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OSCILLATION THEOREMS FOR A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

Yuri V. Rogovchenko* and Fatoş Tuncay

Abstract. For a class of second order nonlinear differential equations with damping, efficient oscillation criteria are derived by refining the standard integral averaging technique. Examples are provided to illustrate the relevance of new theorems.

1. Introduction

We study the problem of oscillation of a class of nonlinear second order differential equations with damping

$$[r(t)\psi(x(t))x'(t)]' + p(t)x'(t) + q(t)f(x(t)) = 0, \tag{1}$$

where $t \geq t_0 \geq 0,\ r \in C^1([t_0,\infty);(0,\infty)),\ p,\ q \in C([t_0,\infty);\mathbb{R}),\ f \in C(\mathbb{R};\mathbb{R})$ and $xf(x) > 0$ for $x \neq 0$.

By a solution of equation (1) we understand a function $x : [t_0, t_1) \to \mathbb{R},\ t_1 > t_0$ such that substitution of $x(t)$ in (1) turns it into identity for all $t \in [t_0, t_1)$. In the sequel, we assume that solutions of equation (1) exist for all $t \geq t_0 \geq 0$. A solution $x(t)$ of equation (1) is called oscillatory if it does not have the largest zero, otherwise it is called non-oscillatory. We say that equation (1) is oscillatory if all its solutions are oscillatory.

Oscillation of differential equations with damping has been discussed by many authors, including quite a few papers dealing with equation (1), see, for instance, Elabbasy et al. [3], Grace [4, 6], Grace and Lalli [7-11], Kirane and Rogovchenko

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*Corresponding author.

[16], Lalli and Grace [18], Manojlović [21], Tiryaki and Zafer [31]. Several authors were concerned with equations with nonlinear damping terms as, for example, Baker [1], Bobisud [2], Grace and Lalli [7], Grace et al. [14], S. Rogovchenko and Yu. Rogovchenko [26, 27], Tiryaki and Zafer [32]. Many papers deal with a special case of equation (1) when $\psi(x) \equiv 1$,

$$[r(t)x'(t)]' + p(t)x'(t) + q(t)f(x(t)) = 0, \tag{2}$$

see [26], [29], and the references cited there.

A particular case of equation (1) without damping term, differential equation

$$[r(t)\psi(x(t))x'(t)]' + q(t)f(x(t)) = 0, \tag{3}$$

has been examined by Grace [4, 6], Lalli and Grace [18], Manojlović [22, 23], Tiryaki and Çakmak [30] and others. As mentioned by Mahfoud [20], "in the case $p(t) = 0$, the presence of the factor $\psi(x)$ in (1) does not create a new problem." He asserted that, for $p(t) = 0$, using a simple change of variable, one can reduce (1) to an equation without $\psi(x)$ and apply oscillation criteria known for a simpler equation. Consequently, the importance of the factor $\psi(x)$ is closely related to the presence of a damping term in equation (1), which makes reduction to simpler differential equation either very complicated or impossible. In fact, it has been shown recently by Mustafa et al. [24] that integral transformations can be successfully used to translate oscillation results established for the differential equation (1) to a new class of differential equations,

$$[r(t)y'(t)]' + p(t)\alpha(y(t))y'(t) + q(t)f(y(t)) = 0, \tag{4}$$

although certain difficulties may arise because coefficients in a transformed equation cannot be always computed explicitly. Hence, results specifically designed for different classes are important in their own right and have both weak and strong points. The reader can find in [24] more details regarding the change of variable that reduces (1) to (4) and how this affects oscillation criteria.

In 1986, Yan [34] established an important generalization of the celebrated Kamenev's oscillation criterion [15] for the linear differential equation with a damping term

$$[r(t)x'(t)]' + p(t)x'(t) + q(t)x(t) = 0,$$

which has been later extended by Grace [6] to equations (1) - (3) using advanced integral averaging technique due to Philos [25].

Let $D = \{(t,s)| -\infty < s \le t < +\infty\}$. We say that a function $H(t,s)$ belongs to the class $\mathcal{W}$ if

(i) $H \in C(D, [0, +\infty))$;

(ii) $H(t,t) = 0$ and $H(t,s) > 0$ for $-\infty < s < t < +\infty$;

(iii) H has continuous partial derivative $\partial H/\partial s$ satisfying

$$\frac{\partial H}{\partial s} = -h(t,s)\sqrt{H(t,s)},$$

where $h \in L_{\mathrm{loc}}(D,\mathbb{R})$. In what follows, we use the notation

$$f_+(x) \overset{\mathrm{def}}{=} \max(f(x),0).$$

Theorem 1. [6, Theorem 6, pp. 239-240]. *Suppose that $f'(x)$ exists,*

$$f'(x) \geq \mu, \tag{5}$$

for some constant $\mu > 0$ and for all $x \neq 0$, and there exist functions $H \in \mathcal{W}$, $g \in C^1([t_0,\infty);(0,\infty))$ such that

$$0 < \inf_{s \geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] \leq \infty \tag{6}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} r(s)g(s)\left(h(t,s)-\gamma(s)\sqrt{H(t,s)}\right)^2 ds < \infty, \tag{7}$$

where $\gamma(t) = (r(t)g'(t) - p(t)g(t))/(r(t)g(t))$. Suppose also that there exists a function $\kappa \in C([t_0,\infty);\mathbb{R})$ such that, for every $T \geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left[H(t,s)g(s)q(s) - \frac{r(s)g(s)}{4\mu} \times \left(h(t,s)-\gamma(s)\sqrt{H(t,s)}\right)^2\right] ds \geq \kappa(T)$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\kappa_+^2(s)}{g(s)r(s)}ds = \infty.$$

Then equation (2) is oscillatory.

Oscillation results for equation (1) under assumption of monotonicity of f were obtained by Tiryaki and Zafer [31], and, in the case when $f(x)$ is not monotonically increasing, by Kirane and Rogovchenko [16], who established the following criterion.

Theorem 2. [16, Theorem 2, pp. 125-130]. *Assume that*

$$0 < C \leq \psi(x) \leq C_1, \tag{8}$$

for all $x \in \mathbb{R}$,

$$\frac{f(x)}{x} \geq \mu, \tag{9}$$

for some constant $\mu > 0$ *and all* $x \neq 0$, *whereas* $q(t)$ *satisfies, for all* $t \geq t_0$,

$$q(t) \geq 0, \tag{10}$$

and q *is not identical zero on* $[t_0, \infty)$. *Suppose that there exist functions* $H \in \mathcal{W}$, $\rho \in C^1([t_0, \infty); (0, \infty))$ *and* $\kappa \in C([t_0, \infty); \mathbb{R})$ *such that (6) holds and, for* $t > t_0$ *and* $T \geq t_0$,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\rho(s)Q^2(t,s)ds < \infty, \\ &\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left(H(t,s)\phi(s) - \frac{C_1}{4}\rho(s)r(s)Q^2(t,s) \right) ds \geq \kappa(T), \end{aligned} \tag{11}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{\kappa_+^2(s)}{\rho(s)r(s)} ds = \infty,$$

where

$$\phi(t) = \rho(t) \left(\mu q(t) - \left(\frac{1}{C} - \frac{1}{C_1} \right) \frac{p^2(t)}{4r(t)} \right)$$

and

$$Q(t,s) = h(t,s) + \left(\frac{p(s)}{C_1 r(s)} - \frac{\rho'(s)}{\rho(s)} \right) \sqrt{H(t,s)}.$$

Then equation (1) *is oscillatory.*

The purpose of this paper is to strengthen Theorems 1 and 2, as well as related results concerning oscillation of equation (1) by refining the standard integral averaging technique developed by Grace [6], Philos [25], Rogovchenko [28], Yan [34], and in other papers on the subject.

The major advantages of a modified approach are the following. Firstly, in contrast to many recent results on oscillation of differential equations with damping, we do not need conditions (7), (11) and likewise anymore. Secondly, we simplify significantly the proof of Theorem 3, which is an analogue of Theorems 1 and 2, along with the proofs of similar results. As a by-product of the main result, we immediately obtain a counterpart of Theorem 3, Theorem 4, where $\limsup$ is replaced with $\liminf$ (this required at least a page-long proof till now). It is also important to note that, similarly to [16, 21, 29, 31], we do not impose additional conditions on the damping coefficient $p(t)$.

2. Oscillation for Increasing f

Theorem 3. *Assume that* (5) *and* (8) *hold. Suppose further that there exist functions* $H \in \mathcal{W}$, $g \in C^1([t_0,\infty);\mathbb{R})$, *and* $\kappa \in C([t_0,\infty);\mathbb{R})$ *such that* (6) *is satisfied and, for some* $\beta > 1$, *all* $t > t_0$ *and* $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - \frac{\beta C_1 v(s) r(s)}{4\mu} h^2(t,s) \right) ds \geq \kappa(T), \tag{12}$$

where

$$v(t) = \exp\left(-\frac{2\mu}{C_1} \int^t \left(\frac{g(s)}{r(s)} - \frac{p(s)}{2\mu r(s)} \right) ds \right) \tag{13}$$

and

$$\phi(t) = v(t) \left(q(t) + \frac{\mu g^2(t)}{C_1 r(t)} - \frac{p(t)g(t)}{C_1 r(t)} - g'(t) + \left(\frac{1}{C_1} - \frac{1}{C} \right) \frac{p^2(t)}{4\mu r(t)} \right). \tag{14}$$

If

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{\kappa_+^2(s)}{v(s)r(s)} ds = \infty, \tag{15}$$

equation (1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of the differential equation (1). Then there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ for all $t \geq T_0$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq T_0$. For $t \geq T_0$, define a generalized Riccati transformation by

$$u(t) = v(t) \left[\frac{r(t)\psi(x(t))x'(t)}{f(x(t))} + g(t) \right], \tag{16}$$

where $v(t)$ is given by (13). Differentiating (16) and using (1), we obtain

$$\begin{aligned} u'(t) &= \frac{v'(t)}{v(t)} u(t) + v(t) \left\{ \frac{[r(t)\psi(x(t))x'(t)]'}{f(x(t))} + g'(t) \right. \\ &\quad \left. - \frac{r(t)\psi(x(t))\,[x'(t)]^2 f'(x(t))}{[f(x(t))]^2} \right\} = \left[-2\mu \frac{g(t)}{C_1 r(t)} + \frac{p(t)}{C_1 r(t)} \right] u(t) \\ &\quad + v(t) \left\{ -\frac{p(t)x'(t)}{f(x(t))} - \frac{q(t)f(x(t))}{f(x(t))} + g'(t) - \frac{r(t)\psi(x(t))\,[x'(t)]^2 f'(x(t))}{[f(x(t))]^2} \right\}. \end{aligned}$$

Therefore, for all $t \geq T_0$,

$$
\begin{aligned}
u'(t) \leq & \left[-2\mu\frac{g(t)}{C_1 r(t)} + \frac{p(t)}{C_1 r(t)}\right] u(t) + v(t)\left\{-\frac{p(t)}{\psi(x(t))r(t)}\left[\frac{u(t)}{v(t)} - g(t)\right]\right. \\
& \left. -q(t) + g'(t) - \frac{\mu}{\psi(x(t))r(t)}\left[\frac{u(t)}{v(t)} - g(t)\right]^2\right\}.
\end{aligned} \tag{17}
$$

In view of (8) and (16), for all $t \geq T_0$, (17) yields

$$
u'(t) \leq -\phi(t) - \frac{\mu}{C_1 v(t) r(t)} u^2(t), \tag{18}
$$

where $\phi(t)$ is defined by (14). Multiplying (18) by $H(t,s)$, integrating from T to t, and using the properties of the function $H(t,s)$, we have, for all $t \geq T \geq T_0$,

$$
\begin{aligned}
& \int_T^t H(t,s)\phi(s)ds \leq -\int_T^t H(t,s)u'(s)ds - \int_T^t H(t,s)\frac{\mu}{C_1 v(s) r(s)} u^2(s)ds \\
= & -H(t,s)u(s)|_T^t - \int_T^t \left[-\frac{\partial H}{\partial s}(t,s)u(s) + H(t,s)\frac{\mu}{C_1 v(s) r(s)} u^2(s)\right] ds \\
= & \, H(t,T)u(T) - \int_T^t \left(h(t,s)\sqrt{H(t,s)}u(s) + H(t,s)\frac{\mu}{C_1 v(s) r(s)} u^2(s)\right) ds.
\end{aligned}
$$

Then, for any $\beta > 1$, the latter inequality can be written as

$$
\begin{aligned}
& \int_T^t H(t,s)\phi(s)ds \leq H(t,T)u(T) \\
& - \int_T^t \left(\sqrt{\frac{\mu H(t,s)}{\beta C_1 v(s) r(s)}} u(s) + \sqrt{\frac{\beta C_1 v(s) r(s)}{4\mu}} h(t,s)\right)^2 ds \\
& + \frac{\beta C_1}{4\mu}\int_T^t v(s) r(s) h^2(t,s) ds - \int_T^t \frac{\mu(\beta-1)H(t,s)}{\beta C_1 v(s) r(s)} u^2(s) ds,
\end{aligned} \tag{19}
$$

and, for all $t \geq T \geq T_0$,

$$
\begin{aligned}
& \int_T^t \left[H(t,s)\psi(s) - \frac{\beta C_1}{4\mu} v(s) r(s) h^2(t,s)\right] ds \leq H(t,T)u(T) \\
& - \int_T^t \left(\sqrt{\frac{\mu H(t,s)}{\beta C_1 v(s) r(s)}} u(s) + \sqrt{\frac{\beta C_1 v(s) r(s)}{4\mu}} h(t,s)\right)^2 ds \\
& - \int_T^t \frac{\mu(\beta-1)H(t,s)}{\beta C_1 v(s) r(s)} u^2(s) ds.
\end{aligned} \tag{20}
$$

It follows from (20) that, for $t > T \geq T_0$,

$$\frac{1}{H(t,T)}\int_T^t \left[H(t,s)\phi(s) - \frac{\beta C_1}{4\mu}v(s)r(s)h^2(t,s)\right] ds \leq u(T)$$
$$-\frac{1}{H(t,T)}\int_T^t \left(\sqrt{\frac{\mu H(t,s)}{\beta C_1 v(s)r(s)}}u(s) + \sqrt{\frac{\beta C_1 v(s)r(s)}{4\mu}}h(t,s)\right)^2 ds$$
$$-\frac{1}{H(t,T)}\int_T^t \frac{\mu\,(\beta-1)\,H(t,s)}{\beta C_1 v(s)r(s)}u^2(s)ds$$
$$\leq u(T) - \frac{1}{H(t,T)}\int_T^t \frac{\mu\,(\beta-1)\,H(t,s)}{\beta C_1 v(s)r(s)}u^2(s)ds$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left[H(t,s)\phi(s) - \frac{\beta C_1}{4\mu}v(s)r(s)h^2(t,s)\right] ds\\ &\leq u(T) - \liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \frac{\mu\,(\beta-1)\,H(t,s)}{\beta C_1 v(s)r(s)}u^2(s)ds.\end{aligned} \tag{21}$$

In virtue of (12), for all $T \geq T_0$,

$$u(T) \geq \kappa(T) + \liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \frac{\mu\,(\beta-1)\,H(t,s)}{\beta C_1 v(s)r(s)}u^2(s)ds.$$

Consequently,

$$u(T) \geq \kappa(T), \qquad \text{for all} \quad T \geq T_0, \tag{22}$$

and

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^t \frac{H(t,s)}{v(s)r(s)}u^2(s)ds \;&\leq M\\ &\stackrel{\text{def}}{=} \frac{\beta C_1}{\mu\,(\beta-1)}\left(u(T_0) - \kappa(T_0)\right) \;< \infty.\end{aligned} \tag{23}$$

To show that

$$\int_{T_0}^\infty \frac{u^2(s)}{v(s)r(s)}ds < \infty, \tag{24}$$

assume the contrary, that is,

$$\int_{T_0}^\infty \frac{u^2(s)}{v(s)r(s)}ds = \infty. \tag{25}$$

It follows from (6) that there exists a constant $\nu > 0$ such that

$$\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] > \nu. \tag{26}$$

On the other hand, by virtue of (25), for any positive number δ, there exists a $T_1 > T_0$ such that

$$\int_{T_0}^{t} \frac{u^2(s)}{v(s)r(s)} ds \geq \frac{\delta}{\nu}, \qquad \text{for all} \quad t \geq T_1.$$

Using integration by parts, we conclude that, for all $t \geq T_1$,

$$\begin{aligned} \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s) \frac{u^2(s)}{v(s)r(s)} ds &= \frac{1}{H(t,T_0)} \int_{T_0}^{t} \left[-\frac{\partial H(t,s)}{\partial s} \right] \left[\int_{T_0}^{s} \frac{u^2(\tau)}{v(\tau)r(\tau)} d\tau \right] ds \\ &\geq \frac{1}{H(t,T_0)} \int_{T_1}^{t} \left[-\frac{\partial H(t,s)}{\partial s} \right] \left[\int_{T_0}^{s} \frac{u^2(\tau)}{v(\tau)r(\tau)} d\tau \right] ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s) \frac{u^2(s)}{v(s)r(s)} ds &\geq \frac{\delta}{\nu} \frac{1}{H(t,T_0)} \int_{T_1}^{t} \left[-\frac{\partial H(t,s)}{\partial s} \right] ds \\ &= \frac{\delta}{\nu} \frac{H(t,T_1)}{H(t,T_0)}. \end{aligned} \tag{27}$$

It follows from (26) that

$$\liminf_{t \to \infty} \frac{H(t,s)}{H(t,t_0)} > \nu > 0,$$

and there exists a $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \geq \nu, \qquad \text{for all} \quad t \geq T_2.$$

Consequently, by (27),

$$\frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s) \frac{u^2(s)}{v(s)r(s)} ds \geq \delta, \qquad \text{for all} \quad t \geq T_2.$$

Since δ is an arbitrary constant, we have

$$\liminf_{t \to \infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s) \frac{u^2(s)}{v(s)r(s)} ds = +\infty,$$

which contradicts (23). Thus, (24) should hold, and, by virtue of (22),

$$\int_{T_0}^{\infty} \frac{\kappa_+^2(s)}{v(s)r(s)} ds \leq \int_{T_0}^{\infty} \frac{u^2(s)}{v(s)r(s)} ds < +\infty,$$

but this contradicts (15). Therefore, our initial assumption that $x(t)$ has constant sign on $[T_0, \infty)$ is wrong, and since $x(t)$ is an arbitrary solution, equation (1) is oscillatory. ■

Theorem 4. *Let* (5) *and* (8) *hold. Assume that there exist functions* $H \in \mathcal{W}$, $g \in C^1([t_0, \infty); \mathbb{R})$, *and* $\kappa \in C([t_0, \infty); \mathbb{R})$ *such that* (6) *is satisfied and, for some* $\beta > 1$ *and all* $T \geq t_0$,

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - \frac{\beta C_1 v(s) r(s)}{4\mu} h^2(t,s) \right) ds \geq \kappa(T),$$

where $v(s)$ *and* $\phi(s)$ *are as in Theorem 3. Suppose also that* (15) *holds. Then equation* (1) *is oscillatory.*

Proof. By virtue of the obvious inequality

$$\begin{aligned}\kappa(T) &\leq \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - \frac{\beta C_1 v(s) r(s)}{4\mu} h^2(t,s) \right) ds \\ &\leq \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - \frac{\beta C_1 v(s) r(s)}{4\mu} h^2(t,s) \right) ds,\end{aligned}$$

the conclusion follows immediately from Theorem 3. ■

In the remaining part of this section, we present oscillation criteria analogous to Theorems 3 and 4. They are proved in a similar manner, although a different trick is being used for deriving an analogue of the inequality (18). These results neither include, nor are included in Theorem 3 and Theorem 4 and are interesting in their own right.

Theorem 5. *Let* (5) *and* (8) *hold. Suppose also that there exist functions* $H \in \mathcal{W}$, $g \in C^1([t_0, \infty); \mathbb{R})$, *and* $\kappa \in C([t_0, \infty); \mathbb{R})$ *such that* (6) *is satisfied and, for some* $\beta > 1$, *all* $t > t_0$, *and* $T \geq t_0$,

$$\begin{aligned}&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - \frac{H(t,s)p^2(s)v(s)}{2\mu C_1 r(s)} \right. \\ &\left. - \frac{\beta C_1 v(s) r(s)}{2\mu} h^2(t,s) \right) ds \geq \kappa(T),\end{aligned} \tag{28}$$

where $\phi(s)$ *is defined by* (14) *and*

$$v(t) = \exp\left(-\frac{2\mu}{C_1} \int^t \frac{g(s)}{r(s)} ds \right). \tag{29}$$

If (15) *is satisfied, equation* (1) *is oscillatory.*

Proof. As in the proof of Theorem 3, assume, without loss of generality, that a non-oscillatory solution $x(t)$ of the differential equation (1) is positive for all $t \geq T_0$. Define a generalized Riccati transformation (16) with $v(t)$ given by (29). Differentiating (16) and using (1), we obtain, for all $t \geq T_0$,

$$u'(t) = -2\mu \frac{g(t)u(t)}{C_1 r(t)} + v(t) \left\{ -\frac{p(t)x'(t)}{f(x(t))} - \frac{q(t)f\,(x\,(t))}{f(x(t))} \right.$$
$$\left. - \frac{r(t)\psi(x(t))\,[x'(t)]^2\, f'(x(t))}{[f(x(t))]^2} + g'(t) \right\},$$

which yields

$$u'(t) \leq -\phi(t) - \frac{p(t)}{C_1 r(t)} u(t) - \frac{\mu}{C_1 v(t) r(t)} u^2(t), \tag{30}$$

where $\phi(t)$ is defined by (14). Using, as in [19], an elementary inequality

$$bz - az^2 \leq \frac{b^2}{2a} - \frac{a}{2} z^2, \tag{31}$$

which is valid for all $a > 0$ and all $b, z \in \mathbb{R}$, we deduce from (30) that

$$\phi(t) - \frac{p^2(t)v(t)}{2\mu C_1 r(t)} \leq -u'(t) - \frac{\mu}{2C_1 v(t) r(t)} u^2(t), \tag{32}$$

for all $t \geq T_0$. As in the proof Theorem 3, multiplying (32) by $H(t,s)$ and integrating from T to t, one has, for any $\beta > 1$ and all $t \geq T \geq T_0$,

$$\int_T^t H(t,s) \left(\phi(s) - \frac{p^2(s)v(s)}{2\mu C_1 r(s)} \right) ds \leq H(t,T)u(T)$$
$$-\frac{1}{2} \int_T^t \left(\sqrt{\frac{\mu H(t,s)}{\beta C_1 v(s) r(s)}} u(s) + \sqrt{\frac{\beta C_1 v(s) r(s)}{\mu}} h(t,s) \right)^2 ds$$
$$+ \frac{\beta C_1}{2\mu} \int_T^t v(s) r(s) h^2(t,s) ds - \int_T^t \frac{\mu\,(\beta - 1)\, H(t,s)}{2\beta C_1 v(s) r(s)} u^2(s) ds.$$

Consequently, we conclude that, for all $t \geq T \geq T_0$,

$$\int_T^t \left[H(t,s)\phi(s) - H(t,s) \frac{p^2(s)v(s)}{2\mu C_1 r(s)} - \frac{\beta C_1}{2\mu} v(s) r(s) h^2(t,s) \right] ds$$
$$\leq H(t,T)u(T) - \int_T^t \frac{\mu\,(\beta - 1)\, H(t,s)}{2\beta C_1 v(s) r(s)} u^2(s) ds$$
$$-\frac{1}{2} \int_T^t \left(\sqrt{\frac{\mu H(t,s)}{\beta C_1 v(s) r(s)}} u(s) + \sqrt{\frac{\beta C_1 v(s) r(s)}{\mu}} h(t,s) \right)^2 ds,$$

which is an analogue of (20). Correspondingly, we have the inequality

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)\phi(s)-\frac{H(t,s)p^2(s)v(s)}{2\mu C_1 r(s)}-\frac{\beta C_1}{2\mu}v(s)r(s)h^2(t,s)\right)ds$$

$$\leq u(T)-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\frac{\mu\,(\beta-1)\,H(t,s)}{2\beta C_1 v(s)r(s)}u^2(s)ds, \tag{33}$$

similar to (21). Using (33) and following the same lines as in the proof of Theorem 3, one arrives at the contradiction with the assumption (15) of the theorem. Therefore, equation (1) is oscillatory. ■

The next result follows immediately from Theorem 5 and properties of $\liminf$ and $\limsup$, cf. Theorem 4.

Theorem 6. *Suppose that all assumptions of Theorem 5 are satisfied except that condition* (28) *be replaced with*

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)\phi(s)-\frac{H(t,s)p^2(s)v(s)}{2\mu C_1 r(s)}\right.$$
$$\left.-\frac{\beta C_1 v(s)r(s)}{2\mu}h^2(t,s)\right)ds\geq\kappa(T).$$

Then equation (1) *is oscillatory.*

3. Oscillation for Non-monotonic f

In what follows, we obtain counterparts of the oscillation criteria derived in the previous section without requiring that f satisfies (5), which allows to study oscillation of differential equations with a nonlinearity that may not be strictly increasing. Although the class of equations to which new criteria apply is different from the one discussed above, all results have appearance similar to that of Theorems 3-6. They are established by modifying the proofs of corresponding theorems in Section 2. Therefore, we concentrate our attention on the differences in the proofs and skip similar details.

Theorem 7. *Suppose that* ψ, f *and* q *satisfy* (8), (9), *and* (10). *Assume further that there exist functions* $H\in\mathcal{W}$, $g\in C^1([t_0,\infty);\mathbb{R})$, *and* $\kappa\in C([t_0,\infty);\mathbb{R})$ *such that* (6) *holds and, for some* $\beta>1$, *all* $t>t_0$, *and any* $T\geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)\phi(s)-\frac{\beta C_1 v(s)r(s)}{4}h^2(t,s)\right)ds\geq\kappa(T), \tag{34}$$

where

$$v(t)=\exp\left(-\frac{2}{C_1}\int^t\left(\frac{g(s)}{r(s)}-\frac{p(s)}{2r(s)}\right)ds\right) \tag{35}$$

and

$$\phi(t)=v(t)\left(\mu q(t)+\frac{g^2(t)}{C_1r(t)}-\frac{p(t)g(t)}{C_1r(t)}-g'(t)+\left(\frac{1}{C_1}-\frac{1}{C}\right)\frac{p^2(t)}{4r(t)}\right). \tag{36}$$

Suppose also that (15) *is satisfied. Then equation* (1) *is oscillatory.*

Proof. As above, assume, without loss of generality, that $x(t)$ is a non-oscillatory solution of the differential equation (1) which is positive for all $t\geq T_0$. Define $u(t)$ by

$$u(t)=v(t)\left[\frac{r(t)\psi(x(t))x'(t)}{x(t)}+g(t)\right], \tag{37}$$

where $v(t)$ is given by (35). Differentiating (37) and using (1), we obtain

$$\begin{aligned}
u'(t) &= \left[-2\frac{g(t)}{C_1r(t)}+\frac{p(t)}{C_1r(t)}\right]u(t)+v(t)\left\{-\frac{p(t)x'(t)}{x(t)}\right.\\
&\quad\left.-\frac{q(t)f\left(x\left(t\right)\right)}{x(t)}-\frac{r(t)\psi(x(t))\left[x'(t)\right]^2}{x^2(t)}+g'(t)\right\}\\
&\leq \left[-2\frac{g(t)}{C_1r(t)}+\frac{p(t)}{C_1r(t)}\right]u(t)+v(t)\left\{\frac{-p(t)}{\psi(x(t))r(t)}\left[\frac{u(t)}{v(t)}-g(t)\right]\right.\\
&\quad\left.-\mu q(t)-\frac{1}{\psi(x(t))r(t)}\left[\frac{u(t)}{v(t)}-g(t)\right]^2+g'(t)\right\},
\end{aligned}$$

which yields, for all $t\geq T_0$,

$$u'(t)\leq-\phi(t)-\frac{1}{C_1v(t)r(t)}u^2(t), \tag{38}$$

where $\phi(t)$ is defined by (36). Multiplying (38) by $H(t,s)$, integrating from T to t, and using the properties of the function $H(t,s)$, we conclude that, for any $\beta>1$ and for all $t\geq T\geq T_0$,

$$\begin{aligned}
&\int_T^t\left[H(t,s)\phi(s)-\frac{\beta C_1}{4}v(s)r(s)h^2(t,s)\right]ds\leq H(t,T)u(T)\\
&-\int_T^t\left(\sqrt{\frac{H(t,s)}{\beta C_1v(s)r(s)}}u(s)+\frac{1}{2}\sqrt{\beta C_1v(s)r(s)}h(t,s)\right)^2ds\\
&-\int_T^t\frac{\left(\beta-1\right)H(t,s)}{\beta C_1v(s)r(s)}u^2(s)ds.
\end{aligned} \tag{39}$$

Then, (39) yields the inequality similar to (21), that is, for all $T \geq T_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\phi(s) - \frac{\beta C_1}{4} v(s) r(s) h^2(t,s) \right] ds \leq u(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)\, H(t,s)}{\beta C_1 v(s) r(s)} u^2(s) ds.$$

Proceeding as in the proof of Theorem 3, we complete the proof. ■

Theorem 8. *Let all assumptions of Theorem 7 be satisfied except that* $\limsup$ *in condition* (34) *be replaced with* $\liminf$. *Then conclusion of Theorem 7 remains intact.*

Theorem 9. *Let* (8), (9), *and* (10) *be satisfied, and suppose that there exist functions* $H \in \mathcal{W}$, $g \in C^1([t_0,\infty);\mathbb{R})$, *and* $\kappa \in C([t_0,\infty);\mathbb{R})$ *such that* (6) *holds and, for some* $\beta > 1$, *all* $t > t_0$, *and any* $T \geq t_0$,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - \frac{H(t,s)p^2(s)v(s)}{2C_1 r(s)} \right. \\ &\left. -\frac{\beta C_1 v(s) r(s)}{2} h^2(t,s) \right) ds \geq \kappa(T), \end{aligned} \tag{40}$$

where ψ *is defined by* (36) *and*

$$v(t) = \exp\left(-\frac{2}{C_1} \int^t \frac{g(s)}{r(s)} ds \right). \tag{41}$$

Assume also that (15) *holds. Then equation* (1) *is oscillatory.*

Proof. As usual, suppose, without loss of generality, that a non-oscillatory solution $x(t)$ of the differential equation (1) satisfies $x(t) > 0$ for all $t \geq T_0$. Define a generalized Riccati transformation (37), where this time $v(t)$ is given by (41). Differentiating (37) and using (1), we obtain, for all $t \geq T_0$,

$$\begin{aligned} u'(t) &= -2\frac{g(t)u(t)}{C_1 r(t)} + v(t) \\ &\quad \left\{ -\frac{p(t)x'(t)}{x(t)} - \frac{q(t)f(x(t))}{x(t)} - \frac{r(t)\psi(x(t))\,[x'(t)]^2}{x^2(t)} + g'(t) \right\} \\ &\leq -2\frac{g(t)u(t)}{C_1 r(t)} + v(t) \left\{ -\frac{p(t)}{\psi(x(t))r(t)} \left[\frac{u(t)}{v(t)} - g(t) \right] - \mu q(t) + g'(t) \right. \\ &\quad \left. -\frac{1}{\psi(x(t))r(t)} \left[\frac{u(t)}{v(t)} - g(t) \right]^2 \right\} \\ &\leq -\phi(t) - \frac{p(t)}{C_1 r(t)} u(t) - \frac{1}{C_1 v(t) r(t)} u^2(t), \end{aligned} \tag{42}$$

where $\psi(t)$ is defined by (36). Using the inequality (31), we conclude from (42) that, for all $t \geq T_0$,

$$\phi(t) - \frac{p^2(t)v(t)}{2C_1 r(t)} \leq -u'(t) - \frac{1}{2C_1 v(t) r(t)} u^2(t). \tag{43}$$

As in Theorem 3, multiplying (43) by $H(t,s)$ and integrating from T to t, we have, for any $\beta > 1$ and all $t \geq T \geq T_0$,

$$\begin{aligned}
&\int_T^t \left[H(t,s)\phi(s) - H(t,s)\frac{p^2(s)v(s)}{2C_1 r(s)} - \frac{\beta C_1}{2} v(s) r(s) h^2(t,s) \right] ds \\
&\leq H(t,T)u(T) - \int_T^t \frac{(\beta-1)\, H(t,s)}{2\beta C_1 v(s) r(s)} u^2(s) ds \\
&\quad -\frac{1}{2}\int_T^t \left(\sqrt{\frac{H(t,s)}{\beta C_1 v(s) r(s)}} u(s) + \sqrt{\beta C_1 v(s) r(s)}\, h(t,s) \right)^2 ds,
\end{aligned}$$

and the analogue of the fundamental inequality (21) assumes now the form

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\phi(s) - H(t,s)\frac{p^2(s)v(s)}{2C_1 r(s)} \right. \\
&\left. -\frac{\beta C_1}{2} v(s) r(s) h^2(t,s) \right] ds \leq u(T) \\
&- \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)\, H(t,s)}{2\beta C_1 v(s) r(s)} u^2(s) ds.
\end{aligned}$$

Following the same lines as in the proof of Theorem 3, we conclude that equation (1) is oscillatory. ∎

Theorem 10. *Suppose that all assumptions of Theorem 9 are satisfied, except that* $\limsup$ *in condition* (40) *be replaced with* $\liminf$. *Then equation* (1) *is oscillatory.*

4. Applications

With an appropriate choice of the functions H and h, one can derive from general Theorems 3-10 a number of efficient oscillation criteria for equation (1). One of the most popular choices for the function $H(t,s)$, originally due to Kamenev [15], is

$$H(t,s) = (t-s)^{n-1}, \qquad (t,s) \in D,$$

where $n > 2$ is an integer. Clearly, $H \in \mathcal{W}$, the function

$$h(t,s) = (n-1)(t-s)^{(n-3)/2}, \qquad (t,s) \in D,$$

is continuous on $[t_0, \infty)$ and satisfies condition (iii). Then, by Theorem 3, we obtain the following oscillation criterion.

Corollary 11. *Let assumptions* (5) *and* (8) *hold. Suppose that there exist functions* $g \in C^1([t_0, \infty); \mathbb{R})$ *and* $\kappa \in C([t_0, \infty); \mathbb{R})$ *such that for all* $T \geq t_0$, *some* $\beta > 1$, *and some integer* $n > 2$,

$$\limsup_{t\to\infty} t^{1-n} \int_T^t \left[(t-s)^{n-1} \phi(s) - \frac{\beta C_1 (n-1)^2}{4\mu} (t-s)^{n-3} v(s) r(s) \right] ds \geq \kappa(T), \tag{44}$$

where $v(t)$ *and* $\phi(t)$ *are as in Theorem 3. Assume also that* (15) *is satisfied. Then equation* (1) *is oscillatory.*

Proof. It is only necessary to verify the condition (6), but with our choice of the functions H and h it is fulfilled automatically since

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty} \frac{(t-s)^{n-1}}{(t-t_0)^{n-1}} = 1,$$

for any $s \geq t_0$. ■

Example 12. For $t \geq 1$, consider the nonlinear differential equation

$$\left(t^2 \psi(x(t)) x'(t)\right)' + (t \cos t)\, x'(t) + q(t) f(x(t)) = 0, \tag{45}$$

where $f(x)$ is any function satisfying (5) with $\mu = 1$, $\psi(x)$ is any function that satisfies (8) with $C = 1/4$ and $C_1 = 1$, whereas

$$q(t) = \left(t^2 + 1\right) \cos^2 t - \frac{1}{2}\left(t^2 - \cos t + t \sin t\right) + 2.$$

We apply Corollary 11, letting $\beta = 2$ and $g(t) = (t \cos t)/2$. Then $v(t) = 1$ and $\phi(t) = t^2 \cos^2 t - t^2/2 + 2$. A straightforward computation of the limit in (44) with $n = 3$ gives

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{t^2} \int_T^t \left[(t-s)^2 \left(s^2 \cos^2 s - \frac{1}{2} s^2 + 2 \right) - \frac{2 \cdot 1 \cdot 2^2}{4 \cdot 1} \cdot 1 \cdot s^2 \right] ds \\ &= \frac{1}{4}\left(1 + \sin T \cos T - 2T \cos^2 T - 7T - 2T^2 \sin T \cos T\right) = \kappa(T), \end{aligned}$$

and it follows from the relation

$$\frac{\kappa_+^2(t)}{v(t) r(t)} = O(t^2) \qquad \text{as } t \to \infty \tag{46}$$

that condition (15) is satisfied. Therefore, equation (45) is oscillatory by Corollary 11.

In a similar manner, one can derive efficient tests for oscillation of equation (1) from any theorem in Sections 2 and 3. For instance, with the same choice of H as in Corollary 11, as an immediate consequence of Theorem 7, we have the following result.

Corollary 13. *Let* (9) *and* (10) *hold, and assume that there exist functions* $g \in C^1([t_0,\infty);\mathbb{R})$ *and* $\kappa \in C([t_0,\infty);\mathbb{R})$ *such that for some* $\beta > 1$, *all* $t > t_0$, *and any* $T \geq t_0$,

(47)
$$\limsup_{t\to\infty} t^{1-n}\int_T^t \left[(t-s)^{n-1}\phi(s) - \frac{\beta C_1\,(n-1)^2}{4}(t-s)^{n-3}v(s)r(s)\right]ds \geq \kappa(T),$$

where $v(t)$ *and* $\phi(t)$ *are as in Theorem 7. If* (15) *holds, equation* (1) *is oscillatory.*

Example 14. For $t \geq 1$, consider the nonlinear differential equation

$$\left(\psi(x)x'(t)\right)' + (\sin 2t)\,x'(t) + \left(\frac{2}{t^2} + \sin^2 2t\right) f(x(t)) = 0, \tag{48}$$

where $f(x)$ and $\psi(x)$ are any functions satisfying conditions of Corollary 13 with $\mu = 1$, $C = 1/4$, and $C_1 = 1$. Choosing $\beta = 2$ and $g(t) = (\sin 2t)/2 - t^{-1}$, we conclude that $v(t) = t^2$. Correspondingly, $\phi(t) = t^2 + 2 - 2t^2\cos^2 t$, and, for $n = 3$, the limit in (47) reads as

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left[(t-s)^2\left(2 + s^2 - 2s^2\cos^2 s\right) - \frac{2\cdot 1\cdot 2^2}{4\cdot 1}\cdot 1\cdot s^2\right] ds \\
&= \frac{1}{2}\left(1 - \sin T\cos T - 5T + 2T\cos^2 T + 2T^2\sin T\cos T\right) = \kappa(T).
\end{aligned}$$

Taking into account that (46) holds, it is not hard to see that condition (15) is satisfied. Thus, by Corollary 13, equation (48) is oscillatory.

5. Conclusions

One of the principal advantages of the refined integral averaging technique suggested in this paper is simplification of both assumptions in oscillation results and their proofs. To notice this, it suffices to compare conditions and proofs of Theorems 3-10 with tests for oscillation of differential equations with damping established by Grace [6, Theorems 3, 4, 6, 7], Kirane and Rogovchenko [16, Theorems 2, 3], Li et al. [19, Theorems 2.2, 2.3], Rogovchenko [28, Theorems 2, 3, 5, 6, 8, 9], [29, Theorems 2, 3], or Tiryaki and Zafer [31, Theorems 2.2, 2.3]. Our criteria have one assumption less, and it is not hard to construct examples where, for the same choice of $v(t)$, conditions like (7) or (11) do not hold, but oscillatory nature of given differential equations can be established using theorems proved in this paper. In addition, the proofs of all our results are simpler and shorter.

We note that the inequality (19) and alike hold also for $\beta = 1$. However, in this case (21) reduces to

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\phi(s) - \frac{\beta C_1}{4\mu} v(s)r(s)h^2(t,s) \right] ds \le u(T),$$

and it is necessary to impose conditions analogous to (7) or (11) in Theorem 3, which reduces it to known results. Similar reasoning applies to other theorems proved in this paper.

A significant drawback of many oscillation results for differential equations with damping reported in the literature is a necessity to impose a variety of additional restrictions on the sign of the damping term $p(t)$. We emphasize that our theorems are free of particular restrictions on $p(t)$. On the other hand, oscillation criteria derived by Elabbasy et al. [3], Grace [4]-[6], Grace and Lalli [9]-[13], Grace et al. [14] require, for $t \ge t_0$, specific technical conditions on coefficients like $\rho'(t) \ge 0,$ $(r(t)\rho(t))' \ge 0,$ $(r(t)\rho(t))'' \le 0,$ and $(r(t)\rho(t) - \rho(t)p(t))' \le 0$ [3, Theorems 2.1 and 2.4], $p(t) \le 0$ and $(p(t)\rho(t))' \ge 0$ [4, Theorem 3], [6, Theorem 1], $\rho'(t) \ge 0$ and$(r(t)\rho'(t))' \le 0$ [6, Theorem 2], [14, Theorem 1], $p(t)\rho'(t) \le 0$ [9, Theorem 2], $p'(t) \ge 0$ [9, Theorem 3], $p'(t) \ge 2q(t)$ [9, Theorem 4], $r(t)\rho'(t) - p(t)\rho(t)/C_1 = \gamma(t) \ge 0$ [11, Theorem 1], $\gamma(t) \ge 0$ and $\gamma'(t) \le 0$ [10, Theorem 1], [11, Theorem 5], cf. also [12, Theorems 2.1-2.4], $\rho'(t) \le 0$ [14, Theorems 1 and 2]. If a damping coefficient $p(t)$ oscillates, which is the case in both our examples (45) and (48), conditions mentioned above fail to hold, which does not allow application of the corresponding oscillation criteria, see also comments in [16, 17, 29].

We also note that instead of condition (8), one can require that the function $\psi(x)$ be bounded below by a positive constant, that is,

$$0 < C \le \psi(x) < +\infty, \qquad x \in \mathbb{R}.$$

However, in this case it is possible to discuss oscillatory behavior only for bounded solutions of equation (1) as in [4, Remark 2], or additional sign conditions on the damping coefficient should be imposed as, for example, in [4, Theorem 3] and [11, Theorems 2, 4, and 6]. Furthermore, for a strongly sublinear equation (1), Manojlović [21] removed the assumption of positivity of $\psi(x)$ requesting that, for all $x \ne 0$,

$$\frac{xf(x)}{\psi(x)} > 0 \qquad \text{and} \qquad \frac{d}{dx}\left[\frac{f(x)}{\psi(x)}\right] \ge 0.$$

In addition, other assumptions on f and ψ are imposed like

$$\frac{f(x)\psi'(x)}{\psi^2(x)} \geq \frac{1}{k} > 0 \qquad \text{and} \qquad |\psi(x)| \geq c > 0,$$

or

$$f(x)\psi'(x) \geq k > 0,$$

for all $x \neq 0$. Finally, we mention that another alternative is to consider (8) along with the condition

$$\frac{f'(x)}{\psi(x)} \geq K > 0 \qquad \text{for } x \neq 0, \tag{49}$$

as in the paper by Grace and Lalli [11], but this requires unpleasant additional restriction on the sign of the damping coefficient $p(t)$. On the other hand, Grace [6] used (49) without additional restrictions on $\psi(x)$, but under too stringent hypotheses on $p(t)$. This condition has been also exploited for studying undamped equation, cf. [4, 10]. We do not consider such conditions on ψ here because they restrict classes of differential equations to which new oscillation criteria may apply. Further details regarding the effect of imposing different conditions on the coefficients and nonlinearities can be found, for instance, in the recent papers [24] and [26].

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Yuri V. Rogovchenko
School of Pure and Applied Natural Sciences,
University of Kalmar,
SE-391 82, Kalmar, Sweden
and
Department of Mathematics,
Eastern Mediterranean University,
Famagusta, TRNC, Mersin 10, Turkey
E-mail: yuri.rogovchenko@emu.edu.tr
yuri.rogovchenko@hik.se

Fatoş Tuncay
Department of Mathematics,
Eastern Mediterranean University,
Famagusta, TRNC, Mersin 10,
Turkey