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STABILITY OF A CLASS OF QUADRATIC PROGRAMS WITH A CONIC CONSTRAINT

G. M. Lee, N. N. Tam and N. D. Yen

Dedicated to Professor Boris Mordukhovich in celebration of his 60th birthday

Abstract. Stability of a general indefinite quadratic program whose constraint set is the intersection of an affine subspace and a closed convex cone is investigated. We present a systematical study of several stability properties of the Karush-Kuhn-Tucker point map, the global solution map, and the optimal value function, assuming that the problem data undergoes small perturbations. Some techniques from our preceding work on stability of indefinite quadratic programs under linear constraints have found further applications and extensions in this paper.

1. INTRODUCTION

It is well known that mathematical programmming problems with indefinite quadratic objective functions play an important role in optimization theory. For quadratic programs (QPs for brevity) under linear constraints, various continuity and differentiability properties of the (global) solution map, the local solution map, the Karush-Kuhn-Tucker point set map, and the optimal value function have been established; see for instance [1, 2 ,4, 6-9, 11-17], and the references therein. It would be interesting to investigate stability properties of QPs under quadratic constraints. As far as we know, very little has been done in this direction. Recently, a stability study of linear-quadratic minimization over Euclidean balls - called *the trust region subproblem* - has been given in [10]. From the results of [10] it is clear that stability criteria for QPs under quadratic constraints may be very different from

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the corresponding criteria for QPs under linear constraints. Besides, quite new arguments may be needed for proving the desired results. We refer to [3] for a comprehensive theory of stability and sensitivity analysis of general optimization problems.

The purpose of this paper is to obtain some stability properties of the problem of minimizing a linear-quadratic function on the intersection of an affine subspace and a finite dimensional closed convex cone with nonempty interior.

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional real Hilber space and $K \subset V$ be a closed convex cone with the positive dual

$$K^* := \{y \in V \,:\, \langle y, x\rangle \geq 0 \;\; \forall x \in K\}.$$

It is assumed that the interior $\mathrm{int}K$ of K is nonempty and K is pointed, i.e., $K \cap (-K) = \{0\}$. By $\mathcal{L}_S(V)$ we denote the set of symmetric linear operators $Q : V \to V$. Thus, for any $Q \in \mathcal{L}_S(V)$ and $x, y \in V$, it holds $\langle Qx, y\rangle = \langle x, Qy\rangle$. Consider the following *quadratic programming problem with a conic constraint*

$$(P) \qquad \inf\Big\{f(x,Q,c) := \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle \,:\, x \in a + X,\; x \in K\Big\},$$

where $Q \in \mathcal{L}_S(V)$, $c, a \in V$, and $X \subset V$ is a linear subspace. Recently, a potential reduction algorithm for (P) has been proposed under an Euclidean Jordan algebra setting in [5]. It is a simple matter to show that quadratic programs under linear constraints are special cases of (P). The above mentioned trust region subproblem is also a sepecial case of (P). Indeed, let $V = \mathbb{R}^n$ and K be the Lorentz cone in V, that is

$$K = \Big\{x = (x_1, \ldots, x_n) \,:\, x_n^2 \geq \sum_{i=1}^{n-1} x_i^2\Big\}.$$

Taking $X = \{x = (x_1, \ldots, x_n) \,:\, x_n = 0\}$ and $a = (0, \ldots, 0, \mu)$ with $\mu > 0$, one has

$$(a + X) \cap K = \Big\{x = (x_1, \ldots, x_{n-1}, \mu) \,:\, \sum_{i=1}^{n-1} x_i^2 \leq \mu^2\Big\}.$$

Hence (P) becomes the problem of minimizing a linear-quadratic function over the closed ball with center 0 and radius $\mu > 0$ in the space $\mathbb{R}^{n-1}$. This means that one has deal with the trust region subproblem.

First-order necessary optimality conditions for (P) (see [5, p. 70, 71] for a proof) can be stated as follows.

Theorem 1.1. *If $x \in K \cap (a + X)$ is a local minimum of (P), then there exist $r \geq 0$ and $s \in K^*$ such that $(r, s) \neq (0, 0)$, $r(Qx + c) - s \in X^\perp$, and $\langle x, s\rangle = 0$, where $X^\perp$ stands for the orthogonal subspace of X. If it is assumed additionally that $(a + X) \cap (\mathrm{int}K)$ is nonempty, then one can take $r = 1$.*

The preceding theorem leads to the following

Definition 1.1. We say that $x \in V$ is a KKT (Karush-Kuhn-Tucker) point of (P) if there exists $s \in V$ satisfying

(1) $$s \in K^*, \quad x \in K \cap (a+X), \quad Qx + c - s \in X^{\perp}, \quad \langle x, s\rangle = 0.$$

The sets of KKT points, of local solutions, and of (global) solutions of (P) are denoted respectively by $S(Q,c,a)$, $\mathrm{loc}(Q,c,a)$, and $\mathrm{Sol}(Q,c,a)$. According to Theorem 1.1, if $\mathrm{int}K \cap (a+X) \neq \emptyset$ then $\mathrm{Sol}(Q,c,a) \subset \mathrm{loc}(Q,c,a) \subset S(Q,c,a)$. Let $\varphi(Q,c,a) := \inf\{f(x,Q,c) : x \in (a+X) \cap K\}$ be the optimal value of (P).

By definition, a multifunction $F : W \rightrightarrows V$, where W is a subset of an Euclidean space, is said to be upper semicontinuous (usc) at $\omega \in W$ if for each open set $\Omega \subset V$ satisfying $F(\omega) \subset \Omega$, there exists $\delta > 0$ such that $F(\omega') \subset \Omega$ for every $\omega' \in W$ with the property that $\|\omega' - \omega\| < \delta$. We say that F is lower semicontinuous (lsc) at $\omega \in W$ if for each open set $\Omega \subset W$ satisfying $F(\omega) \cap \Omega \neq \emptyset$, there exists $\delta > 0$ such that $F(\omega') \cap \Omega \neq \emptyset$ for every $\omega' \in W$ with the property that $\|\omega' - \omega\| < \delta$. Upper semicontinuity of F at ω indicates that the value sets of the restriction of F to a neighborhood of ω have an *external stability* (they do not 'explode'), while the presence of the lower semicontinuity of F at ω assures us that the value sets possess an *internal stability* (they do not 'disappear'). If F is simultaneously usc and lsc at ω, we say that it is continuous at ω.

By introducing some modifications to the proof schemes in [7-9], [12-17], we will be able to derive necessary as well as sufficient conditions for the upper or lower semicontinuity of the KKT point map $(Q,c,a) \mapsto S(Q,c,a)$, the global solution map $(Q,c,a) \mapsto \mathrm{Sol}(Q,c,a)$, and conditions for the continuity of the optimal value function $(Q,c,a) \mapsto \varphi(Q,c,a)$.

We give necessary and sufficient conditions for the upper semicontinuity of the KKT point map in the next section. Then, in the subsequent section, we discuss the continuity of the global solution map. The last section is devoted to the continuity of the optimal value function.

2. Stability of the KKT Point Map: Necessary and Sufficient Conditions

Let $\Sigma_Q := \{x \in X \cap K \,:\, Qx \in X^{\perp}\}$. Using the idea of proving Theorem 2.1 in [16] we can establish necessary conditions for the usc property of the KKT point map of problem (P) as follows.

Theorem 2.1. (Necessary conditions for stability I) *Suppose that* $(a+X) \cap K$ *is nonempty and the KKT point set* $S(Q,c,a)$ *is bounded. If the multifunction* $S(\cdot,\cdot,a)$ *is upper semicontinuous at* (Q,c)*, then* $\Sigma_Q = \{0\}$.

Proof. On the contrary, suppose that $(a + X) \cap K \neq \emptyset$, $S(Q, c, a)$ is bounded, $S(\cdot, \cdot, a)$ is usc at (Q, c), but $\Sigma_Q \neq \{0\}$. Since $0 \in \Sigma_Q$, the last condition implies the existence of $\bar{x} \neq 0$ satisfying

$$\bar{x} \in X \cap K, \quad Q\bar{x} \in X^\perp. \tag{2}$$

Choose $\widetilde{a} \in (a + X) \cap K$ and observe that $(\widetilde{a} + X) \cap K = (a + X) \cap K$. For every $t > 0$, let $x_t = \frac{1}{t}\bar{x} + \widetilde{a}$. Since $\widetilde{a} \in K$, by (2) we have

$$x_t \in (\widetilde{a} + X) \cap K = (a + X) \cap K. \tag{3}$$

We claim that there exists an operator $Q_t \in \mathcal{L}_S(V)$ of the form $Q_t = Q + tQ_0$ and a vector $c_t \in V$ of the form $c_t = c + tc_0$ such that

$$Q_t x_t + c_t \in X^\perp, \tag{4}$$

where the operator $Q_0 \in \mathcal{L}_S(V)$ and vector c_0 are to be constructed in the sequel. Note that

$$\begin{aligned} Q_t x_t + c_t &= (Q + tQ_0)\Big(\frac{1}{t}\bar{x} + \widetilde{a}\Big) + (c + tc_0) \\ &= \frac{1}{t}Q\bar{x} + (Q_0\bar{x} + Q\widetilde{a} + c) + t(Q_0\widetilde{a} + c_0). \end{aligned}$$

Hence, if we have

$$Q_0\bar{x} + Q\widetilde{a} + c = 0 \tag{5}$$

and

$$Q_0\widetilde{a} + c_0 = 0, \tag{6}$$

then the inclusion (4) is valid.

Since V is of finite dimension and $\bar{x} \neq 0$, there exists $Q_0 \in \mathcal{L}_S(V)$ satisfying the condition $Q_0\bar{x} = -(Q\widetilde{a} + c)$. Indeed, choose an orthonormal basis $(\mathcal{B})$ of V. Let $(\bar{x}_1, \ldots, \bar{x}_n)$ be the coordinates of $\bar{x}$, $(b_1, \ldots, b_n)$ be the coordinates of $-(Q\widetilde{a} + c)$ in $(\mathcal{B})$. Let

$$I = \{i : \bar{x}_i \neq 0\} \subset \{1, \ldots, n\}.$$

Since $\bar{x} \neq 0$, $I \neq \emptyset$. Fix any index $i_0 \in I$. Put $\bar{Q}_0 = (q_{ij})$, where q_{ij} $(1 \leq i, j \leq n)$ are defined as follows:

$$q_{ii} = (\bar{x}_i)^{-1} b_i \quad \forall i \in I,$$

$$q_{i_0 j} = q_{j i_0} = (\bar{x}_{i_0})^{-1} b_j \quad \forall j \in \{1, \ldots, n\} \setminus I,$$

and $q_{ij} = 0$ for other pairs (i, j) with $1 \le i, j \le n$. Clearly, $\bar{Q}_0$ is a symmetric matrix. Now, let Q_0 denote the operator from V to itself such that the matrix associated with Q_0 with respect to the basis $(\mathcal{B})$ is the chosen $\bar{Q}_0$. It is easy to check that this Q_0 is the desired operator.

Putting $c_0 = -Q_0\widetilde{a}$, we see at once that (5) and (6) are fulfilled. Hence (4) holds true.

Now, set $s_t = 0$. By (3) and (4) we have

$$s_t \in K^*, \quad x_t \in (a + X) \cap K, \quad Q_t x_t + c_t - s_t \in X^{\perp}, \quad \langle x_t, s_t \rangle = 0.$$

This shows that $x_t \in S(Q_t, c_t, a)$. Let $\Omega \subset V$ be a bounded open set satisfying $S(Q, c, a) \subset \Omega$. Since $\lim_{t \to 0} Q_t = Q$ and $\lim_{t \to 0} c_t = c$, from the usc property of $S(\cdot, \cdot, a)$ it follows that $x_t \in \Omega$ for all $t > 0$ sufficiently small. This is an absurd, because $\lim_{t \to 0} \|x_t\| = \lim_{t \to 0} \|\frac{1}{t}\bar{x} + a\| = +\infty$. The proof is complete. ■

Theorem 2.2. (Necessary conditions for stability II) *Suppose that $Q \in \mathcal{L}_S(V)$, $c \in V$, $(a + X) \cap K$ is nonempty, and $S(Q, c, a)$ is bounded. If $S(\cdot, \cdot, a)$ is upper semicontinuous at (Q, c), then $S(Q, 0, 0) \cap \mathrm{int}K = \emptyset$.*

Proof. On the contrary, suppose that our assumptions are fulfilled, $S(\cdot, \cdot, a)$ is usc at (Q, c), but $S(Q, 0, 0) \cap \mathrm{int}K \neq \emptyset$. Choose a nonzero vector $\bar{x} \in S(Q, 0, 0) \cap \mathrm{int}K$. By the definition of KKT point, there exists $\bar{s} \in V$ such that

$$\bar{s} \in K^*, \quad \bar{x} \in K \cap X, \quad Q\bar{x} - \bar{s} \in X^{\perp}, \quad \langle \bar{x}, \bar{s} \rangle = 0. \tag{7}$$

Since $\bar{x} \in \mathrm{int}K$, from the equality in (7) it follows that $\bar{s} = 0$. Then, the third inclusion in (7) implies $Q\bar{x} \in X^{\perp}$. Thus (2) is satisfied. Using the latter and arguing similarly as in the proof of Theorem 2.1, we will arrive at a contradiction. ■

If $a = 0$ then (P) is an optimization problem under the conic constraint $x \in X \cap K$. We now state a result on stability of the KKT point set map of such a problem.

Theorem 2.3. (Necessary conditions for stability III). *Let $Q \in \mathcal{L}_S(V)$, $c \in V$, and $S(Q, c, 0)$ be bounded. If $S(\cdot, c, 0)$ is upper semicontinuous at Q, then $S(Q, 0, 0) = \{0\}$.*

Proof. On the contrary, suppose that $S(Q, c, 0)$ is bounded, $S(\cdot, c, 0)$ is usc at Q, but $S(Q, 0, 0) \neq \{0\}$. Fix a nonzero vector $\bar{x} \in S(Q, 0, 0)$. By the definition of KKT point, there exists $\bar{s} \in V$ such that (7) is fulfilled. Similarly as in the proof of Theorem 2.1, we can construct a symmetric linear oprerator $Q_0 : V \to V$ such that $Q_0\bar{x} + c = 0$. For each $t > 0$, let

$$x_t = \frac{1}{t}\bar{x}, \quad s_t = \frac{1}{t}\bar{s}, \tag{8}$$

and $Q_t = Q + tQ_0$. Using (7) we get

$$Q_t x_t + c - s_t \in X^{\perp}, \tag{9}$$

From (7)-(9) it follows that $x_t \in S(Q_t, c, 0)$ for all $t > 0$. Using the last inclusion we can obtain a contradiction. ∎

Theorem 2.4. *Let $Q \in \mathcal{L}_S(V)$, $c \in V$, and $a \in V$. Suppose that $K^* \cap X^{\perp} = \{0\}$ and one of the following conditions is satisfied:*

(a) $\{x \in X \cap K \, : \, \langle x, Qx \rangle = 0\} = \{0\}$,

(b) $S(Q, 0, 0) = \{0\}$.

Then $S(\cdot)$ is upper semicontinuous at (Q, c, a).

Proof. Arguing by contraposition, we suppose that one of the conditions (a), (b) is satisfied but $S(\cdot)$ is not usc at (Q, c, a). Then there exist an open set $\Omega \subset V$ containing $S(Q, c, a)$, a sequence $\{(Q^k, c^k, a^k)\}$ in $\mathcal{L}_S(V) \times V \times V$ converging to (Q, c, a), and a sequence $\{x^k\}$ in V such that $x^k \in S(Q^k, c^k, a^k)$ and $x^k \notin \Omega$ for every k. For each k there is $s^k \in V$ satisfying

$$s^k \in K^*, \quad x^k \in (a^k + X) \cap K, \quad Q^k x^k + c^k - s^k \in X^{\perp}, \quad \langle x^k, s^k \rangle = 0. \tag{10}$$

We first consider the situation where $\{\|(x^k, s^k)\|\}$ is bounded. Then $\{x^k\}$ and $\{s^k\}$ are bounded sequences. Without loss of generality, we may assume that $x^k \to \bar{x}$ and $s^k \to \bar{s}$. From (10) it follows that

$$\bar{s} \in K^*, \quad \bar{x} \in (a + X) \cap K, \quad Q\bar{x} + c - \bar{s} \in X^{\perp}, \quad \langle \bar{x}, \bar{s} \rangle = 0.$$

We then get $\bar{x} \in S(Q, c, a) \subset \Omega$, a contradiction, because $x^k \notin \Omega$ for all k. Thus the sequence $\{(x^k, s^k)\}$ must be unbounded. There is no loss of generality in assuming that $\|(x^k, s^k)\| \to \infty$ and $\|(x^k, s^k)\| \neq 0$ for all k. Therefore, we can admit that $\|(x^k, s^k)\|^{-1}(x^k, s^k) \to (\widehat{x}, \widehat{s})$ with $\|(\widehat{x}, \widehat{s})\| = 1$. Using (10) we obtain

$$\widehat{s} \in K^*, \quad \widehat{x} \in X \cap K, \quad Q\widehat{x} - \widehat{s} \in X^{\perp}, \quad \langle \widehat{x}, \widehat{s} \rangle = 0. \tag{11}$$

Consider the case where the condition (a) is satisfied. From (11) and the assumption $\{x \in K \cap X \, : \, \langle x, Qx \rangle = 0\} = \{0\}$ it is easy to deduce that $\widehat{x} = 0$. Hence, by (11),

$\widehat{s} \in K^* \cap X^\perp$. By the condition $K^* \cap X^\perp = \{0\}$, the latter implies that $\widehat{s} = 0$. In result, we get $(\widehat{x}, \widehat{s}) = (0, 0)$, a contradiction.

We now consider the case where (b) is satisfied. From (11) it follows that $\widehat{x} \in S(Q, 0, 0)$. By our assumption, $\widehat{x} = 0$. Using (11) once again, we have $\widehat{s} \in K^* \cap X^\perp$. From the condition $K^* \cap X^\perp = \{0\}$ it follows that $(\widehat{x}, \widehat{s}) = (0, 0)$. This is impossible. ∎

It would be of interest to find necessary and sufficient conditions for the lsc property of the map $S(\cdot)$ at a given point $(Q, c, a) \in \mathcal{L}_S(V) \times V \times V$. If the constraint set of (P) is a polyhedral convex set or a closed ball in an Euclidean space, then such conditions can be found in [7, 9, 10, 16]. For instance, it has been proved [10] that, for the problem

$$\min\Big\{\frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle \, : \, x \in V, \ \|x\|^2 \le \alpha^2\Big\}$$

with α being a positive real number, if $S(Q, \cdot, a)$ is lsc at c then $S(Q, c, a)$ is a finite set. *The complexity of the proof given in* [10] *indicates that if a similar fact is valid for (P), then the corresponding proof would be nontrivial.*

3. Stability of the Global Solution Map

This section presents several conditions for stability of the global solution map of (P). We begin with a sufficient condition for the usc property of $\mathrm{Sol}(\cdot)$.

Theorem 3.1. (Sufficient conditions for the usc property I). *Let $Q \in \mathcal{L}_S(V)$, $c \in V$, and $a \in V$. If $(a + X) \cap (\mathrm{int}K) \neq \emptyset$ and $\mathrm{Sol}(Q, 0, 0) = \{0\}$, then $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (Q, c, a).*

Proof. If the assertion of the theorem was false, then we would find an open set Ω with $\mathrm{Sol}(Q, c, a) \subset \Omega$, a sequence $\{(Q^k, c^k, a^k)\}$ converging to (Q, c, a), and a sequence $\{x^k\}$ such that $x^k \in \mathrm{Sol}(Q^k, c^k, a^k) \setminus \Omega$ for all k.

If $\{x^k\}$ is bounded, then we may assume that $x^k \to \bar{x}$ for some $\bar{x} \in V$. Since $x^k \in (a^k + X) \cap K$, $\bar{x} \in (a + X) \cap K$. Let $x \in (a + X) \cap K$ be given arbitrarily. Then there exist a subsequence $\{k_j\}$ of $\{k\}$ and a sequence $\{y^{k_j}\}$, $y^{k_j} \in (a^{k_j} + X) \cap (\mathrm{int}K)$, such that $y^{k_j} \to x$. In fact, let $x = a + x^0$ with $x^0 \in X$. Choose an $y \in (a + X) \cap (\mathrm{int}K)$. Let $y = a + y^0$ with $y^0 \in X$. We have

$$\frac{1}{j}y + (1 - \frac{1}{j})x \in \mathrm{int}K \quad \forall j \ge 2.$$

It follows that

$$a + \frac{1}{j}y^0 + (1 - \frac{1}{j})x^0 \in (a + X) \cap (\mathrm{int}K) \quad \forall j \ge 2.$$

Since $a^{k_j} \to a$, for every $j \geq 2$ there exists $k_j > j$ such that

$$y^{k_j} := a^{k_j} + \frac{1}{j}y^0 + (1 - \frac{1}{j})x^0 \in \mathrm{int}K.$$

As $\frac{1}{j}y^0 + (1 - \frac{1}{j})x^0 \in X$, we have $y^{k_j} \in (a^{k_j} + X) \cap (\mathrm{int}K)$ and $y^{k_j} \to x$ as $j \to \infty$. Since $x^k \in \mathrm{Sol}(Q^k, c^k, a^k)$, we have $f(x^{k_j}, Q^{k_j}, c^{k_j}) \leq f(y^{k_j}, Q^{k_j}, c^{k_j})$. Passing to the limit, we obtain $f(\bar{x}, Q, c) \leq f(x, Q, c)$ for all $x \in (a + X) \cap K$. This yields $\bar{x} \in \mathrm{Sol}(Q, c, a) \subset \Omega$, which contradicts the fact that $x^k \notin \Omega$ for all k and $x^k \to \bar{x}$.

Now suppose that $\{x^k\}$ is unbounded. We may assume that

$$\frac{x^k}{\|x^k\|} \to \widehat{x}, \quad \widehat{x} \in \Omega \cap X, \quad \|\widehat{x}\| = 1.$$

Fix any $x \in (a + X) \cap K$. Since $(a + X) \cap (\mathrm{int}K) \neq \emptyset$, as shown above, there exist a subsequence $\{k_j\}$ of $\{k\}$ and a sequence $\{y^{k_j}\}$, $y^{k_j} \in (a^{k_j} + X) \cap K$, such that $y^{k_j} \to x$. We have

$$f(x^{k_j}, Q^{k_j}, c^{k_j}) \leq f(y^{k_j}, Q^{k_j}, c^{k_j}).$$

Dividing this inequality by $\|x^{k_j}\|^2$ and letting $j \to \infty$, we obtain $\langle \widehat{x}, Q\widehat{x} \rangle \leq 0$. From this it follows that either $\mathrm{Sol}(Q, 0, 0) = \emptyset$, or $\widehat{x} \in \mathrm{Sol}(Q, 0, 0)$. In both cases, we have $\mathrm{Sol}(Q, 0, 0) \neq \{0\}$, a contradiction. ∎

Corollary 3.1. *Let $Q \in \mathcal{L}_S(V)$, $c \in V$, and $a \in V$. If $(a + X) \cap (\mathrm{int}K) \neq \emptyset$ and $X \cap K = \{0\}$, then $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (Q, c, a).*

Proof. The assumption $X \cap K = \{0\}$ forces $\mathrm{Sol}(Q, 0, 0) = \{0\}$. Thus $\mathrm{Sol}(\cdot)$ is usc at (Q, c, a) by the above theorem. ∎

In the next statement, the condition $\mathrm{Sol}(Q, 0, 0) = \{0\}$ in Theorem 3.1 is replaced by the requirement that $S(Q, 0, 0) = \{0\}$. But we have to impose the additional assumption $K^* \cap X^\perp = \{0\}$.

Theorem 3.2. (Sufficient conditions for the usc property II). *Let $Q \in \mathcal{L}_S(V)$, $c \in V$, and $a \in V$. If $(a+X) \cap (\mathrm{int}K) \neq \emptyset$, $S(Q, 0, 0) = \{0\}$ and $K^* \cap X^\perp = \{0\}$, then $\mathrm{Sol}(\cdot)$ is upper semicontinuous at (Q, c, a).*

Proof. If $\mathrm{Sol}(\cdot)$ is not usc at (Q, c, a), then there exist an open set $\Omega \subset V$ containing $\mathrm{Sol}(Q, c, a)$, a sequence $\{(Q^k, c^k, a^k)\}$ converging to (Q, c, a), and a sequence $\{x^k\}$ in V such that $x^k \in \mathrm{Sol}(Q^k, c^k, a^k) \setminus \Omega$ for every k. If $\{x^k\}$ is bounded, then we may assume that $x^k \to \bar{x} \in (a + X) \cap K$. Given any $x \in (a + X)\cap K$, by the condition $(a+X)\cap(\text{int}K) \neq \emptyset$ we can find a sequence $\{y^{k_j}\}$, $y^{k_j} \in (a^{k_j} + X)\cap(\text{int}K)$, where $\{k_j\}$ is a subsequence of $\{k\}$, such that $y^{k_j} \to x$ (see the proof of Theorem 3.1). Passing the inequality $f(x^{k_j}, Q^{k_j}, c^{k_j}) \leq f(y^{k_j}, Q^{k_j}, c^{k_j})$ to the limit, we get $f(\bar{x}, Q, c) \leq f(x, Q, c)$. Therefore, $\bar{x} \in \text{Sol}(Q, c, a) \subset \Omega$. This contradicts the fact that $x^k \notin \Omega$ for all k and $x^k \to \bar{x}$. Thus $\{x^k\}$ must be unbounded. We may assume that $\|x^k\| \neq 0$ for all k, and $\|x^k\| \to \infty$ as $k \to \infty$. As mentioned above, by the assumption $(a + X) \cap (\text{int}K) \neq \emptyset$ we can find a sequence $\{y^{k_j}\}$, $y^{k_j} \in (a^{k_j} + X) \cap (\text{int}K)$, where $\{k_j\}$ is a subsequence of $\{k\}$. Hence $(a^{k_j} + X) \cap (\text{int}K) \neq \emptyset$ for each j. By Theorem 1.1 we have $\text{Sol}(Q^{k_j}, c^{k_j}, a^{k_j}) \subset S(Q^{k_j}, c^{k_j}, a^{k_j})$. So there exists $s^{k_j} \in V$ such that

$$(12) \qquad \begin{cases} s^{k_j} \in K^*, \quad x^{k_j} \in (a^{k_j} + X) \cap K, \\ Qx^{k_j} + c^{k_j} - s^{k_j} \in X^{\perp}, \quad \langle x^{k_j}, s^{k_j} \rangle = 0. \end{cases}$$

Since $\|(x^{k_j}, s^{k_j})\| \geq \|x^{k_j}\| \to \infty$, we may assume that

$$\frac{(x^{k_j}, s^{k_j})}{\|(x^{k_j}, s^{k_j})\|} \to (\widehat{x}, \widehat{s}), \quad \|(\widehat{x}, \widehat{s})\| = 1.$$

Dividing the inclusions in (12) by $\|(x^{k_j}, s^{k_j})\|$ and the equality there by $\|(x^{k_j}, s^{k_j})\|^2$, then taking the limits as $j \to \infty$, we arrive at

$$(13) \qquad \widehat{s} \in K^*, \quad \widehat{x} \in X \cap K, \quad Q\widehat{x} - \widehat{s} \in X^{\perp}, \quad \langle \widehat{x}, \widehat{s} \rangle = 0.$$

This shows that $\widehat{x} \in S(Q, 0, 0)$. Hence $\widehat{x} = 0$ by our assumption. Then, by (13) we have $-\widehat{s} \in K^* \cap X^{\perp}$. From the condition $K^* \cap X^{\perp} = \{0\}$ it follows that $\widehat{s} = 0$. This is impossible because $\|(\widehat{x}, \widehat{s})\| = 1$. ■

We now establish a criterion for the lsc property of the solution map of (P).

Theorem 3.3. (Necessary and sufficient conditions for the lsc property). *Assume that* $Q \in \mathcal{L}_S(V)$, $c \in V$, $a \in V$, $(a + X) \cap (\text{int}K) \neq \emptyset$, *and* $X \cap K = \{0\}$. *Then the map* $\text{Sol}(\cdot)$ *is lower semicontinuous at* (Q, c, a) *if and only if* $\text{Sol}(Q, c, a)$ *is a singleton.*

Proof. Necessity: On the contrary, suppose that $\text{Sol}(\cdot)$ is lsc at (Q, c, a), but $\text{Sol}(Q, c, a)$ is not a singleton. Since $X \cap K = \{0\}$ and $(a + X) \cap (\text{int}K) \neq \emptyset$, $(a + X) \cap K$ is nonempty and compact, hence $\text{Sol}(Q, c, a)$ is nonempty. As $\text{Sol}(Q, c, a)$ is not a singleton, there exist $\bar{x}, \bar{y} \in \text{Sol}(Q, c, a)$ such that $\bar{x} \neq \bar{y}$. Let $c_0 \in V$ be such that $\|c_0\| = 1$ and $\langle c_0, \bar{x} - \bar{y} \rangle > 0$. Then we can choose an open neighborhood U of $\bar{x}$ such that

$$\langle c_0, x\rangle > \langle c_0, \bar{y}\rangle \quad \forall x \in U. \tag{14}$$

Let $\delta > 0$ be given arbitrarily. Fix an $\varepsilon \in (0, \delta)$ and $c' = c + \varepsilon c_0$. It holds $\|c' - c\| = \varepsilon < \delta$. We are going to show that $\mathrm{Sol}(Q, c', a) \cap U = \emptyset$. Given any $x \in [(a + X) \cap K] \cap U$, since $\bar{x}, \bar{y} \in$ Sol(Q, c, a), by (14) we have

$$\begin{aligned}\frac{1}{2}\langle x, Qx\rangle + \langle c', x\rangle &= \Big[\frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle\Big] + \varepsilon\langle c_0, x\rangle \\ &\geq \Big[\frac{1}{2}\langle \bar{x}, Q\bar{x}\rangle + \langle c, \bar{x}\rangle\Big] + \varepsilon\langle c_0, x\rangle \\ &> \Big[\frac{1}{2}\langle \bar{y}, Q\bar{y}\rangle + \langle c, \bar{y}\rangle\Big] + \varepsilon\langle c_0, \bar{y}\rangle \\ &= \frac{1}{2}\langle \bar{y}, Q\bar{y}\rangle + \langle c', \bar{y}\rangle.\end{aligned}$$

From this it follows that $x \notin \mathrm{Sol}(Q, c', a)$. Therefore, for every $\delta > 0$ there exists $c' \in V$ satisfying $\|c' - c\| < \delta$, such that $\mathrm{Sol}(Q, c', a) \cap U = \emptyset$. This contradicts the lsc property of $\mathrm{Sol}(Q, \cdot, a)$ at c. We have thus proved that $\mathrm{Sol}(Q, c, a)$ is a singleton.

Sufficiency: Suppose that $\mathrm{Sol}(Q, c, a) = \{\bar{x}\}$ and $U \subset V$ is any open set with $\bar{x} \in U$. Since $(a + X) \cap (\mathrm{int}K) \neq \emptyset$, there exists an open set $W \subset V$ containing a such that $(a' + X) \cap K \neq \emptyset$ for every $a' \in W$. Combining this with the assumption $X \cap K = \{0\}$, we see that $(a' + X) \cap K$ is nonempty and compact for every $a' \in W$. Hence $\mathrm{Sol}(Q', c', a') \neq \emptyset$ for all $(Q', c', a') \in \mathcal{L}_S(V) \times V \times W$. Since $(a + X) \cap (\mathrm{int}K) \neq \emptyset$ and $X \cap K = \{0\}$, $\mathrm{Sol}(\cdot)$ is usc at (Q, c, a) by Corollary 3.1. Thus there is a neighborhood $W_Q \times W_c \times W_a$ of (Q, c, a) in $\mathcal{L}_S(V) \times V \times V$ such that

$$\mathrm{Sol}(Q', c', a') \subset U \quad \forall (Q', c', a') \in W_Q \times W_c \times W_a.$$

Consequently, we have $\mathrm{Sol}(Q', c', a') \cap U \neq \emptyset$ for every $(Q', c', a') \in W_Q \times W_c \times (W_a \cap W)$. This shows that $\mathrm{Sol}(\cdot)$ is lsc at (Q, c, a). ■

4. Continuity of the Optimal Value Function

In this final section, we study the continuity of the real-valued function $\varphi(\cdot)$.

Theorem 4.1. (Sufficient conditions for the upper semicontinuity) *Let* $Q \in \mathcal{L}_S(V)$, $c \in V$, $a \in V$. *If* $(a + X) \cap (\mathrm{int}K) \neq \emptyset$, *then* $\varphi(\cdot)$ *is upper semicontinuous at* (Q, c, a).

Proof. Let $\{(Q^k, c^k, a^k)\}$ be any sequence converging to (Q, c, a). Choose a subsequence $\{k'\}$ of $\{k\}$ such that

$$\limsup_{k\to\infty} \varphi(Q^k, c^k, a^k) = \lim_{k'\to\infty} \varphi(Q^{k'}, c^{k'}, a^{k'}). \tag{15}$$

Since $(a + X) \cap K \neq \emptyset$, $\varphi(Q, c, a) < +\infty$. Then there exists a sequence $\{x^m\}$ in $(a + X) \cap K$ such that $\lim_{m\to\infty} f(x^m, Q, c) = \varphi(Q, c, a)$. Let $x^m = a + v^m$, where $v^m \in X$. Fix some $y \in (a + X) \cap (\text{int}K)$ and put $y = a + v^0$, where $v^0 \in X$. For any $m \geq 1$ and $\ell \geq 2$, it holds

$$\frac{1}{\ell} y + \big(1 - \frac{1}{\ell}\big) x^m \in \text{int}K.$$

Substituting $y = a + v^0$ and $x^m = a + v^m$ into this inclusion, we get

$$a + \frac{1}{\ell} v^0 + \big(1 - \frac{1}{\ell}\big) v^m \in \text{int}K.$$

As $\lim_{k'\to\infty} a^{k'} = a$, for each $\ell \geq 2$ there exists an index $k'(\ell) > \ell$ such that

$$y^{k'(\ell),m} := a^{k'(\ell)} + \frac{1}{\ell} v^0 + \big(1 - \frac{1}{\ell}\big) v^m \in \text{int}K.$$

Then we have $y^{k'(\ell),m} \in \big(a^{k'(\ell)} + X\big) \cap (\text{int}K)$ and

$$\lim_{\ell\to\infty} y^{k'(\ell),m} = a + v^m = x^m.$$

Since $y^{k'(\ell),m} \in (a^{k'(\ell)} + X) \cap K$, we have

$$\varphi\big(Q^{k'(\ell)}, c^{k'(\ell)}, a^{k'(\ell)}\big) \leq f\big(y^{k'(\ell),m}, Q^{k'(\ell)}, c^{k'(\ell)}\big).$$

It follows that

$$\lim_{\ell\to\infty} \varphi\big(Q^{k'(\ell)}, c^{k'(\ell)}, a^{k'(\ell)}\big) \leq f(x^m, Q, c).$$

Taking account of (15), from this we deduce that

$$\limsup_{k\to\infty} \varphi(Q^k, c^k, a^k) \leq f(x^m, Q, c).$$

Passing to the limit as $m \to \infty$, we get

$$\limsup_{k\to\infty} \varphi(Q^k, c^k, a^k) \leq \varphi(Q, c, a).$$

This establishes the usc property of $\varphi(\cdot)$ at (Q, c, a). ∎

Theorem 4.2. (Sufficient conditions for the continuity) *Let $Q \in \mathcal{L}_S(V)$, $c \in V$, $a \in V$. If $(a + X) \cap (\text{int}K) \neq \emptyset$ and $X \cap K = \{0\}$, then $\varphi(\cdot)$ is continuous at (Q, c, a).*

Proof. Suppose that $(a+X)\cap(\text{int}K)\neq\emptyset$ and $X\cap K=\{0\}$. By Theorem 4.1, $\varphi(\cdot)$ is usc at (Q,c,a). In order to obtain the desired lsc property $\varphi(\cdot)$, given a sequence $(Q^k,c^k,a^k)\to(Q,c,a)$, we will show that

$$\liminf_{k\to\infty}\varphi(Q^k,c^k,a^k)\geq\varphi(Q,c,a).$$

Arguing by contraposition, suppose that

$$\liminf_{k\to\infty}\varphi(Q^k,c^k,a^k)<\varphi(Q,c,a). \tag{16}$$

Choose a subsequence $\{k'\}$ of $\{k\}$ such that

$$\liminf_{k\to\infty}\varphi(Q^k,c^k,a^k)=\lim_{k'\to\infty}\varphi(Q^{k'},c^{k'},a^{k'}). \tag{17}$$

By (16) and (17), there exist k'_0 and a real constant $\alpha<\varphi(Q,c,a)$ such that

$$\varphi(Q^{k'},c^{k'},a^{k'})\leq\alpha\quad\forall k'\geq k'_0.$$

Then, for each $k'\geq k'_0$, we have $(a^{k'}+X)\cap K\neq\emptyset$. By the condition $X\cap K=\{0\}$, $(a^{k'}+X)\cap K$ is a compact set. Therefore

$$\text{Sol}(Q^{k'},c^{k'},a^{k'})\neq\emptyset\quad\forall k'\geq k'_0.$$

Thus for every $k'\geq k'_0$ there exists $x^{k'}\in(a^{k'}+X)\cap K$ such that

$$f(x^{k'},Q^{k'},c^{k'})=\varphi(Q^{k'},c^{k'},a^{k'})\leq\alpha.$$

Let $x^{k'}=a^{k'}+v^{k'}$, where $v^{k'}\in X$. If $\{x^{k'}\}$ is unbounded then $\{v^{k'}\}$ is also unbounded and, by considering a subsequence (if necessary), we have

$$\frac{v^{k'}}{\|v^{k'}\|}\to\bar{v}\in X\cap K,\quad\|\bar{v}\|=1.$$

But this contradicts the condition $X\cap K=\{0\}$. Thus $\{x^{k'}\}$ is bounded. Without loss of generality, we may assume that $x^{k'}\to\bar{x}\in(a+X)\cap K$. Hence

$$\lim_{k'\to\infty}f(x^{k'},Q^{k'},c^{k'})=f(\bar{x},Q,c)\leq\alpha.$$

Since $\alpha<\varphi(Q,c,a)$, we obtain $f(\bar{x},Q,c)<\varphi(Q,c,a)$, a contradiction. The proof is complete. ∎

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G. M. Lee
Department of Applied Mathematics,
Pukyong National University,
Pusan 608-737,
Korea
E-mail: gmlee@pknu.ac.kr

N. N. Tam
Department of Mathematics,
Hanoi Pedagogical Institute No. 2,
Xuan Hoa, Me Linh, Vinh Phuc,
Vietnam
E-mail: nnangtam@yahoo.com

N. D. Yen
Institute of Mathematics,
Vietnamese Academy of Science and Technology,
18 Hoang Quoc Viet, 10307 Hanoi,
Vietnam
E-mail: ndyen@math.ac.vn