TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 6A, pp. 1847-1865, December 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

BREGMAN DISTANCES AND KLEE SETS IN BANACH SPACES

Donghui Fang, Wen Song[1,*] and Chong Li[2]

Dedicated to Boris Mordukhovich on his 60th Birthday

Abstract. In this paper, we first present some sufficient conditions for the upper semicontinuity and/or the continuity of the Bregman farthest-point map Q_C^g and the relative farthest-point map S_C^g for a nonempty D-maximally approximately compact subset C of a Banach space X. We next present certain sufficient conditions as well as equivalent conditions for a Klee set to be singleton in a Banach space X. Our results extend and/or improve the corresponding ones of [Bauschke, et al., J. Approx. Theory, 158 (2009), pp. 170-183] to infinite dimensional spaces.

1. INTRODUCTION

Let X be a real normed space with the dual space X^*. Let $C \subset X$ be a nonempty subset of X. As usual, the norm farthest-point map on C is denote by $Q_C : X \rightrightarrows C$ and defined by

$$Q_C(x) := \{z \in C : \|x - z\| = \sup_{y \in C} \|x - y\|\} \quad \text{for each } x \in X.$$

We recall (cf. [16]) that C is a Klee set if $Q_C(x)$ is singleton for each $x \in X$.

One of the oldest questions (dating back to the 1960s) in real analysis and approximation is the so-called singleton problem of Klee sets on farthest-points, which is formulated as follows:

"Is a Klee set in a Hilbert space necessarily a singleton?"

Received August 10, 2009.
2000 *Mathematics Subject Classification*: 41A65, 90C48, 47H04.
Key words and phrases: Bregman farthest-point map, Klee set, D-maximally approximate compactness, Totally convex function.
[1]Research of this author was partially supported by the National Natural Sciences Grant (No. 10671050) and the Distinguish Young Scholar foundation of Heilongjiang Province of China (JC200707).
[2]Research of this author was supported in part by the National Natural Science Foundation of China (grant 10671175; 10731060).
*Corresponding author.

This problem is closely related to the famous convexity problem of Chebyshev sets, and has attracted a lot of attention of mathematicians; see, e.g., [5, 13-16, 22] and the references therein. Although the answer to this problem is affirmative in $\mathbb{R}^n$, as was shown originally by Klee [16], only partial results are known in infinite-dimensional Banach space settings; see, e.g., [18, 21, 22].

Recent interests are focused on some similar problems but with the Bregman distance instead of the norm distance on X. Let $g\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a proper convex function with its domain $\mathrm{dom}g$. The right hand side derivative of g at $x \in \mathrm{dom}g$ in the direction h is given by

$$g'_+(x,h)\colon = \lim_{t\to 0^+} \frac{g(x+th)-g(x)}{t}. \tag{1.1}$$

The Bregman distance with respect to g between the points $x, y \in \mathrm{dom}g$ is defined as

$$D_g(y,x)\colon = g(y) - g(x) - g'_+(x, y-x). \tag{1.2}$$

In 1976, Bregman discovered an elegant and effective technique for the use of the function D_g in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithms for solving not only feasibility and optimization problems, but also algorithms for solving variational inequalities and computing fixed points of nonlinear mappings and more (see [3, 8-12, 19] and the references therein).

Let $C \subset \mathrm{dom}g$ be a nonempty subset. The Bregman farthest-point map, denoted by Q^g_C, is defined as the set of the solutions of the optimization problem $\max\limits_{y\in C} D_g(y,x)$, i.e.,

$$Q^g_C(x) := \arg\max_{y\in C} D_g(y,x) \quad \text{for each } x \in \mathrm{dom}g.$$

In 2009, Bauschke et al. started in [5] to consider the singleton problem of a Klee set in the sense of Bregman distance in Euclidean space $\mathbb{R}^n$. Under the assumption that g is a convex function of Legendre type and 1-coercive, they proved that each Klee set (in the sense of Bregman distance) is a singleton. The corresponding convexity problem of D-Chebyshev sets in $\mathbb{R}^n$ was explored in [4]. The techniques used there are closely dependent upon the properties possessed by the Euclidean space.

Very recently, the results on characterization of convexity of D-Chebyshev sets in $\mathbb{R}^n$ are extended in [17] to the infinite-dimensional Banach space setting. The purpose of the present paper is to consider the singleton problem of a Klee set (in the sense of Bregman distance) in general Banach spaces. Our approach is based on the study of the Bregman farthest-point map Q_C^g as well as the relative farthest-point map $S_C^g : X^* \to C$, which is defined by

$$S_C^g(x^*) := \arg\max_{y\in C} W^g(y, x^*),$$

where W^g is the function defined by

$$W^g(x, x^*) := g(x) - \langle x^*, x\rangle + g^*(x^*) \quad \text{for each pair } (x, x^*) \in X \times X^*.$$

In this paper, we first present some sufficient conditions ensuring the upper semicontinuity and the continuity of the Bregman farthest-point map Q_C^g and the relative farthest-point map S_C^g for a nonempty compact subset C of a Banach space X. We next present certain sufficient conditions as well as equivalent conditions for the pointwise of a Klee subset of a Banach space X. Our results extend and/or improve the corresponding results of [5] to infinite dimensional spaces.

2. Preliminaries

Let X be a Banach space and $g\colon X \to \overline{\mathbb{R}}$ be a proper convex function. As usual, the closed unit ball and unit sphere of X are denoted by $\mathbf{B}$ and $\mathbf{S}$, respectively. We also denote by $\mathbf{B}(x, r)$ the closed ball centered at x with radius r. Moreover, we use $\mathrm{dom} g$ to denote the domain of g. Let $x \in \mathrm{dom} g$. The *subdifferential* of g at x is the convex set defined by

$$\partial g(x) := \{x^* \in X^* : \ g(x) + \langle x^*, y - x\rangle \le g(y) \quad \text{for each } y \in X\};$$

while the *conjugate function* of g is the function $g^* : X^* \to \overline{\mathbb{R}}$ defined by

$$g^*(x^*) := \sup\{\langle x^*, x\rangle - g(x) : \ x \in X\}.$$

Then, by [24, Theorem 2.4.2(iii)], the Young-Fenchel inequality holds

$$\langle x^*, x\rangle \le g(x) + g^*(x^*) \quad \text{for each pair} (x^*, x) \in X^* \times X, \tag{2.1}$$

and the equality holds if and only if $x^* \in \partial g(x)$, i.e.,

$$\langle x^*, x\rangle = g(x) + g^*(x^*) \Longleftrightarrow x^* \in \partial g(x) \quad \text{for each pair} (x^*, x) \in X^* \times X. \tag{2.2}$$

The domain and the image of ∂g are denoted by $\mathrm{dom}(\partial g)$ and $\mathrm{Im}(\partial g)$, respectively, which are defined by

$$\mathrm{dom}(\partial g) = \{x \in \mathrm{dom}\ g : \partial g(x) \ne \emptyset\}$$

and

$$\mathrm{Im}(\partial g) = \{x^* \in X^* : x^* \in \partial g(x),\, x \in \mathrm{dom}\, \partial g\}.$$

Recall that the Bregman distance with respect to g is defined by

$$D_g(y,x) = g(y) - g(x) - g'_+(x, y-x) \quad \text{for any } x, y \in \mathrm{dom} g. \tag{2.3}$$

According to [10], we define the modulus of total convexity at x by

$$\nu_g(x,t) := \inf\{D_g(y,x) : \; y \in \mathrm{dom} g, \|y-x\| = t\} \quad \text{for each } t \geq 0. \tag{2.4}$$

Defintion 2.1. Let $x \in \mathrm{dom} g$. The function $g \colon X \to \mathbb{R}$ is said to be

(a) totally convex at x if its modulus is positive on $(0,\infty)$, i.e. $\nu_g(x,t) > 0$ for each $t > 0$;

(b) essentially strictly convex if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of $\mathrm{dom}(\partial g)$.

Remark 2.1. (a) The notion of total convexity at a point was first introduced in [8] but using the terminology "very convex"; while the notion of the essentially strict convexity was introduced in [2].

(b) It was proved in [10] (see also [12, Proposition 2.2]) that if g is totally convex at any point of $\mathrm{dom} g$, then it is strictly convex on $\mathrm{dom} g$, and in [12, Proposition 2.13] that if X is reflexive and g is totally convex at any point of $\mathrm{dom}(\partial g)$, then it is essentially strictly convex.

(c) By [19, Proposition 2.2], the function g is totally convex at $x \in \mathrm{dom} g$ if and only if for any sequence $\{y_n\} \subset \mathrm{dom} g$, the following implication holds:

$$\lim_{n\to\infty} D_g(y_n, x) = 0 \Longrightarrow \lim_{n\to\infty} \|y_n - x\| = 0. \tag{2.5}$$

(d) Recall from [23] that g is uniformly convex at $x \in \mathrm{dom} g$, if the function

$$\begin{aligned} &\mu_g(x,t) : \\ &= \inf\left\{ \frac{\lambda g(x) + (1-\lambda) g(y) - g(\lambda x + (1-\lambda) y)}{\lambda(1-\lambda)} : \begin{array}{c} \lambda \in (0,1), y \in \mathrm{dom} g, \\ \|y-x\| = t \end{array} \right\} \end{aligned} \tag{2.6}$$

is positive whenever $t > 0$. Then $\nu_g(x,t) \geq \mu_g(x,t)$ for all $t \geq 0$. Hence, if g is uniformly convex at $x \in \mathrm{dom} g$, then it is totally convex at the same point.

By [12, Theorem 2.14], we have the following proposition, which shows that all convexities are equivalent for a real-valued convex function g on the Euclidean space $\mathbb{R}^n$.

Proposition 2.1. *Let* $g\colon \mathbb{R}^n \to \mathbb{R}$ *be a convex function. Then the following conditions are equivalent.*

(i) *The function* g *is strictly convex.*

(ii) *The function* g *is essentially strictly convex.*

(iii) *The function* g *is totally convex at any* $x \in \mathbb{R}^n$.

(iv) *The function* g *is uniformly convex at any* $x \in \mathbb{R}^n$.

In an infinite dimensional setting, for reasons related to effective computability, the functions g_p defined by $g_p(\cdot) := \frac{1}{p}\|\cdot\|^p$ with $p > 1$ are among the most likely functions to be used in the build up of Bregman type algorithms. It was shown by Zalinescu (see [24]) that if X is locally uniformly convex then g_p with $p > 1$ is uniformly convex at any $x \in X$, and so totally convex at any $x \in X$. Following [19], a Banach space X is called locally totally convex, if the function g_2 is totally convex at each $x \in \mathbf{S}$. It is clear that locally uniformly convex spaces are locally totally convex. The following proposition on characterizing the locally total convexity of X was given in [12].

Proposition 2.2. *Suppose that* X *is a Banach space. Then the following conditions are equivalent.*

(i) *The sapce* X *is locally totally convex.*

(ii) *For any* $x \in \mathbf{S}$ *and for any real number* $\varepsilon > 0$, *there exists* $\delta = \delta(x, \varepsilon) > 0$ *such that, for any* $y \in \mathbf{S}$ *with* $\|y - x\| \geq \varepsilon$, *there exists* $\lambda_0 \in (0,1)$ *such that*

$$\|(1-\lambda_0)x + \lambda_0 y\| < 1 - \lambda_0\delta.$$

We end this section with two propositions on some properties of convex functions, which will be frequently used in next sections; see [6, Proposition 2.11] and [1, Corollary 3.1, Corollary 3.2] for the first one, and [11, Proposition 3.4] for the second one.

Proposition 2.3. *Suppose that* $g\colon X \to \overline{\mathbb{R}}$ *is a lower semicontinuous proper convex function which is Gâteax differentiable (resp. Fréchet differentiable) on* $\mathrm{int}(\mathrm{dom}g)$. *Then* g *is continuous and its Gâteax derivative* ∇g *is norm-weak** *continuous (resp. continuous) on* $\mathrm{int}(\mathrm{dom}g)$.

Proposition 2.4. *Suppose that* $g\colon X \to \overline{\mathbb{R}}$ *is a lower semicontinuous proper convex function. Let* $x \in \mathrm{dom}g$ *and suppose that* g *is totally convex at* x. *Then* $\partial g(x) \subseteq \mathrm{int}(\mathrm{dom}g^*)$ *and* g^* *is Fréchet differentiable at each point* $x^* \in \partial g(x)$. *Furthermore, there exists a nondecreasing function* $\theta\colon [0,+\infty) \to [0,+\infty)$ *with* $\lim_{t\to 0+}\theta(t) = 0$ *such that, for any pair* $(y, y^*) \in X \times X^*$ *with* $y^* \in \partial g(y)$, *one has*

$$\|y - x\| \leq \theta(\|y^* - x^*\|).$$

3. Approximate Compactness and Continuity of the Bregman Farthest Point Mapping

For the remainder, we always assume that X, Y are Banach spaces and that $g\colon X \to \overline{\mathbb{R}}$ is lower semicontinuous proper convex and Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}g)$ with its Gâteaux derivative denoted by $\bigtriangledown g$. Suppose that $C \subseteq \mathrm{int}(\mathrm{dom}g)$ is a nonempty bounded set. Then the Bregman distance D_g with respect to g can be reexpressed as

$$D_g(y,x) = g(y) - g(x) - \langle \bigtriangledown g(x), y - x\rangle \quad \text{for each pair } (y,x) \in X \times \mathrm{int}(\mathrm{dom}g). \tag{3.1}$$

Clearly, $D_g(\cdot, x)$ is convex for each $x \in \mathrm{int}(\mathrm{dom}g)$ and the following equality holds for any $\hat{y},\ x \in \mathrm{int}(\mathrm{dom}g)$ and $y \in X$:

$$D_g(y,\hat{y}) = D_g(y,x) - D_g(\hat{y},x) + \langle \nabla g(\hat{y}) - \nabla g(x), \hat{y} - y\rangle. \tag{3.2}$$

We define the Bregman farthest function of C by

$$F_C^g(x) := \sup_{y \in C} D_g(y,x) \quad \text{for each } x \in \mathrm{int}(\mathrm{dom}g). \tag{3.3}$$

Then the Bregman farthest point map onto C is

$$Q_C^g(x) = \{y \in C:\ D_g(y,x) = F_C^g(x)\} \quad \text{for each } x \in \mathrm{int}(\mathrm{dom}g). \tag{3.4}$$

One key tool for our study is the function $W^g : X \times X^* \to \overline{\mathbb{R}}$ associated with g, which was first introduced by Butnariu and Resmerita [12] and is defined by

$$W^g(x,x^*) = g(x) - \langle x^*, x\rangle + g^*(x^*) \quad \text{for each pair } (x,x^*) \in X \times X^*.$$

Clearly, the function W^g is nonnegative, convex and continuous on $\mathrm{int}(\mathrm{dom}g) \times \mathrm{int}(\mathrm{dom}g^*)$. Moreover the following equality holds:

$$W^g(y,y^*) = W^g(y,x^*) + g^*(y^*) - g^*(x^*) - \langle y^* - x^*, y\rangle \quad \text{for any } x^*, y^* \in X^* \text{ and } y \in X. \tag{3.5}$$

Similar to the case of Bregman farthest functions, we define

$$V_C^g(x^*) := \sup_{y \in C} W^g(y,x^*) \quad \text{for each } x^* \in \mathrm{dom}g^*. \tag{3.6}$$

Then

$$S_C^g(x^*) = \{y \in C:\ W^g(y,x^*) = V_C^g(x^*)\} \quad \text{for each } x^* \in \mathrm{dom}g^*. \tag{3.7}$$

In the case when X is a Hilbert space and $g(\cdot) = \frac{1}{2}\|\cdot\|^2$, the operators Q_C^g and S_C^g coincide and are equal to the norm farthest point map onto the set C. For the general case, the relationship between the two operators are described in the following proposition, which is a direct consequence of the Young-Fenchel inequality and the definition of subdifferential of convex functions (cf. (2.1) and (2.2)).

Proposition 3.1. *The following assertions hold:*

$$D_g(y, x) = W^g(y, \nabla g(x)) \quad \textit{for each } (y, x) \in \mathrm{int}(\mathrm{dom} g) \times \mathrm{int}(\mathrm{dom} g); \tag{3.8}$$

$$Q_C^g(x) = S_C^g(\nabla g(x)) \quad \textit{for each } x \in \mathrm{int}(\mathrm{dom} g). \tag{3.9}$$

Let $x \in \mathrm{int}(\mathrm{dom} g)$ and $\{y_n\} \subseteq C$. The sequence $\{y_n\} \subseteq C$ is called a D-maximizing sequence of x if

$$\lim_{n\to\infty} D_g(y_n, x) = F_C^g(x). \tag{3.10}$$

Defintion 3.1. The set C is said to be D-maximally approximately compact if, for any $x \in \mathrm{int}(\mathrm{dom} g)$, each D-maximizing sequence of x has a subsequence that converges to an element of C.

Lemma 3.1. *Let $x \in \mathrm{int}(\mathrm{dom} g)$ and let $\{y_n\} \subseteq C$ be a D-maximizing sequence of x. If $\bar{y} \in C$ is a cluster point of $\{y_n\}$, then $\bar{y} \in Q_C^g(x)$.*

Proof. We may assume, without loss of generality, that $y_n \to \bar{y}$ as $n \to \infty$. Since g is continuous at $\bar{y}$ by Proposition 2.3, we have $g(\bar{y}) = \lim_{n\to\infty} g(y_n)$; consequently

$$\begin{aligned} D_g(\bar{y}, x) &= g(\bar{y}) - g(x) - \langle \nabla g(x), \bar{y} - x \rangle \\ &= \lim_{n\to\infty} (g(y_n) - g(x) - \langle \nabla g(x), y_n - x \rangle) \\ &= \lim_{n\to\infty} D_g(y_n, x) \\ &= F_C^g(x). \end{aligned} \tag{3.11}$$

Hence $\bar{y} \in Q_C^g(x)$. ∎

Proposition 3.2. *Suppose that C is D-maximally approximately compact. Then $Q_C^g(x) \neq \emptyset$ for any $x \in \mathrm{int}(\mathrm{dom} g)$.*

Proof. Let $x \in \mathrm{int}(\mathrm{dom} g)$ be arbitrary and take a sequence $\{y_n\} \subseteq C$ such that (3.10) holds. Since C is D-maximally approximately compact, $\{y_n\}$ has a subsequence which is convergent to an element of C. Without loss of generality, we may assume that $y_n \to \bar{y} \in C$; hence $\bar{y}$ is a cluster point of $\{y_n\}$. Thus $\bar{y} \in Q_C^g(x)$ by Lemma 3.1. ■

For the remainder, we need the notion of the 1-coercivity, or super-coercivity (cf.[2]). We say that g is 1-coercive if

$$\lim_{\|y\|\to\infty} \frac{g(y)}{\|y\|} = \infty.$$

It is easy to see (cf. [2]) that g is 1-coercive if and only if

$$\mathrm{int}(\mathrm{dom} g^*) = \mathrm{dom} g^* = X^*. \tag{3.12}$$

Theorem 3.1. *Suppose that g is 1-coercive or totally convex at any point of $\mathrm{int}(\mathrm{dom} g)$. Then the following statements hold.*

(*i*) *The function V_C^g is continuous on $\nabla g(\mathrm{int}(\mathrm{dom} g))$.*

(*ii*) *If g is Fréchet differentiable on $\mathrm{int}(\mathrm{dom} g)$, then F_C^g is continuous on $\mathrm{int}(\mathrm{dom} g)$.*

Proof. (i) Let $x^* \in \nabla g(\mathrm{int}(\mathrm{dom} g))$ and $\{x_n^*\} \subset \mathrm{dom} g^*$ be such that $x_n^* \to x^*$. Then we have the assertion $x^* \in \mathrm{int}(\mathrm{dom} g^*)$, which is true by (3.12) in the case when g is 1-coercive and by Proposition 2.4 in the case when g is totally convex at any point of $\mathrm{int}(\mathrm{dom} g)$. Since g^* is continuous at x^*, one has

$$g^*(x_n^*) \to g^*(x^*). \tag{3.13}$$

Fix $n \in \mathbb{N}$. By (3.5), we have

$$W^g(y, x_n^*) = W^g(y, x^*) + g^*(x_n^*) - g^*(x^*) - \langle x_n^* - x^*, y\rangle \quad \text{for each } y \in C. \tag{3.14}$$

It follows that

$$W^g(y, x_n^*) \to W^g(y, x^*) \quad \text{for each } y \in C. \tag{3.15}$$

Thus,

$$V_C^g(x_n^*) \geq W^g(y, x_n^*) \longrightarrow W^g(y, x^*) \quad \text{for each } y \in C;$$

hence

$$\liminf_{n\to\infty} V_C^g(x_n^*) \geq V_C^g(x^*). \tag{3.16}$$

Below we verify that

$$V_C^g(x^*) \geq \limsup_{n\to\infty} V_C^g(x_n^*). \tag{3.17}$$

Granting this together with (3.16), we complete the proof of assertion (i).

To show (3.17), let $\epsilon > 0$ be arbitrary and let $\{y_n\} \subseteq C$ be such that

$$W^g(y_n, x_n^*) \geq V_C^g(x_n^*) - \epsilon \quad \text{for each } n = 1, 2, \cdots .$$

By (3.5),

$$W^g(y_n, x^*) = W^g(y_n, x_n^*) + g^*(x^*) - g^*(x_n^*) + \langle x_n^* - x^*, y_n \rangle. \tag{3.18}$$

Noticing that $\{y_n\}$ is bounded, and letting $n \to +\infty$ in (3.18) and using (3.13), we get that

$$V_C^g(x^*) \geq \limsup_{n\to\infty} W^g(y_n, x_n^*) \geq \limsup_{n\to\infty} V_C^g(x_n^*) - \epsilon.$$

This completes the proof of (3.17).

(ii) Since g is Fréchet differentiable on $\mathrm{int}(\mathrm{dom} g)$, by Proposition 2.3, one has that ∇g is continuous on $\mathrm{int}(\mathrm{dom} g)$. By Proposition 3.1, we have $F_C^g(x) = V_C^g(\nabla g(x))$ for every $x \in \mathrm{int}(\mathrm{dom} g)$. Hence the assertion follows directly from assertion (i). ∎

Let Z be a Bananch space and let $T : Z \rightrightarrows X$ be a set-valued mapping. The domain of T is denoted by $\mathcal{D}(T)$ and defined by

$$\mathcal{D}(T) := \{z \in Z : T(z) \neq \emptyset\}.$$

Defintion 3.2. The set-valued mapping T is said to be

(a) upper semicontinuous at $z_0 \in \mathcal{D}(T)$ if for every open set $U \supset T(z_0)$, there exists $\delta > 0$ such that $T(z) \subset U$ for every $z \in \mathbf{B}(z_0, \delta) \cap \mathcal{D}(T)$;

(b) upper semicontinuous on a subset $Z_0 \subset \mathcal{D}(T)$ if it is upper semicontinuous at each $z \in Z_0$.

Theorem 3.2. *Suppose that g is 1-coercive or totally convex at any point of $\mathrm{int}(\mathrm{dom} g)$. If C is D-maximally approximately compact, then the following statements hold.*

(i) *The operator S_C^g is upper semicontinuous on $\nabla g(\mathrm{int}(\mathrm{dom} g))$.*

(ii) *If g is Fréchet differentiable on $\mathrm{int}(\mathrm{dom} g)$, then the operator Q_C^g is upper semicontinuous on $\mathrm{int}(\mathrm{dom} g)$.*

Proof. (i) By Proposition 3.2, $S_C^g(\nabla g(x)) = Q_C^g(x) \neq \emptyset$ for each $x \in \mathrm{int}(\mathrm{dom} g)$. Suppose on the contrary that S_C^g is not upper semicontinuous at $x^* = \nabla g(x)$ for some $x \in \mathrm{int}(\mathrm{dom} g)$. Then there exist an open set $U \supset S_C^g(x^*)$, sequences $\{x_n^*\} \subset \mathrm{dom}g$ and $\{y_n\}$ with each $y_n \in S_C^g(x_n^*)$ such that $x_n^* \to x^*$ and $y_n \in X \setminus U$ for each n. Then, by (3.5),

$$W^g(y_n, x^*) = W^g(y_n, x_n^*) + g^*(x^*) - g^*(x_n^*) + \langle x_n^* - x^*, y_n \rangle. \tag{3.19}$$

Since $x^* \in \mathrm{int}(\mathrm{dom}g^*)$ as noted earlier, it follows that $g^*(x_n^*) \to g^*(x^*)$. Furthermore, by Theorem 3.1, we have that $W^g(y_n, x_n^*) = V_C^g(x_n^*) \to V_C^g(x^*)$. Taking limit in (3.19) , we get that

$$D_g(y_n, x) = W^g(y_n, x^*) \to V_C^g(x^*) = F_C^g(x).$$

Since C is D-maximally approximately compact, we have $\{y_n\}$ has a subsequence which converges to some point $\bar{y} \in C$. Hence $W^g(\bar{y}, x^*) = V_C^g(x^*)$ by the continuity of g at $\bar{y}$, that is, $\bar{y} \in S_C^g(x^*)$. However, since $y_n \in X \setminus U$ for each n and $X \setminus U$ is closed, it follows that $\bar{y} \in X \setminus U$, which is a contradiction. Therefore S_C^g is upper semicontinuous at x^*.

(ii) Suppose that g is Fréchet differentiable on $\mathrm{int}(\mathrm{dom}g)$. Then, by Proposition 2.3, ∇g is continuous on $\mathrm{int}(\mathrm{dom}g)$. Hence the assertion follows from (i) and (3.9). ∎

Defintion 3.3. Let $C \subset X$. The set C is said to be Klee with respect to the Bregman distance, or simply D-Klee, if for every $x \in \mathrm{int}(\mathrm{dom}g)$, $Q_C^g(x)$ is singleton.

Remark 3.1. By the definition of D-Klee set and the formula (3.9), one sees that if C is D-Klee, then $S_C^g(x^*)$ is single-valued for each $x^* \in \nabla g(\mathrm{int}(\mathrm{dom}g))$.

Observe that if a set-valued mapping is single-valued, then its upper semicontinuity is equivalent to its continuity. Thus, by Theorem 3.2, we obtain the following continuity results for operators S_C^g and Q_C^g whenever C is a D-Klee subset of $\mathrm{int}(\mathrm{dom}g)$.

Corollary 3.1. *Suppose that g is 1-coercive or totally convex at any point of $\mathrm{int}(\mathrm{dom}g)$. If C is D-maximally approximately compact and D-Klee subset of $\mathrm{int}(\mathrm{dom}g)$. Then the following statements hold.*

(i) *The operator S_C^g is continuous on $\nabla g(\mathrm{int}(\mathrm{dom}g))$.*

(ii) *If g is Fréchet differentiable on $\mathrm{int}(\mathrm{dom}g)$, then the operator Q_C^g is continuous on $\mathrm{int}(\mathrm{dom}g)$.*

Applying Theorem 3.2 and Corollary 3.1 to the special convex function $g = g_2$ (noting that g_2 is clearly 1-coercive), we obtain the following corollary.

Corollary 3.2. *Let C be a D-maximally approximately compact subset of X. Then the following statements hold.*

(i) *If X is a smooth Banach space, then the operator $S_C^{g_2}$ is upper semicontinuous on X^*. If, in addition, C is D-Klee, then the operator $S_C^{g_2}$ is continuous on X^*.*

(ii) *If the norm of X is Fréchet differentiable, then the operator $Q_C^{g_2}$ is upper semicontinuous on X. If, in addition, C is D-Klee, then the operator $Q_C^{g_2}$ is continuous on X.*

4. Singleton of D-Klee Sets

As assumed in the previous section, let $g\colon X \to \overline{\mathbb{R}}$ be a lower semicontinuous proper convex function and Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} g)$, and let $C \subseteq \operatorname{int}(\operatorname{dom} g)$ be a nonempty bounded subset. This section is devoted to the study of the singleton problem of D-Klee sets. For this purpose, we need to introduce the notions of essentially smooth convex functions and Legendre convex functions, which have been studied extensively in [2].

Defintion 4.1. The function g is said to be

(a) essentially smooth if ∂g is both locally bounded and single-valued on its domain;

(b) Legendre if g is both essentially strictly convex and essentially smooth.

The following proposition is useful and known in [2, Theorems 5.4 and 5.6].

Proposition 4.1. *The following assertions hold.*

(i) *The function g is essentially smooth if and only if $\operatorname{dom}(\partial g) = \operatorname{int}(\operatorname{dom} g) \neq \emptyset$ and ∂g is single-valued on its domain.*

(ii) *If X is reflexive, then g is essentially smooth if and only if g^* is essentially strictly convex.*

Remark 4.2. (a) By (2.2), the following equivalence holds:

$$x \in \partial g^*(x^*) \Longleftrightarrow x^* \in \partial g(x) \quad \text{for each pair } (x, x^*) \in X \times X^*; \tag{4.1}$$

hence

$$x \in (\partial g^* \circ \nabla g)(x) \quad \text{for each } x \in \operatorname{int}(\operatorname{dom} g). \tag{4.2}$$

(b) By (a) and Proposition 4.1(i), if g is essentially smooth, then

$$\mathrm{Im}(\partial g^*) = \mathrm{dom}(\partial g) = \mathrm{int}(\mathrm{dom} g). \tag{4.3}$$

(c) If g is 1-coercive and g is essentially smooth, then

$$\nabla g(\mathrm{int}(\mathrm{dom} g)) = \mathrm{dom}(\partial g^*) = X^* \tag{4.4}$$

and

$$F_C^g \circ \partial g^* = V_C^g \quad \text{and} \quad Q_C^g \circ \partial g^* = S_C^g. \tag{4.5}$$

In fact, by (4.3) and the 1-coercivity assumption, one has that $\mathrm{dom}(\partial g^*) = \mathrm{dom} g^* = X^*$ and $\mathrm{Im}(\partial g^*) = \mathrm{int}(\mathrm{dom} g)$. Thus (4.4) follows from (4.1); while (4.5) holds because of Proposition 3.1 and (4.2).

(d) If both g and g^* are essentially smooth (e.g., if X is reflexive and g is Legendre), then $\nabla g : \mathrm{int}(\mathrm{dom} g) \to \mathrm{int}(\mathrm{dom} g^*)$ is a bijection satisfying

$$(\nabla g)^{-1} = \nabla g^*. \tag{4.6}$$

Let $T : X^* \rightrightarrows X$ be a set-valued mapping. Recall that T is monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0 \quad \text{for any } x^*,\, y^* \in \mathcal{D}(T) \text{ and } x \in T(x^*), y \in T(y^*). \tag{4.7}$$

A monotone set-valued mapping T is maximal monotone if, for any monotone mapping $T' : X^* \rightrightarrows X$, the condition that $T(x^*) \subset T'(x^*)$ for each $x^* \in \mathcal{D}(T)$ implies that $T = T'$.

Proposition 4.2. *The operator* $-S_C^g$ *is monotone.*

Proof. Let $x^*, y^* \in \mathrm{dom} S_C^g$ and $x \in S_C^g(x^*), y \in S_C^g(y^*)$ be arbitrary elements. Then, by the definition of S_C^g, one has that

$$W^g(x, x^*) \geq W^g(y, x^*)$$

and

$$W^g(y, y^*) \geq W^g(x, y^*).$$

Adding these inequalities, one obtains

$$\langle -x + y, x^* - y^* \rangle \geq 0,$$

which shows that $-S_C^g$ is monotone. ∎

Proposition 4.3. *Suppose that g is 1-coercive and essentially smooth. Then the following statements hold.*

(i) *If C is D-Klee, then S_C^g is single-valued on X^*.*

(ii) *If C is D-Klee and if S_C^g is continuous, then $-S_C^g$ is maximal monotone.*

(iii) *If X is reflexive and $-S_C^g$ is maximal monotone, then C is singleton.*

Proof. (i) By Remark 4.2(c), $\nabla g(\text{int}(\text{dom}g)) = \text{dom}(\partial g^*) = X^*$. If C is D-Klee, then $\text{dom}S_C^g = X^*$ and $S_C^g(x^*)$ is a singleton for each $x^* \in X^*$ by Remark 3.1.

(ii) By (i), $-S_C^g$ is single-valued on X^*. Thus, if S_C^g is continuous, then $-S_C^g$ is maximal monotone by a well known fact about maximal monotonicity (cf. [7], Lemma 2).

(iii) Consider the following two maximal monotone operators ∂g^* and $-S_C^g$, by Brézis-Haraux range theorem (see [20, Corollary 31.6]), we have

$$\begin{aligned}\text{int}[\text{range}(\partial g^* - S_C^g)] &= \text{int}[\text{range}(\partial g^*) - \text{range}(S_C^g)]\\ &= \text{int}[\text{int}(\text{dom}g) - \text{range}(S_C^g)].\end{aligned}$$

Since $\text{range}(S_C^g) \subset C \subset \text{int}(\text{dom}g)$, we have $0 \in \text{int}[\text{int}(\text{dom}g) - \text{range}(S_C^g)]$. It follows that $0 \in \text{int}[\text{range}(\partial g^* - S_C^g)]$. Thus there exists $x^* \in \text{int}(\text{dom}g^*)$ such that $\partial g^*(x^*) = S_C^g(x^*)$, this together with (4.5) implies that $Q_C^g(x) = x$, where $x = \partial g^*(x^*)$. Hence C must be a singleton. ■

Let I_C denote the indicate function of the set C, that is,

$$I_C(x) := \begin{cases} 0 & \text{for each } x \in C,\\ +\infty & \text{for each } x \in X \setminus C.\end{cases}$$

It will be convenient to define the function $g^{\#}$ on X defined by $g^{\#}(x) = g(-x)$ for each $x \in X$.

Lemma 4.1. *Let $x^* \in \text{dom}g^*$. Then the following assertions hold.*

(4.8) $$(-g^{\#} + I_{-C})^*(x^*) = V_C^g(x^*) - g^*(x^*);$$

(4.9) $$-S_C^g(x^*) \subset \partial(-g^{\#} + I_{-C})^*(x^*).$$

Consequently, if g is 1-coercive, then (4.8) *holds for each $x^* \in X^*$.*

Proof. We observe that

$$V_C^g(x^*) = \sup_{x\in C}\{g(x) + g^*(x^*) - \langle x^*, x\rangle\} \\ = g^*(x^*) + \sup_{x\in X}\{\langle x^*, -x\rangle - (-g(x) + I_C(x))\} \\ = g^*(x^*) + \sup_{x\in X}\{\langle x^*, x\rangle - (-g(-x) + I_{-C}(x))\} \\ = g^*(x^*) + (-g^\# + I_{-C})^*(x^*). \tag{4.10}$$

Hence (4.8) is proved.

To show (4.9), we first note that the following equivalences for $x \in -C$:

$$x \in (\partial(-g^\# + I_{-C}))^{-1}(x^*) \Leftrightarrow x^* \in \partial(-g^\# + I_{-C})(x) \\ \Leftrightarrow -g(-x)+I_{-C}(x)+(-g^\# + I_{-C})^*(x^*) = \langle x^*, x\rangle \\ \Leftrightarrow -g(-x) + I_{-C}(x) + V_C^g(x^*) - g^*(x^*) = \langle x^*, x\rangle \\ \Leftrightarrow W^g(-x, x^*) = V_C^g(x^*) \\ \Leftrightarrow -x \in S_C^g(x^*).$$

Now, let $x \in -S_C^g(x^*)$. Then $x \in (\partial(-g^\# + I_{-C}))^{-1}(x^*)$ and hence $x^* \in \partial(-g^\# + I_{-C})(x)$. By (2.2),

$$(-g^\# + I_{-C})^*(x^*) + (-g^\# + I_{-C})(x) = \langle x^*, x\rangle.$$

since $(-g^\# + I_{-C})^{**}(x) \le (-g^\# + I_{-C})(x)$, we have

$$(-g^\# + I_{-C})^{**}(x) + (-g^\# + I_{-C})^*(x^*) \le \langle x^*, x\rangle \tag{4.11}$$

and hence

$$(-g^\# + I_{-C})^{**}(x) + (-g^\# + I_{-C})^*(x^*) = \langle x^*, x\rangle$$

because the converse inquality of (4.11) holds automatically. This implies that $x \in \partial(-g^\# + I_{-C})^*(x^*)$ and completes the proof. ∎

The main results are given in the following theorems.

Theorem 4.1. *Suppose that X is a reflexive Banach space and g is a 1-coercive, essentially smooth function. Let $C \subset \mathrm{int}(\mathrm{dom} g)$ be such that $Q_C^g(x) \ne \emptyset$ for each $x \in \mathrm{int}(\mathrm{dom} g)$. Then following conditions are equivalent.*

(i) *The set C is singleton.*

(ii) *The set C is D-maximally approximately compact and D-Klee.*

(iii) *The operator $-Q_C^g \circ \nabla g^*(= -S_C^g)$ is single-valued and continuous on X^*.*

(iv) *The operator* $-Q_C^g \circ \nabla g^*$ *is maximal monotone.*

Furthermore, if g *is additionally essentially strictly convex, then each of assertions* (i)-(iv) *is equivalent to the following one:*

(v) $F_C^g \circ \nabla g^* (= V_C^g)$ *is Gâteaux differentiable on* $\mathrm{int}(\mathrm{dom} g^*)$.

Proof. The implication (i)$\Longrightarrow$(ii) is trivial. The implication (ii)$\Longrightarrow$(iii) follows from Corollary 3.1 and Proposition 4.3(i); while the implications (iii)$\Longrightarrow$(iv)$\Longrightarrow$(i) follow from Proposition 4.2 and Proposition 4.3(iii).

Suppose that g is additionally essentially strictly convex. Then g^* is essentially smooth by Proposition 4.1 and so Gâteaux differentiable on $\mathrm{dom}(\partial g^*)$. Moreover, by Remark 4.2(c), $\nabla g(\mathrm{int}\,\mathrm{dom} g) = \mathrm{dom}(\partial g^*) = X^*$ and $F_C^g \circ \nabla g^* = V_C^g$. By hypothesis $Q_C^g(x) \neq \emptyset$ for each $x \in \mathrm{int}(\mathrm{dom} g)$, it follows that $S_C^g(x^*) = Q_C^g \circ \partial g^*(x^*) \neq \emptyset$ for every $x^* \in X^*$ by (4.2). Note that (4.8) holds for each $x^* \in X^*$ by Lemma 4.1. One has that V_C^g is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} g^*) = X^*$ if and only if so is $(-g^{\#} + I_{-C})^*$. This together with (4.9) implies that (v) is equivalent to $-S_C^g = \partial(-g^{\#} + I_{-C})^*$, which is in turn equivalent to $-S_C^g$ is maximal monotone because $\partial(-g^{\#} + I_{-C})^*$ is maximal monotone extension of $-S_C^g(x^*)$ by (4.9) and Proposition 4.2. Hence the equivalence (iv) $\Longleftrightarrow$ (v) is proved and completes the proof. ■

Theorem 4.2. *Suppose that* X *is a reflexive Banach space and* g *is 1-coercive, essentially smooth and totally convex at any point of* $\mathrm{int}(\mathrm{dom} g)$. *Let* $C \subseteq \mathrm{int}(\mathrm{dom} g)$ *be such that* $Q_C^g(x) \neq \emptyset$ *for each* $x \in \mathrm{int}(\mathrm{dom} g)$.

(1) *The following conditions are equivalent.*

(i) *The set* C *is singleton.*

(ii) *The set* C *is* D*-maximally approximately compact and* D*-Klee.*

(iii) *The operator* $-Q_C^g \circ \nabla g^* (= -S_C^g)$ *is single-valued and continuous on* X^*.

(iv) *The operator* $-Q_C^g \circ \nabla g^*$ *is maximal monotone.*

(v) $F_C^g \circ \nabla g^* (= V_C^g)$ *is Gâteaux differentiable on* $\mathrm{int}(\mathrm{dom} g^*)$.

(vi) *The function* $F_C^g \circ \nabla g^*$ *is Fréchet differentiable on* X^*.

(2) *If* g *is Fréchet differentiable on* $\mathrm{int}(\mathrm{dom} g)$, *then each of* (i)-(vi) *is equivalent to the following one:*

(vii) *The operator* Q_C^g *is single-valued and continuous on* $\mathrm{int}(\mathrm{dom} g)$.

(3) *If* ∇g *and* ∇g^* *are Fréchet differentiable respectively on* $\mathrm{int}(\mathrm{dom} g)$ *and* $\mathrm{int}(\mathrm{dom} g^*)$, *then each of* (i)-(vii) *is equivalent to the following one:*

(viii) *The function* F_C^g *is Fréchet differentiable on* $\mathrm{int}(\mathrm{dom} g)$.

Proof. (1) By Theorem 4.1, the assertions (i)-(v) are equivalent. To show the equivalence between (iv) and (vi), we note that g^* is Fréchet differentiable on $\mathrm{dom}(\partial g^*)$ by Proposition 2.4. Thus the same argument for the proof of the equivalence (iv)$\Longleftrightarrow$(v) in Theorem 4.1 shows that (iv) and (vi) are equivalent. Hence (1) is proved.

(2) Suppose that g is Fréchet differentiable on $\mathrm{int}(\mathrm{dom} g)$. Then, by Propositions 2.3 and 2.4, ∇g and ∇g^* are continuous respectively on $\mathrm{int}(\mathrm{dom} g)$ and $\mathrm{int}(\mathrm{dom} g^*)$. Since $Q_C^g = (Q_C^g \circ \nabla g^*) \circ \nabla g$, it follows that (iii)$\Longleftrightarrow$(vii) and the proof of (2) is complete.

(3) Finally, suppose that ∇g and ∇g^* are Fréchet differentiable respectively on $\mathrm{int}(\mathrm{dom} g)$ and $\mathrm{int}(\mathrm{dom} g^*)$. Since $F_C^g = (F_C^g \circ \nabla g^*) \circ \nabla g$, we have that (vi)$\Longleftrightarrow$(viii) and complete the proof. ■

Applying above Theorem 4.2 to the Euclidean space $\mathbb{R}^n$, we immediately have the following corollary, which extends and improves the corresponding one in [5].

Corollary 4.1. *Let $X = \mathbb{R}^n$ and suppose that $g : \mathbb{R}^n \to \overline{\mathbb{R}}$ be Legendre and* 1*-coercive. Let $C \subset \mathrm{int}(\mathrm{dom} g)$ be a nonempty closed set. Then the following assertions are equivalent.*

(i) *The set C is singleton.*

(ii) *The set C is D-Klee.*

(iii) *The operator Q_C^g is continuous on* $\mathrm{int}(\mathrm{dom}\, g)$.

(iv) *The operator $-Q_C^g \circ \nabla g^*$ is maximal monotone.*

(v) *The function $F_C^g \circ \nabla g^*$ is differentiable on X^*.*

If, in addition, g is second order continuously differentiable on $\mathrm{int}(\mathrm{dom} g)$, *and $\nabla^2 g(x)$ is positive definite for every $x \in \mathrm{int}(\mathrm{dom} g)$, then each of* (i)-(v) *is equivalent to the following one:*

(vi) *The function F_C^g is differentiable on* $\mathrm{int}(\mathrm{dom} g)$.

Proof. We need only prove that if g is second order continuously differentiable on $\mathrm{int}(\mathrm{dom} g)$, and for every $x \in \mathrm{int}(\mathrm{dom} g)$, $\nabla^2 g(x)$ is positive definite, then ∇g^* is differentiable on $\mathrm{int}(\mathrm{dom} g^*)$. In fact, since g is Legendre, by Remark 4.2(d), $\nabla g \colon \mathrm{int}(\mathrm{dom} g) \to \mathrm{int}(\mathrm{dom} g^*)$ is bijective, and $\nabla g^* = (\nabla g)^{-1}$. Since g is second order continuously differentiable on $\mathrm{int}(\mathrm{dom} g)$, and for every $x \in \mathrm{int}(\mathrm{dom} g)$, $\nabla^2 g(x)$ is positive definite, by the well known inverse theorem, $\nabla g^* = (\nabla g)^{-1}$ is continuously differentiable on $\mathrm{int}(\mathrm{dom} g^*)$. ■

Consider the significant particular case when $g = g_2$. Let $J : X \rightrightarrows X^*$ and $J^* \colon X^* \rightrightarrows X$ be the normlized duality mappings, i.e.,

$$J(x) : = \{x^* \in X^* \colon \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\},$$
$$J^*(x^*) : = \{x \in X \colon \langle x^*, x\rangle = \|x^*\|^2 = \|x\|^2\}.$$

It is well known that when X is a reflexive smooth and strictly convex Banach space, J is bijective and $J^{-1} = J^*$.

Corollary 4.2. *Suppose that X is a reflexive, smooth and strictly convex Banach space. Suppose that C is D-Klee subset of X with respect to the function g_2. Then the following statements are equivalent.*

(i) *The set C is singleton.*

(ii) *The set C is D-maximally approximately compact.*

(iii) *The operator $Q_C^{g_2} \circ J^*$ is continuous on X^*.*

(iv) *The operator $-Q_C^{g_2} \circ J^*$ is maximal monotone.*

(v) *The function $F_C^{g_2} \circ J^*$ is Gâteaux differentiable on X^*.*

Moreover, if X is locally totally convex, then (i)-(v) *are equivalent to the following assertions:*

(iv) *The function $F_C^{g_2} \circ J^*$ is Fréchet differentiable on X^*.*

Proof. Since X is smooth if and only if $\partial g_2 = J$ is single-valued. It is clear that $\mathrm{dom} J = \mathrm{int}(\mathrm{dom} g_2) = X$. Hence g_2 is essentially smooth. By Lemma 5.8 in [2], X is strictly convex if and only if g_2 is essentially strictly convex.

Moreover, the local total convexity of X implies that g_2 is totally convex at any point of X. Hence the result follows from Theorem 4.2. ■

References

1. E. Asplund and R. T. Rockafellar, Gradients of convex functions, *Trans. Amer. Math. Soc.*, **139** (1969), 443-467.

2. H. H. Bauschke, J. M. Borwein and P. L. Combettes, Essential smoothness, essential strictly convexity, and Legendre functions in Banach spaces, *Commun. Contemp. Math.*, **3** (2001), 615-647.

3. H. H. Bauschke, J. M. Borwein and P. L. Combettes, Bregman monotone optimization algorithms, *SIAM J. Control Optim.*, **42** (2003), 596-636.

4. H. H. Bauschke, X. F. Wang, J. J. Ye and X. M. Yuan, Bregman distances and Chebyshev sets, *J. Approx. Theory*, **159** (2009), 3-25.

5. H. H. Bauschke, X. F. Wang, J. J. Ye and X. M. Yuan, Bregman distances and Klee sets, *J. Approx. Theory*, **158** (2009), 170-183.

6. J. F. Bonnanas and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer-Verlag, New York, Inc. 2000.

7. F. E. Browder, Nonlinear maximum monotone operators in Banach spaces, *Math. Ann.*, **175** (1968), 89-113.

8. D. Butnariu, Y. Censor and S. Reich, Iterative averaging of entropic projections for solving stochastic convex feasibility problems, *Comput. Optim. Appl.*, **8** (1997), 21-39.

9. D. Butnariu and A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers, Dordrecht, 2000.

10. D. Butnariu and A. N. Iusem, Local moduli of convexity and their applications to finding almost common fixed points of mesurable families of operators, in: *Recent Developments in Optimization and Nonlinear Analysis, Contemporary Mathematics*, Y. Censor and S. Reich (Eds.), Vol. 204, American Mathematical Society, Providence, Rhode Island, 1997, pp. 33-61.

11. D. Butnariu, A. N. Iusem and C. Zalinescu, On uniform convexity, total convexity and convergence of the proximal point and outer Bregman projection algorithms in Banach spaces, *J. Convex Anal.*, **10** (2003), 35-61.

12. D. Butnariu and E. Resmerita, Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, *Abstr. Appl. Anal.*, (2006), 84919. p. 39.

13. N. W. Efimow and S. B. Stechkin, Approximative compactness and Chebyshev sets, *Soviet Mathematics*, **2** (1961), 1226-1228.

14. J.-B. Hiriart-Urruty, La conjecture des points les plus éloignés revisitée, *Ann. Sci. Math. Québec*, **29** (2005), 197-214.

15. J.-B. Hiriart-Urruty, Potpourri of conjectures and open questions in nonlinear analysis and optimization, *SIAM Review*, **49** (2007), 255-273.

16. V. Klee, Convexity of Chebyshev sets, *Math. Ann.*, **142** (1961), 292-304.

17. C. Li, W. Song and J. C. Yao, Bregman distance, approximate compactness and convexity of Chebyshev sets in Banach spaces, *J. Approx. Theory*, to appear.

18. B. B. Panda and O. P. Kapoor, On farthest points of sets, *J. Math. Anal. Appl.*, **62** (1978), 345-353.

19. E. Resmerita, On total convexity, Bregman projections and stability in Banach spaces, *J. Convex Anal.*, **11** (2004), 1-16.

20. S. Simons, *From Hahn-Banach to Monotonicity*, in Lecture Notes in Mathematics, 1693, Springer-Verlag, 2008.

21. U. Westphal and T. Schwartz, Farthest points and monotone operators, *Bull. Austral. Math. Soc.*, **58** (1998), 75-92.

22. S. Y. Xu, C. Li and W. S. Yang, *Non-linear Approximation Theory in Banach Spaces*, Scientific Press, Beijing, 1997, in Chinese.

23. C. Zălinescu, On uniformly convex functions, *J. Math. Anal. Appl.*, **95** (1983), 344-374.

24. C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, River Edge, N.J., 2002.

Donghui Fang
School of Mathematical and Computer Sciences,
Jishou University,
Jishou 416000,
and
Department of Mathematics,
Zhejiang University,
Hangzhou 310027,
P. R. China

Wen Song
School of Mathematical Sciences,
Harbin Normal University,
Harbin 150025,
P. R. China
wsong218@yahoo.com.cn

Chong Li
Department of Mathematics,
Zhejiang University,
Hangzhou 310027,
P. R. China