \prec \alpha q(z) + \beta \frac{zq'(z)}{q(z)},$$

then
$$\frac{z^{\gamma}L(a+1,c)f(z)}{(L(a,c)f(z))^{1+\gamma}} \prec q(z)$$
and q is th best dominant, where $L(a,c)$ is the familiar Carlson-Shaffer opertor.

By taking $a = n+1$, $c = 1$, $\alpha = 0, \beta = 1$ and $q(z) = 1 + (1-b)z$ in corollary 4.2, then we have the following:

Corollary 4.3. *If $f \in \mathcal{A}$ and*
$$\gamma n + (\gamma - 1) + (n+2)\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - (1+\gamma)(n+1)\frac{D^{n+1}f(z)}{D^{n}f(z)} \prec \frac{(1-b)z}{1+(1-b)z},\ (0 \le b < 1),$$
then
$$\frac{z^{\gamma}D^{n+1}f(z)}{(D^{n}(f(z))^{1+\gamma}} \prec 1 + (1-b)z$$
and $1 + (1-b)z$ is the best subordinant, where D^n is Ruscheweyh derivative operator.

Theorem 4.4. *Let $0 \neq q(z)$ be convex univalent in Δ with $q(0) = 1$ and satisfies (3.8). Suppose $\frac{zq'(z)}{q(z)}$ be starlike univalent in Δ. Let $f \in \mathcal{A}$, $0 \neq \frac{z^{\gamma}H^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^{1+\gamma}} \in \mathcal{H}[1,1] \cap \mathcal{Q}$. If $\phi(\alpha, \beta, l, m; z)$ as defined by (4.1) is univalent in Δ and*
$$\alpha q(z) + \beta\frac{zq'(z)}{q(z)} \prec \phi(\alpha, \beta, l, m; z),$$
then
$$q(z) \prec \frac{z^{\gamma}H^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^{1+\gamma}}$$
and q is the best dominant.

Proof. Theorem 4.4 follows from Theorem 2.2 by taking
$$p(z) := \frac{z^{\gamma}H^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^{1+\gamma}}$$

By combining Theorem 4.1 and Theorem 4.4 we get the following sandwich theorem.

Theorem 4.5. *Let $0 \neq q_1(z)$ and $0 \neq q_2(z)$ be convex univalent satisfying (3.8) and (3.1) respectively. Suppose $\frac{zq_i'(z)}{q_i(z)}$ be starlike univalent in Δ for $i = 1, 2$.*

If $f \in \mathcal{A}$, $0 \neq \left(\frac{z^{\gamma}H^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^{1+\gamma}}\right) \in \mathcal{H}[1,1] \cap \mathcal{Q}$ *and* $\phi(\alpha,\beta,l,m;z)$ *as defined by (4.1) is univalent in* Δ. *Further if*

$$\alpha q_1(z) + \beta\frac{zq_1'(z)}{q_1(z)} \prec \phi(\alpha,\beta,l,m;z) \prec \alpha q_2(z) + \beta\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{z^{\gamma}H^{l,m}[\alpha_1+1]f(z)}{(H^{l,m}[\alpha_1]f(z))^{1+\gamma}} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and best dominant.*

5. Application to Multiplier Transformation

Theorem 5.1. *Let* $0 \neq q(z)$ *be univalent in* Δ *with* $q(0) = 1$ *and satisfies (3.1). Suppose* $\frac{zq'(z)}{q(z)}$ *be starlike univalent in* Δ. *Let* $f \in \mathcal{A}$ *and*

$$\begin{aligned}\eta(\alpha,\beta,r,\lambda;z) := \alpha&\left(\frac{z^{\gamma}I(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^{1+\gamma}}\right)\\ &+\beta\left\{\gamma(1+\lambda)+(1+\lambda)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)}\right.\\ &\left.-(1+\gamma)(1+\lambda)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}\right\}. \end{aligned} \tag{5.1}$$

If

$$\eta(\alpha,\beta,r,\lambda;z) \prec \alpha q(z) + \beta\frac{zq'(z)}{q(z)},$$

then

$$\frac{z^{\gamma}I(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^{1+\gamma}} \prec q(z) \tag{5.2}$$

and $q(z)$ *is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) := \frac{z^{\gamma}I(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^{1+\gamma}} \tag{5.3}$$

By taking logarithmic derivative of $p(z)$ given by (5.3), we get

$$\frac{zp'(z)}{p(z)} := \gamma + \frac{z(I(r+1,\lambda)f(z))'}{I(r+1,\lambda)f(z)} - (1+\gamma)\frac{z(I(r,\lambda)f(z))'}{I(r,\lambda)f(z)}\ . \tag{5.4}$$

By using the identity

$$z(I(r,\lambda)f(z))' = (1+\lambda)I(r+1,\lambda)f(z) - \lambda I(r,\lambda)f(z)$$

and (5.3) in (5.4) we obtain

$$\frac{zp'(z)}{p(z)} := \gamma(1+\lambda) + (1+\lambda)\frac{I(r+2,\lambda)f(z)}{I(r+1,\lambda)f(z)} - (1+\gamma)(1+\lambda)\frac{I(r+1,\lambda)f(z)}{I(r,\lambda)f(z)}.$$

The assertion (5.2) of Theorem 5.1 follows from Theorem 2.1.

Since the superordination results are dual of the subordination, we state the results pertaining to the superordination, using duality.

Theorem 5.2. *Let* $0 \neq q(z)$ *be convex univalent in* Δ *with* $q(0) = 1$ *and satisfies (3.8). Suppose* $\frac{zq'(z)}{q(z)}$ *be starlike univalent in* Δ*. Let* $f \in \mathcal{A}$, $0 \neq \frac{z^\gamma I(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^{1+\gamma}} \in \mathcal{H}[1,1] \cap \mathcal{Q}$*. If* $\eta(\alpha,\beta,r,\lambda;z)$ *as defined by (5.1) is univalent in* Δ *and*

$$\alpha q(z) + \beta\frac{zq'(z)}{q(z)} \prec \eta(\alpha,\beta,r,\lambda;z),$$

then

$$q(z) \prec \frac{z^\gamma I(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^{1+\gamma}}$$

and q *is the best subordinant.*

Combining Theorem 5.1 and Theorem 5.2, we state the following sandwich theorem.

Theorem 5.3. *Let* $0 \neq q_1(z)$ *and* $0 \neq q_2(z)$ *be convex univalent in* Δ *satisfying (3.8) and (3.1) respectively. Suppose* $\frac{zq_i'(z)}{q_i(z)}$ *be starlike univalent in* Δ *for* $i = 1, 2$*. If* $f \in \mathcal{A}$, $0 \neq \left(\frac{z^\gamma I(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^{1+\gamma}}\right) \in \mathcal{H}[1,1] \cap \mathcal{Q}$ *and* $\eta(\alpha,\beta,r,\lambda;z)$ *as defined by (5.1) is univalent in* Δ*. Further if*

$$\alpha q_1(z) + \beta\frac{zq_1'(z)}{q_1(z)} \prec \eta(\alpha,\beta,r,\lambda;z) \prec \alpha q_2(z) + \beta\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{z^\gamma I(r+1,\lambda)f(z)}{(I(r,\lambda)f(z))^{1+\gamma}} \prec q_2(z)$$

and q_1 *and* q_2 *are respectively the best subordinant and best dominant.*

For the choices of $q_1(z) = \frac{1+A_1z}{1+B_1z}$, $q_2(z) = \frac{1+A_2z}{1+B_2z}$ $(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1)$ and $\alpha = \lambda = 0$, $\beta = 1$ in Theorem 5.3, we have the following:

Example 5.1. *Let* $f \in \mathcal{A}$, $\frac{z^\gamma \mathcal{D}^{m+1}(f(z))}{(\mathcal{D}^m(f(z))^{1+\gamma}} \in \mathcal{H}[1,1] \cap Q$ *and* $\left(\frac{\mathcal{D}^{m+2}f(z)}{\mathcal{D}^{m+1}(f(z))} - (1+\gamma)\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^m(f(z))} + \gamma\right)$ *is univalent in* Δ. *Further if*

$$\frac{(A_1 - B_1)z}{(1+A_1z)(1+B_1z)} \prec \left(\gamma + \frac{\mathcal{D}^{m+2}(f(z))}{\mathcal{D}^{m+1}(f(z))} - (1+\gamma)\frac{\mathcal{D}^{m+1}(f(z))}{\mathcal{D}^m(f(z))}\right) \prec \frac{(A_2 - B_2)z}{(1+A_2z)(1+B_2z)}$$

then

$$\frac{1+A_1z}{1+B_1z} \prec \frac{z^\gamma \mathcal{D}^{m+1}f(z)}{(\mathcal{D}^m(f(z))^{1+\gamma}} \prec \frac{1+A_2z}{1+B_2z},$$

where $\mathcal{D}^m$ *is the Salagean operator. The functions* $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are respectively the best subordinant and best dominant.*

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