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DUALITY IN NONDIFFERENTIABLE MULTIOBJECTIVE FRACTIONAL PROGRAMS INVOLVING CONES

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Abstract. In this paper, we introduce nondifferentiable multiobjective fractional programming problems with cone constraints over arbitrary closed convex cones, where every component of the objective function contains a term involving the support function of a compact convex set. For this problem, Wolfe and Mond-Weir type duals are proposed. We establish weak and strong duality theorems for a weakly efficient solution under suitable (V, ρ)-invexity assumptions. As special cases of our duality relations, we give some known duality results.

1. INTRODUCTION

Multiobjective fractional programming duality has been of much interest in the recent past. Duality and optimality for nondifferentiable multiobjective programming problems in which the objective function contains a support function was studied by Mond and Schechter [11]. Bector et al. [2], derived Fritz John and Karush-Kuhn-Tucker necessary and sufficient optimality conditions for a class of nondifferentiable convex multiobjective fractional programming problems and established some duality theorems. Later, Khan and Hanson [5] and Reddy and Mukherjee [14] have used the ratio invexity concept to characterize optimality and duality results in fractional programming. Motivated by various concepts of generalized convexity, Liang et al. [9] introduced a unified formulation of the generalized convexity, which was called (F, α, ρ, d)-convexity, and obtained some corresponding optimality conditions and duality results for the single-objective fractional problem. Also, they extended their results to multiobjective fractional programming problems in [8].

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Very recently, Kim et al. [6] formulated a class of nondifferentiable multiobjective fractional programs and established necessary and sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems. Subsequently, Kim et al. [7] considered two pairs of nondifferentiable multiobjective second order symmetric dual problems with cone constraints over arbitrary closed convex cones, which are Mond-Weir type and Wolfe type. And weak, strong, converse and self-duality theorems were established under the assumptions of second order pseudo-invex functions.

On the other hand, taking motivation from Bazaraa and Goode [1] and Hanson and Mond [4], Nanda and Das [13] attempted to extend the results of Mond and Weir [12] to cone domains with appropriate pseudo-invexity and quasi-invexity assumptions on objective and constraint functions. However, Chandra and Abha [3] pointed out that there are some deficiencies in the work of Nanda and Das [13]. They suggested appropriate modifications for study of duality under pseudo-invexity assumptions.

In this paper, we construct nondifferentiable multiobjective fractional programming problems with cone constraints over arbitrary closed convex cones, where every component of the objective function contains a term involving the support function of a compact convex set. For this problem, Wolfe and Mond-Weir type duals are proposed. And we establish weak and strong duality theorems for a weakly efficient solution by using (V, ρ)-invexity conditions. Moreover, we give some special cases of our duality results.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its non-negative orthant. The following convention for inequalities will be used in this paper.

If $x, u \in \mathbb{R}^n$, then

$x < u \iff u - x \in int\mathbb{R}^n_+$;

$x \leqq u \iff u - x \in \mathbb{R}^n_+$;

$x \leq u \iff u - x \in \mathbb{R}^n_+ \setminus \{0\}$;

$x \nless u$ is the negation of $x < u$.

Definition 2.1. A nonempty set C in $\mathbb{R}^n$ is said to be a cone with vertex zero, if $x \in C$ implies that $\lambda x \in C$ for all $\lambda \geqq 0$. If, in addition, C is convex, then C is called a convex cone.

Definition 2.2. The polar cone C^* of C is defined by

$$C^* = \{z \in \mathbb{R}^n \quad | \quad x^T z \leqq 0 \quad \textit{for all} \quad x \in C\}.$$

Consider the following nondifferentiable multiobjective fractional programming problem:

$$
\begin{aligned}
(\mathbf{MFP}) \quad \text{Minimize} \quad & \frac{f(x)+s(x|D)}{g(x)} \\
& = \Big(\frac{f_1(x)+s(x|D_1)}{g_1(x)}, \cdots, \frac{f_k(x)+s(x|D_k)}{g_k(x)}\Big) \\
\text{subject to} \quad & h(x) \in C_2^*,\ x \in C_1,
\end{aligned}
$$

where X_0 is an open set of $\mathbb{R}^n$, $f : X_0 \to \mathbb{R}^k, g : X_0 \to \mathbb{R}^k$ and $h : X_0 \to \mathbb{R}^m$ are continuously differentiable over X_0. C_1 and C_2 are closed convex cones with nonempty interiors in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.
We assume that

$$f(x) \geqq 0 \text{ and } g(x) > 0,\ \text{ for all } x \in X_0,$$

whenever g is not linear.

Definition 2.3. [6]. PA vector function $f : X_0 \to \mathbb{R}^k$ is said to be (V, ρ)-invex at $u \in X_0$ with respect to the functions η and $\theta : X_0 \times X_0 \to \mathbb{R}^n$ if there exists $\alpha_i : X_0 \times X_0 \to \mathbb{R}_+ \setminus \{0\}$ and $\rho_i \in \mathbb{R}, i = 1, 2, \cdots, k$, such that, for any $x \in X_0$ and for $i = 1, 2, \cdots, k$,

$$\alpha_i(x, u)[f_i(x) - f_i(u)] \geqq \nabla f_i(u)\eta(x, u) + \rho_i\|\theta(x, u)\|^2.$$

The function f is (V, ρ)-invex on X_0 if it is (V, ρ)-invex at every point in X_0

Lemma 2.1. [6]. *Assume that f and g are vector-valued differentiable functions defined on X_0 and that $f(x) + x^T w \geqq 0, g(x) > 0$ for all $x \in X_0$. If $f(\cdot) + (\cdot)^T w$ and $-g(\cdot)$ are (V, ρ)-invex at $u \in X_0$, then $[f(\cdot) + (\cdot)^T w]/g(\cdot)$ is (V, ρ)-invex at u, where*

$$\overline{\alpha}_i(x, u) = [g_i(x)/g_i(u)]\alpha_i(x, u),\ \overline{\theta}(x, u) = [1/g_i(u)]^{1/2}\theta(x, u).$$

Definition 2.4. [11] The support function $s(x|B)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y|B) \geq s(x|B) + z^T(y - x) \ \ f(x) \ \ y \in B.$$

Equivalently,

$$z^T x = s(x|B).$$

The subdifferential of $s(x|B)$ is given by

$$\partial s(x|B) := \{z \in B : z^T x = s(x|B)\}.$$

For any set $S \subset \mathbb{R}^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z-x) \leq 0 \text{ for all } z \in S\}.$$

It is readily verified that for a compact convex set B, y is in $N_B(x)$ if and only if $s(y|B) = x^T y$, or equivalently, x is in the subdifferential of s at y.

3. Mond-weir Type Duality

In this section, we propose the following dual problem $(\mathbf{MMFD})$ to $(\mathbf{MFP})$:

$$(\mathbf{MMFD}) \quad \text{Maximize} \quad \frac{f(u)+u^T w}{g(u)}$$

$$\text{(1)} \quad \text{subject to} \quad \lambda^T \nabla \left[\frac{f(u)+u^T w}{g(u)}\right] + \nabla y^T h(u) = 0,$$

$$\text{(2)} \quad -h(u) \in C_2^*, \ y \in C_2,$$
$$w_i \in D_i, \ i=1,\cdots,k, \ \lambda \geq 0, \ \lambda^T e = 1,$$

where

(i) C_2 is closed convex cone in $\mathbb{R}^m$ with nonempty interiors,
(ii) C_2^* is polar cone of C_2,
(iii) $e = (1,\cdots,1)^T$ is vector in $\mathbb{R}^k$,
(iv) $w_i (i=1,\cdots,k)$ is vector in $\mathbb{R}^n$ and $D_i (i=1,\cdots,k)$ is compact convex set in $\mathbb{R}^n$,
(v) $u^T w = (u^T w_1, \cdots, u^T w_k)^T$.

Now we establish the duality theorems of $(\mathbf{MFP})$ and $(\mathbf{MMFD})$.

Theorem 3.1. (Weak Duality). *Let x and (u,y,λ,w) be feasible solutions of $(\mathbf{MFP})$ and $(\mathbf{MMFD})$, respectively. Assume that $f_i(\cdot)+(\cdot)^T w_i$ and $-g_i(\cdot), i=1,\cdots,k$, are (V,ρ_i)-invex at u and $y^T h(\cdot)$ is (V,σ)-invex at u with respect to the same η with $\rho^T\lambda \geqq 0$ and $\sigma \geqq 0$. Then*

$$\frac{f(x)+s(x|D)}{g(x)} \nless \frac{f(u)+u^T w}{g(u)}.$$

Proof. Assume to the contrary that

$$\frac{f(x)+s(x|D)}{g(x)} < \frac{f(u)+u^Tw}{g(u)}.$$

Since $\overline{\alpha}_i(x,u)>0, i=1,2,\cdots,k$, and $\lambda \geq 0$, we have

$$\sum_{i=1}^{k}\overline{\alpha}(x,u)\lambda_i\Big[\frac{f_i(x)+s(x|D_i)}{g_i(x)}\Big] < \sum_{i=1}^{k}\overline{\alpha}(x,u)\lambda_i\Big[\frac{f_i(u)+u^Tw_i}{g_i(u)}\Big]. \tag{3}$$

By Lemma 2.1, we get

$$\begin{aligned}&\overline{\alpha}_i(x,u)\Big[\frac{f_i(x)+x^Tw_i}{g_i(x)}-\frac{f_i(u)+u^Tw_i}{g_i(u)}\Big]\\ &\geqq \nabla\Big[\frac{f_i(u)+u^Tw_i}{g_i(u)}\Big]\eta(x,u)+\rho_i\|\overline{\theta}(x,u)\|^2,\ i=1,2,\cdots,k.\end{aligned}$$

Since $\lambda \geq 0$, it implies that

$$\begin{aligned}&\overline{\alpha}_i(x,u)\lambda_i\Big[\frac{f_i(x)+x^Tw_i}{g_i(x)}-\frac{f_i(u)+u^Tw_i}{g_i(u)}\Big]\\ &\geqq \lambda_i\nabla\Big[\frac{f_i(u)+u^Tw_i}{g_i(u)}\Big]\eta(x,u)+\rho_i\lambda_i\|\overline{\theta}(x,u)\|^2,\ i=1,2,\cdots,k,\end{aligned}$$

(4) i.e.,

$$\begin{aligned}&\sum_{i=1}^{k}\overline{\alpha}_i(x,u)\lambda_i\Big[\frac{f_i(u)+x^Tw_i}{g_i(x)}-\frac{f_i(u)+u^Tw_i}{g_i(u)}\Big]\\ &\geqq \sum_{i=1}^{k}\lambda_i\nabla\Big[\frac{f_i(u)+u^Tw_i}{g_i(u)}\Big]\eta(x,u)+\sum_{i=1}^{k}\rho_i\lambda_i\|\overline{\theta}(x,u)\|^2.\end{aligned}$$

Also, by (V,σ)-invexity of $y^Th(\cdot)$, we get

$$\beta(x,u)[y^Th(x)-y^Th(u)] \geqq \nabla y^Th(u)\eta(x,u)+\sigma\|\theta(x,u)\|^2. \tag{5}$$

Adding (4) and (5), we obtain

$$\begin{aligned}&\sum_{i=1}^{k}\overline{\alpha}_i(x,u)\lambda_i\Big[\frac{f_i(u)+x^Tw_i}{g_i(x)}-\frac{f_i(u)+u^Tw_i}{g_i(u)}\Big]+\beta(x,u)[y^Th(x)-y^Th(u)]\\ &\geqq \Big[\sum_{i=1}^{k}\lambda_i\nabla\Big(\frac{f_i(u)+u^Tw_i}{g_i(u)}\Big)+\nabla y^Th(u)\Big]\eta(x,u)\\ &\quad+\sum_{i=1}^{k}\rho_i\lambda_i\|\overline{\theta}(x,u)\|^2+\sigma\|\theta(x,u)\|^2.\end{aligned}$$

From the dual constraint (2) and $h(x) \in C_2^*$, we obtain $y^T h(x) \leqq y^T h(u)$. So, the above inequality implies that

$$\begin{aligned}
&\sum_{i=1}^{k} \overline{\alpha}_i(x,u)\lambda_i\Big[\frac{f_i(u)+x^T w_i}{g_i(x)} - \frac{f_i(u)+u^T w_i}{g_i(u)}\Big] \\
&\geqq \Big[\sum_{i=1}^{k} \lambda_i \nabla\Big(\frac{f_i(u)+u^T w_i}{g_i(u)}\Big) + \nabla y^T h(u)\Big]\eta(x,u) \\
&\quad + \sum_{i=1}^{k} \rho_i \lambda_i \|\overline{\theta}(x,u)\|^2 + \sigma\|\theta(x,u)\|^2.
\end{aligned}$$

By the dual constraint (1), it yields

$$\begin{aligned}
&\sum_{i=1}^{k} \overline{\alpha}_i(x,u)\lambda_i\Big[\frac{f_i(u)+x^T w_i}{g_i(x)} - \frac{f_i(u)+u^T w_i}{g_i(u)}\Big] \\
&\geqq \sum_{i=1}^{k} \rho_i \lambda_i \|\overline{\theta}(x,u)\|^2 + \sigma\|\theta(x,u)\|^2 \\
&\geqq 0.
\end{aligned}$$

Using the fact that $s(x|D_i) \geqq x^T w_i, i = 1, 2, \cdots, k$, it follows that

$$\sum_{i=1}^{k} \overline{\alpha}_i(x,u)\lambda_i\Big[\frac{f_i(u)+s(x|D_i)}{g_i(x)} - \frac{f_i(u)+u^T w_i}{g_i(u)}\Big] \geqq 0,$$

which contradicts (3). Thus,

$$\frac{f(x)+s(x|D)}{g(x)} \not< \frac{f(u)+u^T w}{g(u)}. \qquad \blacksquare$$

We obtain the following lemma from [1] and [6] in order to prove strong duality theorem.

Lemma 3.1. *If* $\overline{x}$ *is a weakly efficient solution of* $(\mathbf{MFP})$ *at which constraint qualification [10] be satisfied. Then there exist* $\overline{w}_i \in D_i (i = 1, \cdots, k), \overline{\lambda} \geq 0$ *and* $\overline{y} \in C_2$ *with* $(\overline{\lambda}, \overline{y}) \neq 0$ *such that*

$$\begin{aligned}
&\Big[\overline{\lambda}^T \nabla\Big(\frac{f(\overline{x})+\overline{x}^T\overline{w}}{g(\overline{x})}\Big) + \nabla \overline{y}^T h(\overline{x})\Big]^T (x-\overline{x}) \geqq 0, \ \textit{for all} \ \ x \in C_1, \\
&\overline{y}^T h(\overline{x}) = 0, \\
&\overline{w}_i \in D_i, \ s(\overline{x}|D_i) = \overline{x}^T \overline{w}_i, \ i = 1, \cdots, k.
\end{aligned}$$

Theorem 3.2. (Strong Duality). *If $\overline{x}$ is a weakly efficient solution of* $(\mathbf{MFP})$ *at which constraint qualification [10] be satisfied. Then there exist* $\overline{\lambda} \geq 0, \overline{y} \in C_2$ *and* $\overline{w}_i \in D_i(i = 1, \cdots, k)$ *such that* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ *is feasible for* $(\mathbf{MMFD})$ *and the objective values of* $(\mathbf{MFP})$ *and* $(\mathbf{MMFD})$ *are equal. If the assumption of Theorem 3.1 are satisfied, then* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ *is weakly efficient for* $(\mathbf{MMFD})$.

Proof. Since $\overline{x}$ is a weakly efficient solution of $(\mathbf{MFP})$, by Lemma 3.1, then there exist $\overline{w}_i \in D_i, i = 1, \cdots, k, \overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

$$\left[\overline{\lambda}^T \nabla\Big(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\Big) + \nabla \overline{y}^T h(\overline{x})\right]^T (x - \overline{x}) \geqq 0, \text{ for all } \ x \in C_1, \tag{6}$$

$$\overline{y}^T h(\overline{x}) = 0, \tag{7}$$

$$\overline{w}_i \in D_i, \ s(\overline{x}|D_i) = \overline{x}^T \overline{w}_i, \ i = 1, \cdots, k. \tag{8}$$

Since $x \in C_1, \ \overline{x} \in C_1$ and C_1 is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (6) implies

$$\left[\overline{\lambda}^T \nabla\Big(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\Big) + \nabla \overline{y}^T h(\overline{x})\right]^T x \geqq 0, \text{ for all } \ x \in C_1,$$

i.e.,

$$\overline{\lambda}^T \nabla\Big[\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\Big] + \nabla \overline{y}^T h(\overline{x}) = 0.$$

And (7) implies $\overline{y}^T h(\overline{x}) \geqq 0$, then $-h(\overline{x}) \in C_2^*$. Clearly, using (8), $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for $(\mathbf{MMFD})$ and corresponding values of $(\mathbf{MFP})$ and $(\mathbf{MMFD})$ are equal. If the assumptions of Theorem 3.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is a weakly efficient solution of $(\mathbf{MMFD})$. ■

Remark 3.1. In the dual problem $(\mathbf{MMFD})$, if we replace the condition of $\lambda \geq 0$ by $\lambda > 0$, then Theorems 3.1 and 3.2 hold in the sense of efficient solutions.

4. Wolfe Type Duality

In this section, we propose the following dual problem $(\mathbf{MWFD})$ to $(\mathbf{MFP})$:

$(\mathbf{MWFD})$ Maximize $\dfrac{f(u) + u^T w}{g(u)} + y^T h(u) e$

(9) subject to

$$\lambda^T \nabla\Big[\frac{f(u) + u^T w}{g(u)}\Big] + \nabla y^T h(u) = 0,$$

$$y \in C_2, \ w_i \in D_i, \ i = 1, \cdots, k,$$

$$\lambda \geq 0, \ \lambda^T e = 1,$$

where

(i) C_2 is closed convex cone in $\mathbb{R}^m$ with nonempty interiors,
(ii) C_2^* is polar cone of C_2,
(iii) $e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^k$,
(iv) $w_i (i = 1, \cdots, k)$ is vector in $\mathbb{R}^n$ and $D_i (i = 1, \cdots, k)$ is compact convex set in $\mathbb{R}^n$,
(v) $u^T w = (u^T w_1, \cdots, u^T w_k)^T$.

Now we establish the duality theorems of $(\mathbf{MFP})$ and $(\mathbf{MWFD})$.

Theorem 4.1. (Weak Duality). *Let x and (u, y, λ, w) be feasible solutions of $(\mathbf{MFP})$ and $(\mathbf{MWFD})$, respectively. Assume that $f_i(\cdot) + (\cdot)^T w_i, -g_i(\cdot), i = 1, \cdots, k$ and $y^T h(\cdot)$ are (V, ρ_i)-invex at u with $\rho^T \lambda \geqq 0$. Then*

$$\frac{f(x) + s(x|D)}{g(x)} \not< \frac{f(u) + u^T w}{g(u)} + y^T h(u) e.$$

Proof. Assume to the contrary that

$$\frac{f(x) + s(x|D)}{g(x)} < \frac{f(u) + u^T w}{g(u)} + y^T h(u) e.$$

Since $\overline{\alpha}_i(x, u) > 0, i = 1, 2, \cdots, k$ and $\lambda \geq 0$, we obtain

$$\sum_{i=1}^{k} \overline{\alpha}_i(x, u) \lambda_i \Big[\frac{f_i(x) + s(x|D_i)}{g_i(x)} - \frac{f_i(u) + u^T w_i}{g_i(u)} \Big] < \sum_{i=1}^{k} \overline{\alpha}_i(x, u) \lambda_i y^T h(u). \tag{10}$$

By Lemma 2.1 and $\lambda \geq 0$, it yields

$$\sum_{i=1}^{k} \overline{\alpha}_i(x, u) \lambda_i \Big[\frac{f_i(x) + x^T w_i}{g_i(x)} + y^T h(x) - \frac{f_i(u) + u^T w_i}{g_i(u)} - y^T h(u) \Big]$$
$$\geqq \sum_{i=1}^{k} \lambda_i \nabla \Big[\frac{f_i(u) + u^T w_i}{g_i(u)} + y^T h(u) \Big] \eta(x, u) + \sum_{i=1}^{k} \rho_i \lambda_i \|\overline{\theta}(x, u)\|^2.$$

Also, by $y^T h(x) \leqq 0$ and the dual constraint (9), it follows that

$$\sum_{i=1}^{k} \overline{\alpha}_i(x, u) \lambda_i \Big[\frac{f_i(x) + x^T w_i}{g_i(x)} - \frac{f_i(u) + u^T w_i}{g_i(u)} - y^T h(u) \Big]$$

$$\geqq \sum_{i=1}^{k} \rho_i \lambda_i \|\overline{\theta}(x,u)\|^2$$
$$\geqq 0.$$

Using the fact that $s(x|D_i) \geqq x^T w_i, i = 1, 2, \cdots, k$, the above inequality becomes

$$\sum_{i=1}^{k} \overline{\alpha}_i(x,u)\lambda_i \Big[\frac{f_i(x) + s(x|D_i)}{g_i(x)} - \frac{f_i(u) + u^T w_i}{g_i(u)} - y^T h(u)\Big] \geqq 0,$$

which contradicts (10). ■

Theorem 4.2. (Strong Duality). *If* $\overline{x}$ *is a weakly efficient solution of* $(\mathbf{MFP})$ *at which constraint qualification [10] be satisfied. Then there exist* $\overline{\lambda} \geq 0, \overline{y} \in C_2$ *and* $\overline{w}_i \in D_i (i = 1, \cdots, k)$ *such that* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ *is feasible for* $(\mathbf{MWFD})$ *and the objective values of* $(\mathbf{MFP})$ *and* $(\mathbf{MWFD})$ *are equal. If the assumption of Theorem 4.1 are satisfied, then* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ *is weakly efficient for* $(\mathbf{MWFD})$.

Proof. Since $\overline{x}$ is a weakly efficient solution of $(\mathbf{MFP})$, by Lemma 3.1, then there exist $\overline{w}_i \in D_i, i = 1, \cdots, k, \overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

(11) $$\Big[\overline{\lambda}^T \nabla\Big(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\Big) + \nabla \overline{y}^T h(\overline{x})\Big]^T (x - \overline{x}) \geqq 0, \text{ for all } x \in C_1,$$

(12) $$\overline{y}^T h(\overline{x}) = 0,$$

(13) $$\overline{w}_i \in D_i,\ s(\overline{x}|D_i) = \overline{x}^T \overline{w}_i,\ i = 1, \cdots, k.$$

Since $x \in C_1,\ \overline{x} \in C_1$ and C_1 is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (11) implies

$$\Big[\overline{\lambda}^T \nabla\Big(\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\Big) + \nabla \overline{y}^T h(\overline{x})\Big]^T x \geqq 0, \text{ for all } x \in C_1,$$

i.e.,

$$\overline{\lambda}^T \nabla\Big[\frac{f(\overline{x}) + \overline{x}^T \overline{w}}{g(\overline{x})}\Big] + \nabla \overline{y}^T h(\overline{x}) = 0.$$

Clearly, using (12) and (13), $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for $(\mathbf{MWFD})$ and corresponding values of $(\mathbf{MFP})$ and $(\mathbf{MWFD})$ are equal. If the assumptions of Theorem 4.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is a weakly efficient solution of $(\mathbf{MWFD})$. ■

Remark 4.2. In the dual problem $(\mathbf{MWFD})$, if we replace the condition of $\lambda \geq 0$ by $\lambda > 0$, then Theorems 4.1 and 4.2 hold in the sense of efficient solutions.

5. Special Cases

We give some special cases of our dual programming. Let $C_1 = \mathbb{R}^n_+$, $C_2 = \mathbb{R}^m_+$.

(i) If $D_i = \{0\}$, $i = 1, \cdots, k$, and $k = 1$, then $(\mathbf{MFP})$ and $(\mathbf{MMFD})$ reduced to the problems considered in [5], [9] and [14].

(ii) If $D_i = \{0\}$, $i = 1, \cdots, k$, then our primal and dual models become dual programs considered in [2] and [8].

(iii) If $C_1 = \mathbb{R}^n_+$, $C_2 = \mathbb{R}^m_+$, then our dual programs become the nondifferentiable programming problems studied by [6].

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