TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 6A, pp. 1787-1810, December 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON ϵ-OPTIMALITY CONDITIONS FOR CONVEX SET-VALUED OPTIMIZATION PROBLEMS

Gue Myung Lee and Le Anh Tuan

Dedicated to Professor Boris Mordukhovich in celebration of his 60th birthday

Abstract. In this paper, ϵ-subgradients for convex set-valued maps are defined. We prove an existence theorem for ϵ-subgradients of convex set-valued maps. Also, we give necessary ϵ- optimality conditions for an ϵ-solution of a convex set-valued optimization problem (CSP). Moreover, using the single-valued function induced from the set-valued map, we obtain theorems describing the ϵ-subgradient sum formula for two convex set-valued maps, and then give necessary and sufficient ϵ-optimality conditions for the problem (CSP).

1. INTRODUCTION

Recently, there have been intensive researches for set-valued optimization problems ([1, 2, 4-7, 10, 13, 17]), which consist of set-valued maps and sets. To get optimality conditions for solutions of set-valued optimization problems, we need generalized derivatives (epiderivatives) for set-valued maps and so, most of researchers have used contingent derivatives (epiderivatives) which are defined by contigent cones.

From computational view, most of algorithms give us ϵ-solutions (approximate solutions) of optimization problems. Thus many researchers have studied optimality conditions for ϵ-solutions for scalar optimization problems and vector optimization problems ([8, 11, 12, 14, 15, 18, 19]). However, there are very little results for optimality conditions for ϵ-solution (approximate solution) of set-valued optimization problems. Moreover, it seems that contigent derivatives (epiderivatives) are not

Received June 30, 2009.
2000 *Mathematics Subject Classification*: 90C25, 90C46.
Key words and phrases: Convex set-valued map, ϵ-Subgradient, Convex set-valued optimization problem, ϵ-Solution, ϵ-Optimality condition.
This work was supported by the Korea Science and Engineering Foundation (KOSEF) NRL Program grant funded by the Korea government(MEST) (No. ROA-2008-000-20010-0).

suitable for getting optimality conditions for ϵ-solutions of set-valued optimization problems.

The purpose of this paper is to define ϵ-subgradients for set-valued maps with the closed convex cones generated by their epigraphs and to establish optimality conditions for ϵ-solutions of a convex set-valued optimization.

Now we recall some notations and preliminary results, which will be used throughout the paper.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then for $\epsilon \geqq 0$, the ϵ-subgradient of f at $\bar{x} \in \mathrm{dom} f$ is defined as the set

$$\partial_\epsilon f(\bar{x}) := \{v \in \mathbb{R}^n \mid f(x) \geqq f(\bar{x}) + v^T(x - \bar{x}) - \epsilon \text{ for any } x \in \mathrm{dom} f\},$$

where the effective domain of f, dom$f$, is given by

$$\mathrm{dom} f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}.$$

When $\epsilon = 0$, $\partial_0 f(\bar{x})$ is denoted by $\partial f(\bar{x})$ and is called the subgradient of f at $\bar{x}$ (see [8, 9, 16]). We define the indicator function of a convex subset C of $\mathbb{R}^n$ as follows:

$$\delta_C(x) = \begin{cases} 0 & \text{if } \ x \in C \\ +\infty & \text{if } \ x \notin C. \end{cases}$$

Hence, if $\bar{x} \in C$ and $\epsilon \geqq 0$, then

$$\partial_\epsilon \delta_C(\bar{x}) = \{v \in \mathbb{R}^n \mid v^T(x - \bar{x}) \leqq \epsilon \text{ for any } x \in C\}.$$

We denote $\partial_\epsilon \delta_C(x)$ by $N_C^\epsilon(\bar{x})$, which is called the ϵ-normal set of C at $\bar{x}$. When $\epsilon = 0$, $\partial \delta_C(\bar{x}) = \partial_0 \delta_C(\bar{x}) = \{v \in \mathbb{R}^n \mid v^T(x - \bar{x}) \leqq 0 \text{ for any } x \in C\}$. We denote $\partial \delta_C(\bar{x})$ by $N_C(\bar{x})$, which is called the normal cone of C at $\bar{x}$. If C is a closed convex cone in $\mathbb{R}^n$, then for any $\epsilon \geqq 0$,

$$N_C^\epsilon(0) = N_C(0).$$

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ be a set-valued map. The domain of F, dom$F$, and the epigraph of F, epi$F$, are defined as follows:

$$\mathrm{dom} F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\},$$
$$\mathrm{epi} F := \{(x, y + \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \mathrm{dom} F,\ y \in F(x),\ \alpha \geqq 0\}.$$

Definition 1.1. A set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ is said to be convex if for any $x, y \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$,

$$\lambda F(x) + (1 - \lambda) F(y) \subset F(\lambda x + (1 - \lambda) y) + \mathbb{R}_+,$$

where $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geqq 0\}$ ($\mathbb{R}_+$ is called the nonnegative real half-line).

Obviously, a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is also a convex set-valued map.

If $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ is a convex set-valued map, then epi$F$ is a convex subset of $\mathbb{R}^{n+1}$(see Lemma 1 in [10]). The cone generated by a nonempty subset M of $\mathbb{R}^{n+1}$ is denoted by

$$\text{cone}(M) := \{\lambda x \mid \lambda \geqq 0,\ x \in M\},$$

and the closure of cone(M) is denoted by $\overline{\text{cone}}(M)$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Recall that the conjugate function of f, $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ defined by for any $v \in \mathbb{R}^n$

$$f^*(v) = \sup\{v^T x - f(x) \mid x \in \mathbb{R}^n\}.$$

Similarly, for a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$, we define the conjugate function of F, $F^* : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ by for any $v \in \mathbb{R}^n$,

$$F^*(v) = \sup\{v^T x - y \mid x \in \mathbb{R}^n,\ y \in F(x)\}.$$

For the proper lower semicontinuous convex functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the infimal convolution of f_1 with f_2 is denoted by $f_1 \Box f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$, and is defined by

$$(f_1 \Box f_2)(x) = \inf_{x_1 + x_2 = x} \{f_1(x_1) + f_2(x_2)\}.$$

Definition 1.2. Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ be a convex set-valued map, and $\bar{x} \in \text{dom}F$ and $\bar{y} \in F(\bar{x})$. Let $\epsilon \geqq 0$. Define, for any $x \in \mathbb{R}^n$,

$$D_\epsilon F(\bar{x}; \bar{y})(x) := \inf\{\lambda \mid (x, \lambda) \in \overline{\text{cone}}\ [\text{epi}F - (\bar{x}, \bar{y} - \epsilon)]\},$$
$$\partial_\epsilon F(\bar{x}; \bar{y}) := \{v \in \mathbb{R}^n \mid D_\epsilon F(\bar{x}; \bar{y})(x) \geqq D_\epsilon F(\bar{x}; \bar{y})(0) + v^T x \text{ for any } x \in \mathbb{R}^n\}.$$

If $x \notin \text{Pr}_{\mathbb{R}^n} \overline{\text{cone}}\ [\text{epi}F - (\bar{x}, \bar{y} - \epsilon)]$, where Pr is the projection onto $\mathbb{R}^n$, then we let $D_\epsilon F(\bar{x}; \bar{y})(x) = +\infty$. We say that $D_\epsilon F(\bar{x}; \bar{y})$ is the radial ϵ-epiderivative of F at $(\bar{x}, \bar{y})$ and that $\partial_\epsilon F(\bar{x}; \bar{y})$ is the ϵ-subgradient of F at $(\bar{x}, \bar{y})$. Moreover, we denote $D_0 F(\bar{x}; \bar{y})$ by $DF(\bar{x}; \bar{y})$, and $\partial_0 F(\bar{x}; \bar{y})$ by $\partial F(\bar{x}; \bar{y})$. We say that $DF(\bar{x}; \bar{y})$ is the radial epiderivative of F at $(\bar{x}, \bar{y})$ (see [6] for the definition of the radial epiderivative) and that $\partial F(\bar{x}; \bar{y})$ is the subgradient of F at $(\bar{x}, \bar{y})$.

Now we give the set-valued version of the indicator function δ_C as follows:

$$\widetilde{\delta}_C(x) = \begin{cases} \{0\} & \text{if } x \in C \\ \emptyset & \text{if } x \notin C. \end{cases}$$

Then we can check that if $\bar{x} \in C$ and $\epsilon \geqq 0$, $\partial_\epsilon \widetilde{\delta}_C(\bar{x}; 0) = N_C^\epsilon(\bar{x})$. Indeed, let $\bar{x} \in C$. Clearly, $D_\epsilon \widetilde{\delta}_C(\bar{x}; 0)(0) \leqq 0$. Moreover, we can easily check that $0 \leqq D_\epsilon \widetilde{\delta}_C(\bar{x}; 0)(0)$.

So, $D_\epsilon\widetilde{\delta}_C(\bar{x};0)(0) = 0$. Notice that $v \in \partial_\epsilon\tilde{\delta}_C(\bar{x};0)$ if and only if for any $x \in \mathbb{R}^n$, $D_\epsilon\widetilde{\delta}_C(\bar{x};0)(x) \geqq v^T x$. Since epi $D_\epsilon\widetilde{\delta}_C(\bar{x};0) = \overline{\text{cone}}(C \times \mathbb{R}_+ - (\bar{x}, -\epsilon))$, $v \in \partial_\epsilon\widetilde{\delta}_C(\bar{x};0)$ if and only if for any $(x,\alpha) \in C \times \mathbb{R}_+ - (\bar{x}, -\epsilon)$,

$$(v, -1)^T(x, \alpha) \leqq 0.$$

Thus, $v \in \partial_\epsilon\widetilde{\delta}_C(\bar{x};0)$ if and only if for any $x \in C$ and any $\alpha \geqq 0$,

$$v^T(x - \bar{x}) \leqq \alpha + \epsilon.$$

Hence, $\partial_\epsilon\widetilde{\delta}_C(\bar{x};0) = N_C^\epsilon(\bar{x})$.

Using the above argument used for proving that $\partial_\epsilon\widetilde{\delta}_C(\bar{x};0) = N_C^\epsilon(\bar{x})$, we can prove that if F is a single-valued map, then $\partial_\epsilon F(\bar{x};\bar{y})$ becomes the usual ϵ-subgradient $\partial_\epsilon F(\bar{x})$ at $\bar{x}$.

In this paper, we consider the following convex set-valued optimization problem:

$$\text{(CSP)} \qquad \text{Minimize} \quad F(x) \qquad \text{subject to} \quad x \in C,$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ is a convex set-valued map and C is a nonempty closed convex subset of $\mathbb{R}^n$. Let $\epsilon \geqq 0$, $\bar{x} \in C$ and $\bar{y} \in F(\bar{x})$. Then $(\bar{x}, \bar{y})$ is said to be an ϵ-solution of (CSP) if for any $x \in C \cap \text{dom}F$ and any $y \in F(x)$,

$$\bar{y} - \epsilon \leqq y,$$

and $(\bar{x}, \bar{y})$ is called a solution of (CSP) if for any $x \in C \cap \text{dom}F$ and any $y \in F(x)$,

$$\bar{y} \leqq y.$$

This paper is organized as follows. In Section 2, we prove existence theorems for ϵ-subgradients of convex set-valued maps. We give a necessary optimality condition for an ϵ-solution of Problem (CSP) in Section 3 and introduce necessary and sufficient ϵ-optimality conditions for an ϵ-solution of (CSP) in Section 4. In particular, the ϵ-solution set of (CSP) is characterized at Theorem 4.5 in Section 4.

2. Existence of ϵ-Subgradients

In this section, we prove propositions which tell about the existence for ϵ-subgradients of convex set-valued maps.

Proposition 2.1. *Let* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ *be a convex set-valued map. Let* $\epsilon \geqq 0$, *and* $\bar{x} \in \text{int dom}F$ *and* $\bar{y} \in F(\bar{x})$. *Assume that* $(\bar{x}, \bar{y} - \epsilon) \notin \text{int epi}F$. *Then we have,*

(i) $D_\epsilon F(\bar{x};\bar{y}) : \mathbb{R}^n \to \mathbb{R}$ *is finite-valued, and sublinear, that is, for any* $x, y \in \mathbb{R}^n$,

$$D_\epsilon F(\bar{x};\bar{y})(x+y) \leqq D_\epsilon F(\bar{x};\bar{y})(x) + D_\epsilon F(\bar{x};\bar{y})(y)$$

and for any $x \in \mathbb{R}^n$ *and any* $\alpha \geqq 0,\ D_\epsilon F(\bar{x};\bar{y})(\alpha x) = \alpha D_\epsilon F(\bar{x};\bar{y})(x)$.

(ii) $\partial_\epsilon F(\bar{x};\bar{y})$ *is a nonempty convex compact subset of* $\mathbb{R}^n$.

Proof. Since $(\bar{x}, \bar{y}-\epsilon) \notin \text{int epi}F,\ (0,0) \notin \text{int epi}F - (\bar{x}, \bar{y}-\epsilon)$. Let $\Omega := \text{epi}F - (\bar{x}, \bar{y}-\epsilon)$. From the convexity of the set $\text{int epi}F - (\bar{x}, \bar{y}-\epsilon)$ and from separation theorem, there exists $(a,b) \in \mathbb{R}^n \times \mathbb{R},\ (a,b) \neq (0,0)$ such that for any $(x,y) \in \Omega,\quad a^T x + by \geqq 0$, and hence for any $(x,y) \in \overline{\text{cone}}(\Omega)$,

$$a^T x + by \geqq 0. \tag{2.1}$$

If $b = 0,$ then $a^T x \geqq 0$ for any $x \in \text{Pr}_{\mathbb{R}^n}\overline{\text{cone}}(\Omega)$. This shows that $a^T x \geqq 0$ for any $x \in \text{dom}F - \bar{x}$, and hence

$$a^T(x-\bar{x}) \geqq 0 \ \text{ for any } \ x \in \text{dom}F. \tag{2.2}$$

Since $\bar{x} \in \text{int dom}F$, we can find $\delta > 0$ such that $\bar{x} + B_\delta(0) \subset \text{dom}F$, where $B_\delta(0) = \{x \in \mathbb{R}^n \mid \|x\| < \delta\}$. Thus, from (2.2), for any $x \in B_\delta(0)$, $a^T x \geqq 0$ and so, $a = 0$. Therefore, $b \neq 0$. Moreover, for any $r \geqq 0$, $(0, r+\epsilon) = (\bar{x}, \bar{y}+r) - (\bar{x}, \bar{y}-\epsilon) \in \Omega$. From (2.1), $b > 0$, and hence for any $(x,y) \in \overline{\text{cone}}(\Omega)$, $y \geqq -\frac{1}{b}a^T x$. This means that for any $x \in \text{Pr}_{\mathbb{R}^n}\overline{\text{cone}}(\Omega)$, $D_\epsilon F(\bar{x}, \bar{y})(x) \geqq -\frac{1}{b}a^T x$. Since $\bar{x} \in \text{int dom}F$, we can check that for any $x \in \mathbb{R}^n$,

$$D_\epsilon F(\bar{x};\bar{y})(x) \geqq -\frac{1}{b}a^T x.$$

Moreover, we can easily check that

$$\text{epi}D_\epsilon F(\bar{x};\bar{y}) = \overline{\text{cone}}(\Omega).$$

This means that $D_\epsilon F(\bar{x};\bar{y})$ is sublinear. Thus the function $D_\epsilon F(\bar{x};\bar{y}) : \mathbb{R}^n \to \mathbb{R}$ is finite-valued and sublinear. Since $\partial_\epsilon F(\bar{x};\bar{y}) = \partial D_\epsilon F(\bar{x};\bar{y})(0)$, $\partial_\epsilon F(\bar{x},\bar{y})$ is a nonempty compact convex set (see [16]). ∎

Remark 2.1. Observe that by Proposition 2.1, for any $x \in \mathbb{R}^n$, $D_\epsilon F(\bar{x};\bar{y})(0) = 0$ and $D_\epsilon F(\bar{x};\bar{y})(x) > -\infty$ and so, $D_\epsilon F(\bar{x};\bar{y})$ is proper and sublinear. Moreover, since $\partial_\epsilon F(\bar{x};\bar{y}) = \partial D_\epsilon F(\bar{x};\bar{y})(0)$, $v \in \partial_\epsilon F(\bar{x};\bar{y})$ if and only if for any $x \in \mathbb{R}^n$, $D_\epsilon F(\bar{x};\bar{y})(x) \geqq v^T x$. Thus we can easily check that $v \in \partial_\epsilon F(\bar{x};\bar{y})$ if and only if for any $(x,\lambda) \in \text{epi}F - (\bar{x}, \bar{y}-\epsilon)$, $v^T x \leqq \lambda$. This shows that $(\bar{x},\bar{y})$ is an ϵ-solution of (CSP) in the case $C = \mathbb{R}^n$ if and only if $0 \in \partial_\epsilon F(\bar{x};\bar{y})$.

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ is a set-valued map. Let us define $F_{\inf}(x) := \inf\{y \mid y \in F(x)\}$ if $x \in \mathrm{dom}F$ and $F_{\inf}(x) = +\infty$ if $x \notin \mathrm{dom}F$, and $\widetilde{F}(x) := F(x) \cup \{F_{\inf}(x)\}$ for all $x \in \mathbb{R}^n$.

Proposition 2.2. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ be a convex set-valued map.*

(i) *If $F_{\inf}(x) > -\infty$ for all $x \in \mathrm{dom}F$, then $F_{\inf}$ is a proper convex function. If we assume furthermore that $\mathrm{dom}F$ and $\mathrm{epi}F_{\inf}$ are closed, then $F_{\inf}$ is lower semicontinuous on $\mathbb{R}^n$.*

(ii) *For any $\epsilon \geqq 0$, and any $\bar{x} \in \mathrm{int}\ \mathrm{dom}F$, $\partial_\epsilon \widetilde{F}(\bar{x}; F_{\inf}(\bar{x})) \neq \emptyset$ and*

$$\partial_\epsilon \widetilde{F}(\bar{x}; F_{\inf}(\bar{x})) = \partial_\epsilon F_{\inf}(\bar{x}).$$

If in addition that $F_{\inf}(x) \in F(x)$ for all $x \in \mathrm{dom}F$, then for any $\epsilon \geqq 0$, and any $\bar{x} \in \mathrm{int}\ \mathrm{dom}F$,

$$\partial_\epsilon F(\bar{x}; F_{\inf}(\bar{x})) = \partial_\epsilon F_{\inf}(\bar{x}).$$

Proof. (i) Obviously, we only need to prove that $F_{\inf}$ is a convex function on $\mathrm{dom}F$. Assume to the contrary that there exist $x_1, x_2 \in \mathrm{dom}F$ and $\lambda \in (0,1)$ such that

$$F_{\inf}(x_\lambda) > \lambda F_{\inf}(x_1) + (1-\lambda)F_{\inf}(x_2), \tag{2.3}$$

where $x_\lambda = \lambda x_1 + (1-\lambda)x_2$. Let us choose δ such that $0 < \delta < F_{\inf}(x_\lambda) - (\lambda F_{\inf}(x_1) + (1-\lambda)F_{\inf}(x_2))$. By the definitions of $F_{\inf}(x_1)$ and $F_{\inf}(x_2)$, we can find $y_1 \in F_{\inf}(x_1)$, $y_2 \in F_{\inf}(x_2)$ such that

$$\begin{cases} F_{\inf}(x_1) > y_1 - \delta \\ F_{\inf}(x_2) > y_2 - \delta. \end{cases}$$

From these and from (2.3), it yields

$$F_{\inf}(x_\lambda) > \lambda(y_1 - \delta) + (1-\lambda)(y_2 - \delta) + \delta = \lambda y_1 + (1-\lambda)y_2 =: y_\lambda. \tag{2.4}$$

Observe that $\mathrm{epi}F$ is a convex set since F is convex. So,

$$(x_\lambda, y_\lambda) = \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in \mathrm{epi}F.$$

This implies that there exist $y \in F(x_\lambda)$ and $r \geqq 0$ such that

$$y_\lambda = y + r \geqq y.$$

From this and from (2.4), we have

$$F_{\inf}(x_\lambda) > y.$$

This is impossible since $F_{\inf}(x_\lambda) \leqq y$, for all $y \in F(x_\lambda)$. Therefore, $F_{\inf}$ is a convex function. Also, it is clear that under given assumptions, $F_{\inf}$ is proper and lower semicontinuous.

(ii) To apply Proposition 2.1 we need to prove that $(\bar{x}, F_{\inf}(\bar{x}) - \epsilon) \notin \text{int epi}F$. Indeed, otherwise that there exists a $\delta > 0$ such that

$$\{\bar{x}\} \times (F_{\inf}(\bar{x}) - \epsilon - \delta, F_{\inf}(\bar{x}) - \epsilon + \delta) \subset \text{epi}F.$$

This means that $(F_{\inf}(\bar{x}) - \epsilon - \delta, F_{\inf}(\bar{x}) - \epsilon + \delta) \subset F(\bar{x}) + \mathbb{R}_+$. Then, for some δ' satisfying $0 < \delta' < \delta$, we can find $y \in F(\bar{x})$ and $r \geqq 0$ such that $F_{\inf}(\bar{x}) - \epsilon - \delta' = y + r$. So, $F_{\inf}(\bar{x}) = y + r + \epsilon + \delta' > y$. This contradicts to the definition of $F_{\inf}(\bar{x})$. Therefore, $(\bar{x}, F_{\inf}(\bar{x}) - \epsilon) \notin \text{int epi}F$. Applying Proposition 2.1, we conclude that $\partial_\epsilon \widetilde{F}(\bar{x}; F_{\inf}(\bar{x})) \neq \emptyset$.

Observe that

$$\begin{aligned} v \in \partial_\epsilon \widetilde{F}(\bar{x}; F_{\inf}(\bar{x})) &\iff \forall (x, \lambda) \in \text{epi}\widetilde{F} - (\bar{x}, F_{\inf}(\bar{x}) - \epsilon),\ v^T x \leqq \lambda \\ &\iff \forall x \in \text{dom}\widetilde{F},\ \forall y \in \widetilde{F}(x),\ \forall r \geqq 0, \\ &\qquad v^T(x - \bar{x}) \leqq y + r - (F_{\inf}(\bar{x}) - \epsilon) \\ &\iff \forall x \in \text{dom}F,\ \forall y \in \widetilde{F}(x), \\ &\qquad v^T(x - \bar{x}) \leqq y - (F_{\inf}(\bar{x}) - \epsilon) \\ &\iff \forall x \in \text{dom}F,\ v^T(x - \bar{x}) \leqq F_{\inf}(x) - (F_{\inf}(\bar{x}) - \epsilon) \\ &\iff v \in \partial_\epsilon F_{\inf}(\bar{x}). \end{aligned}$$

Therefore, $\partial_\epsilon \widetilde{F}(\bar{x}, F_{\inf}(\bar{x})) = \partial_\epsilon F_{\inf}(\bar{x})$. ■

Remark 2.2. Observe that if $\text{dom}F$ and $\text{epi}F$ are closed and if $F_{\inf} > -\infty$ for any $x \in \text{dom}F$, then $F_{\inf}$ is lower semicontinuous. Indeed, we should prove that $\text{epi}F_{\inf}$ is closed. Let $(x_n, \alpha_n) \in \text{dom}F \times \mathbb{R}$ with $F_{\inf}(x_n) \leqq \alpha_n$ and let (x_n, α_n) converge to $(\bar{x}, \bar{\alpha})$. Then there exist $\epsilon_n > 0$ and $y_n \in F(x_n)$ such that ϵ_n converges to 0 and $F_{\inf}(x_n) \leqq y_n < \alpha_n + \epsilon_n$. Thus $(x_n, \alpha_n + \epsilon_n) \in \text{epi}F$ converges to $(\bar{x}, \bar{\alpha})$. Since $\text{epi}F$ is closed, $(\bar{x}, \bar{\alpha}) \in \text{epi}F$. Hence, $(\bar{x}, \bar{\alpha}) \in \text{epi}F_{\inf}$.

A set-valued map F, which is satisfied all of the conditions: $\text{dom}F$ is closed, $F_{\inf} > -\infty$ for any $x \in \text{dom}F$, and $F_{\inf}$ is lower semicontinuous, may not be satisfied the condition: $\text{epi}F$ is closed. Indeed, it is clear that the set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $F(x) = x^2 + \text{int } \mathbb{R}_+$ for all $x \in \mathbb{R}$, is satisfied all of the previous conditions except the closedness of $\text{epi}F$.

Using the same proof way as the proof of Proposition 2.2(ii), we obtain the following proposition.

Proposition 2.3. *Let* $F^1, F^2 : \mathbb{R}^n \rightrightarrows \mathbb{R}$ *be convex such that* $\mathrm{dom}F^1 \cap \mathrm{dom}F^2 \neq \emptyset$. *Assume that* $F^i_{\inf}(x) > -\infty$ *for all* $x \in \mathrm{dom}F^i, i = 1, 2$. *Then for all* $\epsilon \geqq 0$ *and for all* $\bar{x} \in \mathrm{int}\ \mathrm{dom}F^1 \cap \mathrm{int}\ \mathrm{dom}F^2$, *we have*

$$\partial_\epsilon(\widetilde{F^1} + \widetilde{F^2})(\bar{x}; F^1_{\inf}(\bar{x}) + F^2_{\inf}(\bar{x})) = \partial_\epsilon(F^1_{\inf} + F^2_{\inf})(\bar{x}).$$

If in addition that $F^i_{\inf}(x) \in F^i(x), i = 1, 2$, *for all* $x \in \mathrm{int}\ \mathrm{dom}F^1 \cap \mathrm{int}\ \mathrm{dom}F^2$, *then*

$$\partial_\epsilon(F^1 + F^2)(\bar{x}; F^1_{\inf}(\bar{x}) + F^2_{\inf}(\bar{x})) = \partial_\epsilon(F^1_{\inf} + F^2_{\inf})(\bar{x}).$$

3. Necessary ϵ-Optimality Conditions

In this section, we give necessary ϵ-optimality conditions for ϵ-solutions and solutions of the convex optimization problem (CSP) formulated in Section 1. First, following the proof method for Theorem 23.8 in [16], we prove a sum formula for convex set-valued maps which will be used for getting necessary ϵ-optimality conditions for (CSP).

Theorem 3.1. *Let* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ *be a convex set-valued map and* C *a closed convex subset of* $\mathbb{R}^n$. *Let* $\bar{x} \in C \cap \mathrm{int}\ \mathrm{dom}F$ *and* $\bar{y} \in F(\bar{x})$, *and* $\epsilon \geqq 0$. *Suppose that* $(\bar{x}, \bar{y} - \epsilon) \notin \mathrm{int}\ \mathrm{epi}F$. *Then we have*

$$\partial_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y}) \subset \partial_\epsilon F(\bar{x}; \bar{y}) + N^\epsilon_C(\bar{x}).$$

Proof. Since $\mathrm{epi}(F + \widetilde{\delta}_C) \subset \mathrm{epi}F$, $D_\epsilon F(\bar{x}; \bar{y})(x) \leqq D_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y})(x)$ for any $x \in \mathbb{R}^n$. Thus, by Proposition 2.1, $D_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y})(0) = 0$ and $D_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y})(x) > -\infty$ for any $x \in \mathbb{R}^n$, and so, $D_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y})$ is proper and sublinear. Moreover, since $\partial_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y}) = \partial D_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y})(0)$, $v \in \partial_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y})$ if and only if for any $x \in \mathbb{R}^n$, $D_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y})(x) \geqq v^T x$. Thus we can easily check that $v \in \partial_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y})$ if and only if for any $(x, \lambda) \in \mathrm{epi}(F + \widetilde{\delta}_C) - (\bar{x}, \bar{y} - \epsilon)$, $v^T x \leqq \lambda$. Moreover, we can check that $v \in \partial_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y})$ if and only if for any $x \in C \cap \mathrm{domF}$ and any $y \in F(x)$,

$$0 \leqq y - \bar{y} + \epsilon - v^T(x - \bar{x}). \tag{3.1}$$

Let $G(x) = F(x) - \bar{y} + \epsilon - v^T(x - \bar{x})$, $C_1 = \mathrm{epi}G$ and $C_2 = \{(x, \lambda) \in C \times \mathbb{R} \mid \lambda \leqq 0\}$. Then $G(\bar{x}) = F(\bar{x}) - \bar{y} + \epsilon$. Since $\bar{y} \in F(\bar{x})$, $\epsilon \in G(\bar{x})$, and since $\bar{x} \in \mathrm{int}\ \mathrm{dom}F$, $\mathrm{int}C_1 \neq \emptyset$. It is clear that C_1 and C_2 are convex. Moreover int$C_1 \cap C_2 = \emptyset$. Indeed, suppose to the contrary that int$C_1 \cap C_2 \neq \emptyset$. Then there exists $(\bar{z}, \bar{\lambda}) \in \mathrm{int} C_1 \cap C_2$. Thus $\bar{z} \in C \cap \mathrm{dom} F$ and $\bar{\lambda} \leqq 0$, and there exists $\delta > 0$ such that $\{\bar{z}\} \times (\bar{\lambda} - \delta, \bar{\lambda} + \delta) \subset C_1$. Let λ be such that $\bar{\lambda} - \delta < \lambda < \bar{\lambda}$. Since $(\bar{z}, \lambda) \in C_1$, we can find $\bar{\bar{y}} \in F(\bar{z})$ and $\lambda_1 \geqq 0$ such that $\lambda = \bar{\bar{y}} - \bar{y} + \epsilon - v^T(\bar{z} - \bar{x}) + \lambda_1$, that is, $\bar{\bar{y}} - \bar{y} + \epsilon - v^T(\bar{z} - \bar{x}) = \lambda - \lambda_1 < \bar{\lambda} \leqq 0$, which contradicts (3.1) since $\bar{z} \in C \cap \mathrm{dom} F$ and $\bar{y} \in F(\bar{z})$. Hence int$C_1 \cap C_2 = \emptyset$. By separation theorem, there exist $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, $(a, b) \neq (0, 0)$ and $\beta \in \mathbb{R}$ such that for any $(x, \lambda) \in C_1$ and any $(\widetilde{x}, \widetilde{\lambda}) \in C_2$,

$$a^T x + b\lambda \leqq \beta \leqq a^T \widetilde{x} + b\widetilde{\lambda}. \tag{3.2}$$

From (3.2), $a^T \bar{x} + b(\lambda + \epsilon) \leqq a^T \bar{x}$ for any $\lambda \geqq 0$, and hence, $b \leqq 0$. If $b = 0$, it follows from (3.2) that $a^T(x - \bar{x}) \leqq 0$ for any $x \in \mathrm{dom} F$. Since $\bar{x} \in \mathrm{int}\ \mathrm{dom} F$, $a = 0$. This is impossible since $(a, b) \neq (0, 0)$. Hence, $b < 0$. From (3.2), $a^T(x - \bar{x}) + b\lambda \leqq 0$ for any $(x, \lambda) \in C_1$, and hence, for any $x \in \mathrm{dom} F$ and any $y \in F(x)$,

$$a^T(x - \bar{x}) + b[y - \bar{y} + \epsilon - v^T(x - \bar{x})] \leqq 0.$$

So, for any $x \in \mathrm{dom} F$ and any $y \in F(x)$,

$$(v - \frac{1}{b}a)^T(x - \bar{x}) \leqq y - \bar{y} + \epsilon.$$

This means that $(v - \frac{1}{b}a, -1)^T((x, y) - (\bar{x}, \bar{y} - \epsilon)) \leqq 0$ for any $(x, y) \in \mathrm{epi} F$. Hence, $v - \frac{1}{b}a \in \partial_\epsilon F(\bar{x}; \bar{y})$. From (3.2), $a^T \bar{x} + b\epsilon \leqq a^T \widetilde{x}$ for any $\widetilde{x} \in C$. This shows that $\frac{1}{b} a^T(\widetilde{x} - \bar{x}) \leqq \epsilon$ for any $\widetilde{x} \in C$. Thus, we have

$$\frac{1}{b} a \in N_C^\epsilon(\bar{x}).$$

Therefore, $v = (v - \frac{1}{b}a) + \frac{1}{b}a \in \partial_\epsilon F(\bar{x}; \bar{y}) + N_C^\epsilon(\bar{x})$. Consequently, we have,

$$\partial_\epsilon(F + \widetilde{\delta}_C)(\bar{x}; \bar{y}) \subset \partial_\epsilon F(\bar{x}; \bar{y}) + N_C^\epsilon(\bar{x}). \qquad \blacksquare$$

Corollary 3.1. *Let* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ *be a convex set-valued map and* $\bar{x} \in C \cap \mathrm{int}\ \mathrm{dom} F$. *Let* $\bar{y} \in F(\bar{x})$ *and suppose that* $(\bar{x}, \bar{y}) \notin \mathrm{int}\ \mathrm{epi} F$. *Then we have,*

$$\partial(F + \widetilde{\delta}_C)(\bar{x}; \bar{y}) = \partial F(\bar{x}; \bar{y}) + N_C(\bar{x}).$$

Proof. By Theorem 3.1, $\partial(F + \widetilde{\delta}_C)(\bar{x}; \bar{y}) \subset \partial F(\bar{x}; \bar{y}) + N_C(\bar{x})$. Now we prove that the converse inclusion holds. Let $v \in \partial F(\bar{x}; \bar{y}) + N_C(\bar{x})$. Then there exist

$v_1 \in \partial F(\bar{x};\bar{y})$ and $v_2 \in N_C(\bar{x})$ such that $v = v_1 + v_2$. Thus for any $x \in \mathrm{dom}F$ and any $y \in F(x)$, $v_1^T(x-\bar{x}) + \bar{y} \leqq y$, and for any $x \in C$, $v_2^T(x-\bar{x}) \leqq 0$. Hence, for any $x \in C \cap \mathrm{dom}F$ and any $y \in F(x)$,

$$(v_1 + v_2)^T(x-\bar{x}) + \bar{y} \leqq y.$$

Thus $v = (v_1+v_2) \in \partial(F + \widetilde{\delta}_C)(\bar{x};\bar{y})$. Hence, the converse inclusion holds. ■

Now we give ϵ-optimality conditions for the convex set-valued optimization problem (CSP) which was formulated in Section 1.

Theorem 3.2. *Let* $\bar{x} \in C \cap \mathrm{int}\ \mathrm{dom}F$ *and* $\bar{y} \in F(\bar{x})$. *Suppose that* $(\bar{x}, \bar{y} - \epsilon) \notin \mathrm{int}\ \mathrm{epi}F$. *If* $(\bar{x},\bar{y})$ *is an* ϵ*-solution of* (CSP)*, then we have,*

$$0 \in \partial_\epsilon F(\bar{x};\bar{y}) + N_C^\epsilon(\bar{x}).$$

Proof. Let $(\bar{x},\bar{y})$ be an ϵ-solution of (CSP). Then for any $x \in C \cap \mathrm{dom}F$ and any $y \in F(x)$, $y \geqq \bar{y} - \epsilon$, and hence, for any $x \in \mathrm{dom}(F + \widetilde{\delta}_C)$ and any $y \in (F + \widetilde{\delta}_C)(x)$, $y \geqq \bar{y} - \epsilon$. Thus for any $(x,\lambda) \in \mathrm{epi}(F + \widetilde{\delta}_C) - (\bar{x}, \bar{y} - \epsilon)$,

$$0 \leqq \lambda.$$

This shows that for any $(x,\lambda) \in \overline{\mathrm{cone}}\ [(F + \widetilde{\delta}_C) - (\bar{x}, \bar{y} - \epsilon)]$,

$$0 \leqq \lambda.$$

This implies that for any $x \in \mathrm{dom}(F + \widetilde{\delta}_C)$,

$$0 \leqq D_\epsilon(F + \widetilde{\delta}_C)(\bar{x};\bar{y})(x).$$

In the proof of Theorem 3.1, we showed that

$$D_\epsilon(F + \tilde{\delta}_C)(\bar{x};\bar{y})(0) = 0.$$

So, $0 \in \partial_\epsilon(F + \widetilde{\delta}_C)(\bar{x};\bar{y})$, and hence by Theorem 3.1, $0 \in \partial_\epsilon F(\bar{x};\bar{y}) + N_C^\epsilon(\bar{x})$. ■

When C is a closed convex cone in (CSP), we can get a necessary and sufficient ϵ-optimality condition for (CSP) as follows.

Corollary 3.2. *Let* C *be a closed convex cone in* $\mathbb{R}^n$ *and suppose that* $0 \in C \cap \mathrm{int}\ \mathrm{dom}F$. *Let* $\bar{y} \in F(0)$. *Assume that* $(0, \bar{y} - \epsilon) \notin \mathrm{int}\ \mathrm{epi}F$. *Then* $(0,\bar{y})$ *is an* ϵ*-solution of* (CSP) *if and only if* $0 \in \partial_\epsilon F(0;\bar{y}) + N_C(0)$.

Proof. Suppose that $(0, \bar{y})$ is an ϵ-solution of (CSP). Then, since C is a convex cone, $N_C^{\epsilon}(0) = N_C(0)$, and hence it follows from Theorem 3.2 that

$$0 \in \partial_{\epsilon} F(\bar{x}; \bar{y}) + N_C(0).$$

Assume that $0 \in \partial_{\epsilon} F(\bar{x}; \bar{y}) + N_C(0)$. Then there exists $v \in \partial_{\epsilon} F(\bar{x}; \bar{y})$ such that $-v \in N_C(0)$. Thus for any $(x, \lambda) \in \mathrm{epi} F - (0, \bar{y} - \epsilon)$,

$$v^T x \leqq \lambda, \tag{3.3}$$

and for any $x \in C$, $v^T x \geqq 0$. So, from (3.3), for any $x \in \mathrm{dom} F$ and any $y \in F(x)$,

$$0 \leqq v^T x \leqq y - \bar{y} + \epsilon.$$

Hence, for any $x \in C \cap \mathrm{dom} F$ and any $y \in F(x)$,

$$\bar{y} - \epsilon \leqq y.$$

So, $(0, \bar{y})$ is an ϵ-solution of (CSP). ∎

From Theorem 3.2, we can obtain the following corollary.

Corollary 3.3. *Let* $\bar{x} \in C \cap \mathrm{int}\, \mathrm{dom} F$ *and* $\bar{y} \in F(\bar{x})$. *Suppose that* $(\bar{x}, \bar{y}) \notin \mathrm{int}\, \mathrm{epi} F$. *Then* $(\bar{x}, \bar{y})$ *is a solution of* (CSP) *if and only if* $0 \in \partial F(\bar{x}; \bar{y}) + N_C(\bar{x})$.

Proof. If $(\bar{x}, \bar{y})$ is a solution of (CSP), it follows from Theorem 3.2 that $0 \in \partial F(\bar{x}; \bar{y}) + N_C(\bar{x})$. Suppose that $0 \in \partial F(\bar{x}; \bar{y}) + N_C(\bar{x})$. Then there exists $v \in \partial F(\bar{x}; \bar{y})$ such that $-v \in N_C(\bar{x})$. Thus for any $x \in \mathrm{dom} F$ and any $y \in F(x)$,

$$v^T (x - \bar{x}) \leqq y - \bar{y},$$

and for any $x \in C$,

$$-v^T (x - \bar{x}) \leqq 0.$$

Hence, for any $x \in C \cap \mathrm{dom} F$ and any $y \in F(x)$,

$$\bar{y} \leqq y.$$

So, $(\bar{x}, \bar{y})$ is a solution of (CSP). ∎

Let $G : \mathbb{R}^n \rightrightarrows \mathbb{R}$ be convex and $C = \{x \in \mathbb{R}^n \mid G(x) \cap (-\mathbb{R}_+) \neq \emptyset\}$ and assume that $\bar{x} \in C$ and $0 \in G(\bar{x})$. Now we calculate the normal cone $N_C(\bar{x})$. Of course, if $\bar{x} \in \mathrm{int} C$, then $N_C(\bar{x}) = \{0\}$.

We need the following Slater condition for calculating the normal cone of C at some $\bar{x} \in C \setminus \mathrm{int}C$, which is a set-valued version of the usual Slater condition:

Slater Condition: there exists $\hat{x} \in \mathbb{R}^n$ such that

$$G(\hat{x}) \cap (-\mathrm{int}\ \mathbb{R}_+) \neq \emptyset.$$

Then we have the following proposition:

Proposition 3.1. *Let* $G : \mathbb{R}^n \rightrightarrows \mathbb{R}$ *be convex and* $C = \{x \in \mathbb{R}^n \mid G(x) \cap (-\mathbb{R}_+) \neq \emptyset\}$. *Suppose that the Slater condition holds. Then we have,*

(i) $\mathrm{int}\ C \ \subset \ \{x \in \mathbb{R}^n \mid G(x) \cap (-\mathrm{int}\ \mathbb{R}_+) \neq \emptyset\}$.

(ii) *if* G *is lower semicontinuous, then*

$$\{x \in \mathbb{R}^n \mid G(x) \cap (-\mathrm{int}\ \mathbb{R}_+) \neq \emptyset\} \ \subset \ \mathrm{int}C.$$

(iii) *if* G *is lower semicontinuous, then*

$$C \setminus \mathrm{int}C = \{x \in \mathbb{R}^n \mid G(x) \cap (-\mathbb{R}_+) = \{0\}\}.$$

(iv) *if* $0 \in G(\bar{x}),\ \bar{x} \in \mathrm{int\ dom}\ G$ *and* $(\bar{x}, 0) \notin \mathrm{int\ epi}G$, *then*

$$\bar{x} \in C \setminus \mathrm{int}C.$$

Proof. (i) Let $S = \{x \in \mathbb{R}^n \mid G(x) \cap (-\mathrm{int}\mathbb{R}_+) \neq \emptyset\}$. Let x be any point in $\mathrm{int}C$. If $x = \hat{x}$, then $x \in S$. Assume that $x \neq \hat{x}$. Then we can find $\delta > 0$ such that $x + B_\delta(0) \subset C$, where $B_\delta = \{z \in \mathbb{R}^n \mid ||z|| < \delta\}$, and $\hat{x} \notin x + B_\delta(0)$. Moreover, since $x \neq \hat{x}$, we can find $v \in B_\delta(0) \setminus \{0\}$ such that $x - v,\ x + v \in \mathrm{aff}\{x, \hat{x}\} := \{\alpha x + (1-\alpha)\hat{x} \mid \alpha \in \mathbb{R}\}$, $\hat{x} \notin [x-v, x+v] := \{\lambda(x-v) + (1-\lambda)(x+v) \mid \lambda \in [0,1]\}$, $x \in (x-v, x+v) := \{\lambda(x-v) + (1-\lambda)(x+v) \mid \lambda \in (0,1)\}$ and $x + v \in (x, \hat{x})$. Then there exists $\hat{\lambda} \in (0,1)$ such that $x + v = \hat{\lambda}\hat{x} + (1-\hat{\lambda})(x-v)$. So, since G is convex, we have,

$$\hat{\lambda}G(\hat{x}) + (1-\hat{\lambda})G(x-v) \subset G(x+v) + \mathbb{R}_+. \tag{3.4}$$

From Slater Condition, we can take $\hat{y} \in G(\hat{x})$ such that $\hat{y} < 0$. Moreover, since $x - v \in x + B_\delta(0) \subset C$, we can find $y_1 \in G(x-v)$ such that $y_1 \leqq 0$. Assume to the contrary that $G(x+v) \cap (-\mathrm{int}\mathbb{R}_+) = \emptyset$. Then, from (3.4),

$$\hat{\lambda}G(\hat{x}) + (1-\hat{\lambda})G(x-v) \subset \mathbb{R}_+. \tag{3.5}$$

Thus, from (3.5), $0 \leqq \hat{\lambda}\hat{y} + (1-\hat{\lambda})y_1 < 0$. This is a contradiction. Hence, $G(x+v) \cap (-\text{int}\mathbb{R}_+) \neq \emptyset$. So, there exists $y_2 \in G(x+v)$ such that $y_2 < 0$. Since G is convex, we have

$$\begin{aligned} \frac{1}{2}y_1 + \frac{1}{2}y_2 &\in \frac{1}{2}G(x-v) + \frac{1}{2}G(x+v) \\ &\subset G(x) + \mathbb{R}_+. \end{aligned}$$

Hence there exist $y \in G(x)$ and $r \geqq 0$ such that $y + r = \frac{1}{2}(y_1 + y_2) < 0$. Thus $y < 0$ and so $G(x) \cap (-\text{int}\mathbb{R}_+) \neq \emptyset$. Hence $x \in S$. Therefore, we have

$$\text{int}C \subset S.$$

(ii) If G is lower semicontinuous, then $\{x \in \mathbb{R}^n \mid G(x) \cap (-\text{int}\mathbb{R}_+) \neq \emptyset\}$ is open and hence $\{x \in \mathbb{R}^n \mid G(x) \cap (-\text{int}\mathbb{R}_+) \neq \emptyset\} \subset \text{int}C$.

(iii) Since G is lower semicontinuous, it follows from (i) and (ii) that

$$C \setminus \text{int}C = C \setminus \{x \in \mathbb{R}^n \mid G(x) \cap (-\text{int}\mathbb{R}_+) \neq \emptyset\}.$$

Let $x \in C \setminus \text{int}C$. Then $G(x) \cap (-\mathbb{R}_+) \neq \emptyset$ and $G(x) \cap (-\text{int}\mathbb{R}_+) = \emptyset$. Thus $G(x) \cap (-\mathbb{R}_+) = \{0\}$. Hence we have

$$C \setminus \text{int}C \subset \{x \in \mathbb{R}^n \mid G(x) \cap (-\text{int}\mathbb{R}_+) \neq \emptyset\}.$$

Conversely, we assume that $G(x) \cap (-\mathbb{R}_+) = \{0\}$. Then $x \in C$ and $x \neq \hat{x}$, where $\hat{x}$ is the point in the definition of Slater condition. For any fixed $\lambda \in (0,1)$, we let $x_\lambda = x + \lambda(\hat{x} - x)$ and $x'_\lambda = x - \lambda(\hat{x} - x)$. Since G is convex, $\lambda G(\hat{x}) + (1-\lambda)G(x) \subset G(x_\lambda) + \mathbb{R}_+$, and hence, taking $\hat{y} \in G(\hat{x})$ with $\hat{y} < 0$ and $0 \in G(x)$, we can find $y_\lambda \in G(x_\lambda)$ such that $y_\lambda < 0$. Since $\frac{1}{2}x_\lambda + \frac{1}{2}x'{}_\lambda = x$ and $G(x) \subset \mathbb{R}_+$,

$$\frac{1}{2}G(x_\lambda) + \frac{1}{2}G(x'_\lambda) \subset G(x) + \mathbb{R}_+ \subset \mathbb{R}_+.$$

So, for any $y'_\lambda \in G(x'_\lambda)$, $\frac{1}{2}y_\lambda + \frac{1}{2}y'_\lambda \geqq 0$ and hence, $y'_\lambda > 0$. Hence $G(x'_\lambda) \cap (-\mathbb{R}_+) = \emptyset$ for any $\lambda \in (0,1)$, and so, $(x, 2x - \hat{x}) \cap C = \emptyset$. This means that $x \notin \text{int}C$. Thus, we have

$$\{x \in \mathbb{R}^n \mid G(x) \cap \mathbb{R}_+ = \{0\}\} \subset C \setminus \text{int}C.$$

(iv) Suppose that $\bar{x} \in C \cap \text{int dom}G$, $0 \in G(\bar{x})$ and $(\bar{x}, 0) \notin \text{int epi}G$. Then from the proof of Proposition 2.1, we can check that there exist $\tilde{a} \in \mathbb{R}^n$ and $\tilde{b} > 0$ such that for any $(x, y) \in \overline{\text{cone}}(\text{epi}G - (\bar{x}, 0))$,

$$\tilde{a}^T x + \tilde{b}y \geqq 0. \tag{3.6}$$

Let $\bar{y} \in G(\bar{x})$ be any point in $\mathbb{R}$. Then for any $\alpha \geqq 0$, $(\bar{x}, \bar{y}+\alpha) \in \mathrm{epi}G$, that is, $(0, \bar{y}+\alpha) \in \mathrm{epi}G - (\bar{x}, 0)$. Thus from (3.6), $\tilde{b}(\bar{y}+\alpha) \geqq 0$ for any $\alpha \geqq 0$. Since $\tilde{b} > 0$, $\bar{y} \geqq 0$. This means that $G(\bar{x}) \cap (-\mathrm{int}\mathbb{R}_+) = \emptyset$. So, by (i), $\bar{x} \notin \mathrm{int}C$. Since $0 \in G(\bar{x})$, $\bar{x} \in C$. Thus $\bar{x} \in C \setminus \mathrm{int}C$. ■

Proposition 3.2. *Let* $G : \mathbb{R}^n \rightrightarrows \mathbb{R}$ *be a upper semicontinuous and convex set-valued map. Let* $\bar{x} \in \mathrm{int\ dom}G$ *and* $0 \in G(\bar{x})$, *and assume that* $(\bar{x}, 0) \notin \mathrm{int\ epi}G$. *Let* $C = \{x \in \mathbb{R}^n \mid G(x) \cap (-\mathbb{R}_+) \neq \emptyset\}$ *and suppose that Slater condition holds. Then* $N_C(\bar{x}) = \mathrm{cone}\ \partial G(\bar{x}; 0)$.

Proof. Since G is upper semicontinuous and convex, then C is a closed and convex subset of $\mathbb{R}^n$. If $v \in \partial G(\bar{x}; 0)$, then for any $x \in \mathrm{dom}G$ and any $y \in G(x)$, $v^T(x - \bar{x}) \leqq y$. So, for any $x \in C$, $v^T(x - \bar{x}) \leqq 0$. Hence, $\partial G(\bar{x}; 0) \subset N_C(\bar{x})$. Since $N_C(\bar{x})$ is a convex cone,

$$\mathrm{cone}\ \partial G(\bar{x}; 0) \subset N_C(\bar{x}). \tag{3.7}$$

By Slater condition, $0 \notin \partial G(\bar{x}; 0)$ and hence it follows from definition of $\partial G(\bar{x}; 0)$ and the fact that $DG(\bar{x}; 0)(0) = 0$ that $\{v \in \mathbb{R}^n \mid DG(\bar{x}; 0)(v) < 0\} \neq \emptyset$. Let $K = \overline{\mathrm{cone}}(C - \bar{x})$. Then $N_C(\bar{x}) = K^0$, where K^0 is the nonpositive dual cone of K. If $DG(\bar{x}; 0)(v) < 0$, then $v \neq 0$, and so, it follows from definition of $DG(\bar{x}; 0)$ that there exist $\lambda_n > 0$ and $x_n \in C$, $n \in \mathbb{N}$, such that $v = \lim_{n\to\infty} \lambda_n(x_n - \bar{x})$ and so, $v \in K$. Thus $\{v \in \mathbb{R}^n \mid DG(\bar{x}; 0)(v) < 0\} \subset K$. Moreover, from Proposition 2.1, $DG(\bar{x}; 0)(\cdot)$ is sublinear and continuous, and so,

$$\{v \in \mathbb{R}^n \mid DG(\bar{x}; 0)(v) \leqq 0\} \subset K. \tag{3.8}$$

Noticing that $DG(\bar{x}; 0)(v) = \sup_{y \in \partial G(\bar{x}; 0)} y^T v$, we get

$$\{v \in \mathbb{R}^n \mid DG(\bar{x}; 0)(v) \leqq 0\} = (\partial G(\bar{x}; 0))^0. \tag{3.9}$$

Moreover, since $\partial G(\bar{x}; 0)$ is compact and $0 \notin \partial G(\bar{x}; 0)$,

$$\overline{\mathrm{cone}}\ \partial G(\bar{x}; 0) = \mathrm{cone}\ \partial G(\bar{x}; 0). \tag{3.10}$$

So, from (3.8)-(3.10), $K^0 \subset \mathrm{cone}\partial G(\bar{x}; 0)$, i.e., $N_C(\bar{x}) \subset \mathrm{cone}\ \partial G(\bar{x}; 0)$. Hence, from (3.7), we have

$$N_C(\bar{x}) = \mathrm{cone}\ \partial G(\bar{x}; 0),$$

as required. ■

From Corollary 3.3 and Proposition 3.2, we can get the following optimality theorem for (CSP).

Theorem 3.3. *Let* $F:\mathbb{R}^n \rightrightarrows \mathbb{R}$ *be a convex set-valued map and* $G:\mathbb{R}^n \rightrightarrows \mathbb{R}$ *a upper semicontinuous and convex set-valued map and* $C=\{x\in\mathbb{R}^n \mid G(x)\cap(-\mathbb{R}_+)\neq\emptyset\}$. *Let* $\bar{x}\in C\cap \mathrm{int}(\mathrm{dom}F\cap\mathrm{dom}G)$, $\bar{y}\in F(\bar{x})$ *and* $0\in G(\bar{x})$. *Assume that* $(\bar{x},\bar{y})\notin \mathrm{int}\ \mathrm{epi}F$ *and* $(\bar{x},0)\notin \mathrm{int}\ \mathrm{epi}G$, *and suppose that Slater condition holds. Then* $(\bar{x},\bar{y})$ *is a solution of* (CSP) *if and only if there exists* $\lambda \geqq 0$ *such that*

$$0\in\partial F(\bar{x};\bar{y})+\lambda\partial G(\bar{x};0).$$

4. Necessary and Sufficient ϵ-Optimality Conditions

In this section, using the single-valued function F_{inf} induced from the set-valued map F and defined in Section 2, we obtain theorems describing the ϵ-subgradient sum formula for two convex set-valued maps (see Theorems 4.1 and 4.2 below), and then give necessary and sufficient ϵ-optimality conditions for Problem (CSP). First, we establish the following proposition.

Proposition 4.1. *Let* $F^1,F^2:\mathbb{R}^n\rightrightarrows\mathbb{R}$ *be set-valued maps,* $\mathrm{dom}F^1\cap\mathrm{dom}F^2\neq\emptyset$. *Suppose that for any* $x\in\mathrm{dom}F^1\cap\mathrm{dom}F^2$, $F^1_{\mathrm{inf}}(x)>-\infty$ *and* $F^2_{\mathrm{inf}}(x)>-\infty$. *Then*

$$(F^1+F^2)^*=(F^1_{\mathrm{inf}}+F^2_{\mathrm{inf}})^*.$$

Proof. Let us take an arbitrary $v\in\mathbb{R}^n$. For $x\in\mathrm{dom}F^1\cap\mathrm{dom}F^2$,

$$v^Tx-(F^1_{\mathrm{inf}}(x)+F^2_{\mathrm{inf}}(x))\geqq v^Tx-(y_1+y_2),\ \text{for any } y_1\in F^1(x) \text{ and any } y_2\in F^2(x).$$

Hence,

$$\sup_{x\in\mathrm{dom}F^1\cap\mathrm{dom}F^2}\{v^Tx-(F^1_{\mathrm{inf}}(x)+F^2_{\mathrm{inf}}(x))\}\geqq\sup_{x\in\mathrm{dom}F^1\cap\mathrm{dom}F^2}\{v^Tx-(F^1+F^2)(x)\}.$$

So,

$$(F^1_{\mathrm{inf}}+F^2_{\mathrm{inf}})^*(v)\geqq(F^1+F^2)^*(v).$$

For each $\epsilon>0$ and each $x\in\mathrm{dom}F^1\cap\mathrm{dom}F^2$, by the definition of F^1_{inf} and F^2_{inf}, we can find $y_1\in F^1(x)$ and $y_2\in F^2(x)$ such that

$$\begin{cases} F^1_{\mathrm{inf}}(x)+\dfrac{\epsilon}{2}>y_1\\ F^2_{\mathrm{inf}}(x)+\dfrac{\epsilon}{2}>y_2.\end{cases}$$

This shows that

$$\begin{aligned} v^Tx-(F^1_{\mathrm{inf}}+F^2_{\mathrm{inf}})(x)-\epsilon &< v^Tx-(y_1+y_2)\\ &\leqq \sup_{x\in\mathrm{dom}F^1\cap\mathrm{dom}F^2}\{v^Tx-(y_1+y_2)\mid y_1\in F^1(x),y_2\in F^2(x)\}.\end{aligned}$$

Hence,

$$\sup_{x\in \mathrm{dom}F^1\cap \mathrm{dom}F^2}\{v^Tx-(F^1_{\text{inf}}+F^2_{\text{inf}})(x)\}-\epsilon \leqq \sup_{x\in \mathrm{dom}F^1\cap \mathrm{dom}F^2}\{v^Tx-(F^1+F^2)(x)\}.$$

Since ϵ is arbitrary, we have

$$(F^1_{\text{inf}}+F^2_{\text{inf}})^*(v) \leqq (F^1+F^2)^*(v).$$

Therefore, $(F^1+F^2)^* = (F^1_{\text{inf}}+F^2_{\text{inf}})^*$. ■

Remark 4.1. Observe that
(i) For any set-valued maps $F^1, F^2 : \mathbb{R}^n \rightrightarrows \mathbb{R}$,

$$(F^1+F^2)^* = (\widetilde{F^1}+\widetilde{F^2})^* = (F^1_{\text{inf}}+F^2_{\text{inf}})^*.$$

(ii) For any set-valued maps $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$, $F^* = \widetilde{F}^* = F^*_{\text{inf}}$.
(Recall that $F_{\text{inf}}(x) := \inf\{y \mid y \in F(x)\}$ and $\widetilde{F}(x) := F(x) \cup \{F_{\text{inf}}(x)\}$).

Theorem 4.1. *Let* $F^1, F^2 : \mathbb{R}^n \rightrightarrows \mathbb{R}$ *be convex set-valued maps such that for all* $i = 1, 2$, $\mathrm{dom}F^i$ *and* $\mathrm{epi}F^i_{\text{inf}}$ *are closed, and* $F^i_{\text{inf}}(x) > -\infty$, *for all* $x \in \mathrm{dom}F^i$. *Let* $\epsilon \geqq 0$. *If* $\mathrm{ri}\ \mathrm{dom}F^1 \cap \mathrm{ri}\ \mathrm{dom}F^2 \neq \emptyset$, *then for all* $x \in \mathrm{dom}F^1 \cap \mathrm{dom}F^2$,

$$\begin{aligned}&\partial_\epsilon(\widetilde{F}^1+\widetilde{F}^2)(x; F^1_{\text{inf}}(x)+F^2_{\text{inf}}(x))\\ &= \bigcup_{\substack{\epsilon_1+\epsilon_2=\epsilon\\ \epsilon_1,\epsilon_2\geqq 0}} \partial_{\epsilon_1}\widetilde{F}^1(x; F^1_{\text{inf}}(x)) + \partial_{\epsilon_2}\widetilde{F}^2(x; F^2_{\text{inf}}(x)).\end{aligned} \tag{4.1}$$

Proof. Applying Proposition 2.2, we have that F^1_{inf} and F^2_{inf} are proper lower semicontinuous convex functions. Obviously,

$$\mathrm{ri}\ \mathrm{dom}F^1_{\text{inf}} \cap \mathrm{ri}\ \mathrm{dom}F^2_{\text{inf}} = \mathrm{ri}\ \mathrm{dom}F^1 \cap \mathrm{ri}\ \mathrm{dom}F^2 \neq \emptyset.$$

Thus, from Theorem 3.1.1 in [8], it yields that for all $x \in \mathrm{dom}F^1_{\text{inf}} \cap \mathrm{dom}F^2_{\text{inf}}$,

$$\partial_\epsilon(F^1_{\text{inf}}+F^2_{\text{inf}})(x) = \bigcup_{\substack{\epsilon_1+\epsilon_2=\epsilon\\ \epsilon_1,\epsilon_2\geqq 0}} \partial_{\epsilon_1}F^1_{\text{inf}}(x) + \partial_{\epsilon_2}F^2_{\text{inf}}(x).$$

Using Propositions 2.2 and 2.3, we have the conclusion, as required. ■

Remark 4.2. Theorem 3.1.1 in [8] is a special case of our Theorem 4.1.

Theorem 4.2. *Let* $F^1, F^2 : \mathbb{R}^n \rightrightarrows \mathbb{R}$ *be convex set-valued maps such that* $\mathrm{dom}F^1 \cap \mathrm{dom}F^2 \neq \emptyset$, *for all* $i = 1, 2$, $\mathrm{dom}F^i$ *and* $\mathrm{epi}F^i_{\inf}$ *are closed, and* $F^i_{\inf}(x) > -\infty$, *for all* $x \in \mathrm{dom}F^i$. *Then the following statements are equivalent:*

(i) $(F^1 + F^2)^* = (F^1)^* \square (F^2)^*$.

(ii) $\mathrm{epi}(F^1)^* + \mathrm{epi}(F^2)^*$ *is closed.*

(iii) *For any* $\epsilon \geq 0$ *and any* $x \in \mathrm{dom}F^1 \cap \mathrm{dom}F^2$,

$$\partial_\epsilon(\widetilde{F^1} + \widetilde{F^2})(x; F^1_{\inf}(x) + F^2_{\inf}(x)) = \bigcup_{\substack{\epsilon_1+\epsilon_2=\epsilon \\ \epsilon_1,\epsilon_2\geqq 0}} \partial_{\epsilon_1}\widetilde{F^1}(x; F^1_{\inf}(x)) + \partial_{\epsilon_2}\widetilde{F^2}(x; F^2_{\inf}(x)).$$

Proof. Applying Proposition 2.2, we have that $F^1_{\inf}, F^2_{\inf}$ are proper lower semi-continuous convex functions. It is easy to verify that

$$\mathrm{dom}F^1_{\inf} \cap \mathrm{dom}F^2_{\inf} = \mathrm{dom}F^1 \cap \mathrm{dom}F^2 \neq \emptyset.$$

Thus, from Theorem 1 in [3], it yields that the following statements are equivalent:

(i) $(F^1_{\inf} + F^2_{\inf})^* = (F^1_{\inf})^* \square (F^2_{\inf})^*$.

(ii) $\mathrm{epi}(F^1_{\inf})^* + \mathrm{epi}(F^2_{\inf})^*$ is closed.

(iii) For any $\epsilon \geqq 0$ and any $x \in \mathrm{dom}F^1_{\inf} \cap \mathrm{dom}F^2_{\inf}$,

$$\partial_\epsilon(F^1_{\inf} + F^2_{\inf})(x) = \bigcup_{\substack{\epsilon_1+\epsilon_2=\epsilon \\ \epsilon_1,\epsilon_2\geqq 0}} \partial_{\epsilon_1}F^1_{\inf}(x) + \partial_{\epsilon_2}F^2_{\inf}(x).$$

To complete the proof, let us apply Remark 4.1 and Propositions 2.2, 2.3 and 4.1 to (i)-(iii) by replacing $F^1_{\inf}$ (resp. $F^2_{\inf}$) of statements (i)-(ii) with F^1 (resp. F^2), and $F^1_{\inf}$ (resp. $F^2_{\inf}$) of statements (iii) with $\widetilde{F^1}$ (resp. $\widetilde{F^2}$). ■

Remark 4.3. Observe that by our approach the main results of this paper are still correct if we replace $\mathbb{R}^n$ by a Banach space X. So, our Theorem 4.2 can be seen as a generalized version of Theorem 1 of [3].

Remark 4.4. In Theorems 4.1 and 4.2, if in addition that for any $x \in \mathrm{dom}F^1 \cap \mathrm{dom}F^2$, $F^i_{\inf}(x) \in F^i(x)$, $i = 1, 2$, then the equality (4.1) can be replaced by the following equality:

$$\partial_\epsilon(F^1+F^2)(x; F^1_{\inf}(x)+F^2_{\inf}(x)) = \bigcup_{\substack{\epsilon_1+\epsilon_2=\epsilon \\ \epsilon_1,\epsilon_2\geqq 0}} \partial_{\epsilon_1}F^1(x; F^1_{\inf}(x)) + \partial_{\epsilon_2}F^2(x; F^2_{\inf}(x)).$$

Applying Theorem 4.1, we can obtain the following necessary and sufficient ϵ-optimality condition for Problem (CSP).

Theorem 4.3. *Let* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ *be a convex set-valued maps such that* $\mathrm{dom}F$ *and* $\mathrm{epi}F_{\inf}$ *are closed, and* $F_{\inf}(x) > -\infty$*, for any* $x \in \mathrm{dom}F$ *and such that* $\mathrm{dom}F^i$ *and* $\mathrm{epi}F^i_{\inf}$ *are closed,* $i = 1, 2$*. Let* C *be a closed convex subset of* X *such that* $\mathrm{ri}\ C \cap \mathrm{ri}\ \mathrm{dom}F \neq \emptyset$*. Let* $\bar{x} \in C \cap \mathrm{int}\ \mathrm{dom}F$*, and* $F_{\inf}(\bar{x}) \in F(\bar{x})$*. Let* $\epsilon \geqq 0$*. Then* $(\bar{x}, F_{\inf}(\bar{x}))$ *is an* ϵ*-solution of* (CSP) *if and only if there exist* $\epsilon_1, \epsilon_2 \geqq 0$ *such that* $\epsilon_1 + \epsilon_2 = \epsilon$*, and*

$$0 \in \partial_{\epsilon_1} F(\bar{x}; F_{\inf}(\bar{x})) + N_C^{\epsilon_2}(\bar{x}).$$

Proof. Observe that $(\bar{x}; F_{\inf}(\bar{x}))$ is an ϵ-solution of (CSP) if and only if

$$0 \in \partial_{\epsilon}(F + \widetilde{\delta}_C)(\bar{x}; F_{\inf}(\bar{x})).$$

Hence, apply Theorem 4.1 setting $F^1 = F,\ F^2 = \widetilde{\delta}_C$ and applying Theorem 4.1, we obtain that $(\bar{x}, F_{\inf}(\bar{x}))$ is an ϵ-solution of (CSP) if and only if

$$0 \in \bigcup_{\substack{\epsilon_1+\epsilon_2=\epsilon \\ \epsilon_1,\epsilon_2\geqq 0}} \partial_{\epsilon_1} F(\bar{x}; F_{\inf}(\bar{x})) + \partial_{\epsilon_2}\widetilde{\delta}_C(\bar{x}, 0),$$

i.e., there exist $\epsilon_1, \epsilon_2 \geqq 0$ such that $\epsilon_1 + \epsilon_2 = \epsilon$, and

$$0 \in \partial_{\epsilon_1} F(\bar{x}; F_{\inf}(\bar{x})) + N_C^{\epsilon_2}(\bar{x}). \qquad \blacksquare$$

Applying Theorem 4.2 to $F^1 = \widetilde{\delta}_{C_1},\ F^2 = \widetilde{\delta}_{C_2}$, where C_1, C_2 are closed convex sets, we have the following result about the ϵ-normal cone $N^{\epsilon}_{C_1\cap C_2}(x)$.

Corollary 4.1. *Let* C_1 *and* C_2 *be closed convex subsets of* X *such that* $C_1 \cap C_2 \neq \emptyset$*. Then, the set* $\mathrm{epi}\delta^*_{C_1} + \mathrm{epi}\delta^*_{C_2}$ *is closed if and only if for each* $\epsilon \geqq 0$ *and each* $x \in C_1 \cap C_2$*,*

$$N^{\epsilon}_{C_1\cap C_2}(x) = \bigcup_{\substack{\epsilon_1+\epsilon_2=\epsilon \\ \epsilon_1,\epsilon_2\geqq 0}} N^{\epsilon_1}_{C_1}(x) + N^{\epsilon_2}_{C_2}(x).$$

Now let us consider the following problem $(\widetilde{\mathrm{CSP}})$

$$(\widetilde{\mathrm{CSP}}) \qquad \begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & x \in C := C_1 \cap C_2 \end{array}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}$ is a convex set-valued maps, $G_i : \mathbb{R}^n \rightrightarrows \mathbb{R}$ are upper semicontinuous and convex set-valued maps, $C_i = \{x \in \mathbb{R}^n \mid G_i(x) \cap (-\mathbb{R}_+) \neq \emptyset\},\ i = 1, 2$ are closed convex subsets of $\mathbb{R}^n$ and $C \neq \emptyset$.

Now we give a necessary and sufficient condition for $(\widetilde{\mathrm{CSP}})$.

Theorem 4.4. *Let* $\bar{x} \in \mathrm{int}\ \mathrm{dom}F \cap \mathrm{int}\ \mathrm{dom}G_1 \cap \mathrm{int}\ \mathrm{dom}G_2$ *and* $\bar{y} \in F(\bar{x})$ *such that* $(\bar{x}, \bar{y}) \notin \mathrm{int}\ \mathrm{epi}F$, $0 \in G_i(\bar{x}), (\bar{x}, 0) \notin \mathrm{int}\ \mathrm{epi}G_i,\ i = 1, 2$. *Assume that the set* $\mathrm{epi}\delta^*_{C_1} + \mathrm{epi}\delta^*_{C_2}$ *is closed, and for each* $i \in I$, *there exists* $\widehat{x} \in \mathbb{R}^n$ *such that*

$$G_i(\widehat{x}) \cap (-\mathrm{int}\mathbb{R}_+) \neq \emptyset.$$

Then, $(\bar{x}, \bar{y})$ *is a solution of* $(\widetilde{\mathrm{CSP}})$ *if and only if there exist* $\lambda_1, \lambda_2 \geqq 0$ *such that*

$$0 \in \partial F(\bar{x}; \bar{y}) + \lambda_1 \partial G_1(\bar{x}; 0) + \lambda_2 \partial G_2(\bar{x}; 0).$$

Proof. Using Theorem 3.3, we have that $(\bar{x}, \bar{y})$ is a solution of Problem $(\widetilde{\mathrm{CSP}})$ if and only if

$$0 \in \partial F(\bar{x}; \bar{y}) + N_C(\bar{x}). \tag{4.2}$$

By Corollary 4.1, (4.2) is equivalent to

$$0 \in \partial F(\bar{x}; \bar{y}) + N_{C_1}(\bar{x}) + N_{C_2}(\bar{x}).$$

From Proposition 3.2, this means that

$$0 \in \partial F(\bar{x}; \bar{y}) + \mathrm{cone}\ \partial G_1(\bar{x}; 0) + \mathrm{cone}\ \partial G_2(\bar{x}; 0),$$

i.e., there exist $\lambda_1, \lambda_2 \geqq 0$ such that

$$0 \in \partial F(\bar{x}; \bar{y}) + \lambda_1 \partial G_1(\bar{x}; 0) + \lambda_2 \partial G_2(\bar{x}; 0).$$

Thus the proof is completed. ■

Let us now consider the following theorem which will provide a relation between the ϵ-solution set of Problem (CSP) and the ϵ-solution set of the following auxiliary Problem (CSP)$'$

$$\begin{array}{lll} (\mathrm{CSP})' & \text{Minimize} & F_{\inf}(x) \\ & \text{subject to} & x \in C. \end{array}$$

Theorem 4.5. *If* $\inf_{x' \in C \cap \mathrm{dom} F} F(x')$ *is finite, then*

$$\epsilon\text{−sol(CSP)} = \{(x, y) \mid x \in \epsilon\text{−sol(CSP)}',\ y \in F(x)\} \cap \{(x, y) \mid y - \epsilon \leqq \inf_{x' \in C \cap \mathrm{dom} F} F(x')\},$$

where ϵ−sol(CSP) *and* ϵ−sol(CSP)$'$ *are the set of all* ϵ*-solutions of* (CSP) *and* (CSP)$'$, *respectively.*

Proof. Let us set $E := \{(x, y) \mid y - \epsilon \leqq \inf_{x' \in C \cap \mathrm{dom} F} F(x')\}$. For $(\bar{x}, \bar{y}) \in \epsilon$−sol(CSP),

$$\bar{x} \in C,\ \bar{y} \in F(\bar{x}) \text{ and for any } x \in C \cap \mathrm{dom} F \text{ and any } y \in F(x), \bar{y} - \epsilon \leqq y.$$

Then

$$\bar{x} \in C,\ \bar{y} \in F(\bar{x}) \text{ and for any } x \in C \cap \mathrm{dom} F,$$

$$\begin{cases} F_{\inf}(\bar{x}) - \epsilon \leqq \bar{y} - \epsilon \leqq F_{\inf}(x) \\ \bar{y} - \epsilon \leqq \inf_{x \in C \cap \mathrm{dom} F} F(x). \end{cases}$$

Since $\bar{x} \in \epsilon$−sol(CSP)$'$, $\bar{y} \in F(\bar{x})$ and $(\bar{x}, \bar{y}) \in E$. Therefore, we have

$$(\bar{x}, \bar{y}) \in \{(x, y) \mid x \in \epsilon\text{−sol(CSP)}',\ y \in F(x)\} \cap E.$$

For $(\bar{x}, \bar{y}) \in \{(x, y) \mid x \in \epsilon\text{−sol(CSP)}',\ y \in F(x)\} \cap E$, we have that $\bar{x} \in C$, $\bar{y} \in F(\bar{x})$ such that for all $x \in C \cap \mathrm{dom} F$,

$$\begin{cases} F_{\inf}(\bar{x}) - \epsilon \leqq F_{\inf}(x) \\ \bar{y} - \epsilon \leqq F(x). \end{cases}$$

This implies that $\bar{x} \in C$, $\bar{y} \in F(\bar{x})$ such that for any $x \in C \cap \mathrm{dom} F$, and any $y \in F(x)$, we have $\bar{y} - \epsilon \leqq y$. Therefore, $(\bar{x}, \bar{y}) \in \epsilon$−sol(CSP), and hence,

$$\epsilon\text{−sol(CSP)} = \{(x, y) \mid x \in \epsilon\text{−sol(CSP)}',\ y \in F(x)\} \cap \{(x, y) \mid y - \epsilon \leqq \inf_{x \in C \cap \mathrm{dom} F} F(x)\}.$$ ■

Now we give an example to illustrate Theorems 4.3 and 4.5.

Example 4.1. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$, $F(x) = x^2 + \mathbb{R}_+$, $C = (-\infty, 0]$. Consider the following problem:

$$\begin{array}{lll} \text{(CSP)} & \text{Minimize} & F(x) \\ & \text{subject to} & x \in C. \end{array}$$

Let us establish the auxiliary problem $(\text{CSP})'$:

$$\begin{array}{lll} (\text{CSP})' & \text{Minimize} & F_{\inf}(x) \\ & \text{subject to} & x \in C, \end{array}$$

where $F_{\inf}(x) = x^2$, for any $x \in \mathbb{R}$. For each $\epsilon \geqq 0$, we have that

$$\partial_\epsilon F_{\inf}(\bar{x}) = \begin{cases} \big[-2\sqrt{\epsilon}, 2\sqrt{\epsilon}\big] & \text{if} \quad \bar{x} = 0 \\ \big[2(\bar{x} - \sqrt{\epsilon}), 2(\bar{x} + \sqrt{\epsilon})\big] & \text{if} \quad \bar{x} < 0, \end{cases}$$

$$N_C^\epsilon(\bar{x}) = \begin{cases} [0, +\infty) & \text{if} \quad \bar{x} = 0 \\ \big[0, -\frac{\epsilon}{\bar{x}}\big] & \text{if} \quad \bar{x} < 0. \end{cases}$$

Observe that from Theorem 4.3, $\bar{x} \in C$ is an ϵ-solution of $(\text{CSP})'$ if and only if

$$0 \in \bigcup_{\substack{\epsilon_1 + \epsilon_2 = \epsilon \\ \epsilon_1, \epsilon_2 \geqq 0}} \partial_{\epsilon_1} F_{\inf}(\bar{x}) + N_C^{\epsilon_2}(\bar{x}).$$

This means that there exists $\epsilon_1, \epsilon_2 \geqq 0,\ \epsilon_1 + \epsilon_2 = \epsilon$ such that

$$0 \in \partial_{\epsilon_1} F_{\inf}(\bar{x}) + N_C^{\epsilon_2}(\bar{x}),$$

or equivalently,

$$\partial_{\epsilon_1} F_{\inf}(\bar{x}) \cap -N_C^{\epsilon_2}(\bar{x}) \neq \emptyset. \tag{4.3}$$

Let $\epsilon \geqq 0$. Now we will find the ϵ-sol$(\text{CSP})'$.

Case I. $\bar{x} = 0 \in C$. Taking $\epsilon_1 = \epsilon,\ \epsilon_2 = 0$, we have that (4.3) holds. So, $0 \in \epsilon$-sol$(\text{CSP})'$.

Case II. $\bar{x} \in [-\sqrt{\epsilon}, 0) \subset C$. Taking $\epsilon_1 = \epsilon,\ \epsilon_2 = 0$, by $\bar{x} + \sqrt{\epsilon} \geqq 0$, we have

$$0 \in [2(\bar{x} - \sqrt{\epsilon}), 2(\bar{x} + \sqrt{\epsilon})] = \partial_\epsilon F_{\inf}(\bar{x}) + N_C(\bar{x}).$$

This shows that $[-\sqrt{\epsilon}, 0) \subset \epsilon - \text{sol}(\text{CSP})'$.

Case III. $\bar{x} \in (-\infty, -\sqrt{\epsilon}) \subset C$. We will prove that $\bar{x} \notin \epsilon$-sol$(\text{CSP})'$, i.e., for any $\epsilon_1,\ \epsilon_2 \geqq 0,\ \epsilon_1 + \epsilon_2 = \epsilon$, we have

$$[2(\bar{x} - \sqrt{\epsilon_1}), 2(\bar{x} + \sqrt{\epsilon_1})] \cap \big[\frac{\epsilon_2}{\bar{x}}, 0\big] = \emptyset. \tag{4.4}$$

In the other words,

$$\begin{aligned} 2(\bar{x}+\sqrt{\epsilon_1}) < \frac{\epsilon_2}{\bar{x}} &\iff 2\bar{x}^2 + 2\bar{x}\sqrt{\epsilon_1} - \epsilon_2 > 0 \\ &\iff \left[\begin{array}{l} \bar{x} < \frac{-\sqrt{\epsilon_1}-\sqrt{\epsilon_1+2\epsilon_2}}{2} = \frac{-\sqrt{\epsilon-\epsilon_2}-\sqrt{\epsilon+\epsilon_2}}{2} \\ \bar{x} > \frac{-\sqrt{\epsilon_1}+\sqrt{\epsilon_1+2\epsilon_2}}{2} \geqq \frac{-\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}{2} = 0 \end{array}\right. \\ &\iff \bar{x} < \frac{-\sqrt{\epsilon-\epsilon_2}-\sqrt{\epsilon+\epsilon_2}}{2} \qquad \text{(by } \bar{x} \in C = (-\infty, 0]). \end{aligned}$$

By the Schwartz inequality,

$$(\sqrt{\epsilon-\epsilon_2}+\sqrt{\epsilon+\epsilon_2})^2 \leqq (1^2+1^2)(\epsilon-\epsilon_2+\epsilon+\epsilon_2) = 4\epsilon,$$

or, equivalently,

$$\frac{\sqrt{\epsilon-\epsilon_2}+\sqrt{\epsilon+\epsilon_2}}{2} \leqq \sqrt{\epsilon}. \tag{4.5}$$

In inequality (4.5) the symbol "$=$" is appeared if and only if

$$\sqrt{\epsilon-\epsilon_2} = \sqrt{\epsilon+\epsilon_2} \iff \epsilon_2 = 0.$$

Hence,

(i) If $\epsilon_2 = 0$, then it is clear that (4.4) holds.

(ii) If $\epsilon_2 > 0$, then we have that

$$\frac{-\sqrt{\epsilon-\epsilon_2}-\sqrt{\epsilon+\epsilon_2}}{2} > -\sqrt{\epsilon} > \bar{x}.$$

This shows that (4.4) also holds. Therefore, $\epsilon-\mathrm{sol(CSP)}' = [-\sqrt{\epsilon}, 0]$ and $\mathrm{sol(CSP)}'$ $= \{0\}$. So, $\inf\limits_{x\in C} \bigcup F(x) = \inf\limits_{x\in C} F_{\inf}(x) = F_{\inf}(0) = 0$. Then, by Theorem 4.5, the ϵ-solution set of (CSP) is established as follows:

$$\begin{aligned} \epsilon\text{-sol (CSP)} &= \{(x,y) \mid x \in \epsilon\text{-sol(CSP)}',\ y \in F(x)\} \cap \mathbb{R} \times \{y \mid y - \epsilon \leqq 0\} \\ &= \{(x,y) \mid x \in [-\sqrt{\epsilon}, 0],\ y \in F(x)\} \cap \mathbb{R} \times \{y \mid y \leqq \epsilon\} \\ &= \{(x,y) \mid x \in [-\sqrt{\epsilon}, 0],\ y \in [x^2, \epsilon]\}. \end{aligned}$$

Remark 4.5. In Example 4.1 if F is replaced by the set-valued map defined by $F(x) = x^2 + \mathrm{int}\ \mathbb{R}_+$, then it is worth noticing that although the solution set of Problem (CSP) is empty, for each $\epsilon > 0$ the ϵ-solution set of Problem (CSP) is nonempty. Using our approach, we can see that

$$\epsilon\text{-sol (CSP)} = \{(x,y) \mid x \in [-\sqrt{\epsilon}, 0],\ y \in (x^2, \epsilon]\}.$$

REFERENCES

1. J. Baier and J. Jahn, On subdifferentials of set-valued maps, *J. Optim. Th. Appl.*, **100** (1999), 233-240.
2. G. Bigi and M. Castellani, K-epidrivatives for set-valued functions and optimization, *Math. Meth. Oper. Res.*, **55** (2002), 401-412.
3. R. S. Burachik and V. Jeyakumar, A new geometric condition for Fenchel's duality in infinite dementional spaces, *Math. Program. Ser. B,* **104** (2005), 229-233.
4. G. Y. Chen and J. Jahn, Optimality conditions for set-valued optimization problems, *Math. Meth. Oper. Res.*, **48** (1998), 187-200.
5. H. W. Corley, Optimality conditions for maximizations of set-valued functions, *J. Optim. Th. Appl.*, **58** (1998), 1-10.
6. Fabián Flores-Bazán, Optimality conditions in non-convex set-valued optimization, *Math. Meth. Oper. Res.*, **53** (2001), 403-417.
7. A. Götz and J. Jahn, The Lagrange multiplier rule in set-valued optimization, *SIAM J. Optim.*, **10** (1999), 331-344.
8. J. B. Hiriart-Urruty and C. Lemarechal, *Convex Analysis and Minimization Algorithms*, Vol. I and II, Springer-Verlag, Berlin, Heidelberg, 1993.
9. J. B. Hiriart-Urruty, M. Moussaoui, A. Seeger and M. Volle, Subdifferential calculus without qualification conditions, using approximate subdifferentials: a survey, *Nonlinear Analysis*, **24** (1995), 1727-1754.
10. J. Jahn and R. Rauh, Contingent epiderivatives and set-valued optimization, *Math. Meth. Oper. Res.*, **46** (1997), 193-211.
11. V. Jeyakumar and B. M. Glover, Characterizing global optimality for DC optimization problems under convex inequality constraints, *J. Global Optim.*, **8** (1996), 171-187.
12. G. S. Kim and G. M. Lee, On ϵ-approximate solutions for convex semidefinite optimization problems, *Taiwanese J. Math.*, **11** (2007), 765-784.
13. D. Kuroiwa, On set-valued optimization, *Nonlinear Analysis*, **47** (2001), 1395-1400.
14. P. Loridan, Necessary conditions for ϵ-optimality, *Math. Programming*, **19** (1982), 140-152.
15. P. Loridan, ϵ-Solutions in vector minimization problems, *J. Optim. Th. Appl.*, **43** (1984), 265-276.
16. R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, 1997.
17. W. Song, Weak subdifferential of set-valued mappings, *Optimization*, **52** (2003), 263-276.
18. J. J. Strodiot, V. H. Nguyen and N. Heukemes, ϵ-Optimal solutions in nondifferentiable convex programming and some related questions, *Math. Programming*, **25** (1983), 307-328.

19. K. Yokoyama, ϵ-Optimality criteria for convex programming problems via exact penalty functions, *Math. Programming*, **56** (1992), 233-243.

Gue Myung Lee
Pukyong National University,
Pusan, Korea
E-mail: gmlee@pknu.ac.kr

Le Anh Tuan
Ninh Thuan College of Pedagogy,
Ninh Thuan,
Vietnam
E-mail: latuan02@yahoo.com