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METRIC REGULARITY OF COMPOSITE MULTIFUNCTIONS IN BANACH SPACES

Xi Yin Zheng and Kung Fu Ng

Abstract. We consider metric regularity of composite multifucntions and establish an inequality on the moduli of metric regularity. Refining the original proofs of the Robinson-Ursescu theorem and Lyusternik-Graves theorem, we give an unified analysis for these two theorems, applicable to the composite of a closed convex multifunction and a continuous function.

1. INTRODUCTION

The open mapping theorem on a bounded linear operator between Banach spaces plays a very important role in functional analysis. In 1973, Ng [11] considered an open mapping theorem for a multifunction and proved the following result: Let X be a complete, semi-metrizable topological vector space with the topology induced by a pseud-metric d, Y be a topological vector space and let $F : X \rightrightarrows Y$ be a multifunction whose graph is a closed convex cone. Suppose that the closure $\mathrm{cl}(F(B_d(0,r)))$ of the image of the open ball $B_d(o,r)$ (with center 0 and radius r) in X under F is a neighborhood of 0 in Y for each $r > 0$. Then, $F(B_d(0,\beta)) \supset \mathrm{cl}(F(B_d(0,\alpha)))$ whenever $\beta > \alpha > 0$; consequently, each $F(B_d(0,r))$ is a neighborhood of 0 in Y. In 1975, Ursescu [15] established some open mapping theorems for closed convex multifunctions from a locally convex complete semi-metrizable space to a barelled space. In 1976, Robinson [12] proved the following important metric regularity result: Let F be a closed convex multifunction between two Banach spaces X and Y. Suppose that $(a,b) \in \mathrm{Gr}(F)$ is such that $b + \eta B_Y \subset F(a + B_X)$ for some $\eta > 0$. Then

$$d(x, F^{-1}(y)) \leq \frac{(1 + \|x-a\|)d(y,F(x))}{\eta - \|y-b\|} \quad \forall x \in X \text{ and } \forall y \in B(b,\eta). \tag{1.1}$$

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It is clear that (1.1) implies the following metric regularity: there exist $\tau, \delta \in (0,\ +\infty)$ such that

$$d(x, F^{-1}(y)) \le \tau d(y, F(x)) \quad \forall (x, y) \in B(a, \delta) \times B(b, \delta).$$

Metric regularity plays a very important role in nonlinear optimization and has been well studied (see [1-6, 7-10, 12,14] and references therein). In particular, it is known that F is metrically regular at (a, b) if and only if F is open at a linear rate around (a, b), that is, there exist $\eta, r_0 \in (0,\ +\infty)$ such that

$$B(y, \eta r) \subset F(B(x, r)) \quad \forall (x, y) \in \mathrm{gph}(F) \cap (B(a, \delta) \times B(b, \delta)) \text{ and } r \in (0,\ r_0).$$

Another important metric regularity result is the Lyusternik-Graves theorem (see [1,2,5,8]): Let X, Y be Banach spaces, $f : X \to Y$ be a continuous function and $T : X \to Y$ be a bounded linear operator, and let $a \in X$. Suppose that there exist $r, L, M \in (0,\ +\infty)$ such that $L < M^{-1}$,

$$\|f(x_1) - f(x_2) - T(x_1 - x_2)\| \le L\|x_1 - x_2\| \quad \forall x_1, x_2 \in B(a, r)$$

and for any $y \in Y$ there exists $x \in T^{-1}(y)$ with $\|x\| \le M\|y\|$. Then there exist $\tau, \delta \in (0,\ +\infty)$ such that

$$d(x, f^{-1}(y)) \le \tau\|y - f(x)\| \quad \forall (x, y) \in B(a, \delta) \times B(f(a), \delta).$$

In this paper, we consider metric regularity of composite multifucntions. In particular, we present an unified analysis of the Robinson-Ursescu theorem and the Lyusternik-Graves theorem by considering a multifunction F of the form $G \circ f$ where $f : X \to Y$ is a continuous function and $G : Y \rightrightarrows Z$ is a closed convex multifunction.

Let Y be a normed space. For a subset A of Y, let $\mathrm{aff}(A)$ denote the affine manifold generated by A and let $\mathrm{ri}(A)$ denote the relative interior of A, that is,

$$\mathrm{ri}(A) := \{a \in A :\ \text{there exists } r > 0 \text{ such that } B(a, r) \cap \mathrm{aff}(A) \subset A\},$$

where $B(a, r)$ denotes the open ball with center a and radius r, while $\bar{B}(a, r)$ denotes the corresponding closed ball. It is well known that $\mathrm{ri}(A)$ is nonempty whenever Y is finite dimensional and A is convex (cf. [13, Theorem 6.2]).

Let X, Y be normed spaces and $F : X \rightrightarrows Y$ a multifunction. Let $\mathrm{gph}(F)$ denote the graph of F, that is,

$$\mathrm{gph}(F) := \{(x, y) \in X \times Y :\ y \in F(x)\}.$$

Recall that F is convex (resp. closed) if $\mathrm{gph}(F)$ is a convex (resp. closed) subset of $X \times Y$. Clearly, F is convex if and only if

$$tF(x_1) + (1 - t)F(x_2) \subset F(tx_1 + (1 - t)x_2) \quad \forall t \in [0,\ 1] \text{ and } x_1, x_2 \in X.$$

The following lemma is known (cf. [16, Corollary 1.3.6]) and useful for us.

Lemma 1.1. *Let X, Y be Banach spaces, $F : X \rightrightarrows Y$ be a closed convex multifunction and let $b \in \mathrm{ri}(F(X))$ and $a \in F^{-1}(b)$. Then, there exist $\delta, r \in (0,\ +\infty)$ such that*

$$B(b, \sigma r) \cap \mathrm{aff}(F(X)) \subset F(B(a, \sigma\delta)) \quad \forall \sigma \in [0,\ 1].$$

2. Metric Regularity of Composite Multifunction

Let $F : X \rightrightarrows Y$ be a multifunction and $(a, b) \in \mathrm{gph}(F)$ and recall that F is metrically regular at a for b if there exist $\tau, \delta \in (0,\ +\infty)$ such that

(2.1) $$d(x, F^{-1}(y)) \le \tau d(y, F(x)) \quad \forall (x, y) \in B(a, \delta) \times B(b, \delta).$$

Let $\mathrm{reg}F(a|b)$ denote the metric regularity modulus of F for a at b defined by

$$\mathrm{reg}F(a|b) := \inf\{\tau > 0 :\ (2.1) \text{ holds for some } \delta > 0\}.$$

For a single-valued function $f : X \to Y$ and $\bar{x} \in X$, let $\mathrm{lip}f(\bar{x})$ denote the lipschitz modulus of f at $\bar{x}$ and be defined as

$$\mathrm{lip}f(\bar{x}) := \limsup_{x \to \bar{x}, x' \to \bar{x}} \frac{|f(x) - f(x')|}{\|x - x'\|}.$$

The metric regularity has been extensively studied and a series of interesting and important results has been established. Recently, Dontchev, Lewis and Rockafellar [3] studied the metric regularity of a sum of a multifunction and a single-valued function and proved the following interesting result.

Theorem DLR. *Let $F : X \rightrightarrows Y$ be a closed multifunction and $(\bar{x}, \bar{y}) \in \mathrm{gph}(F)$ be such that $0 < \mathrm{reg}F(\bar{x}|\bar{y}) < +\infty$. Then, for any single-valued function $f : X \to Y$ with $\mathrm{lip}f(\bar{x}) < \frac{1}{\mathrm{reg}F(\bar{x}|\bar{y})}$,*

$$\mathrm{reg}(F + f)(\bar{x}|\bar{y} + f(\bar{x})) < (\mathrm{reg}F(\bar{x}|\bar{y})^{-1} - \mathrm{lip}f(\bar{x}))^{-1}.$$

Motivated by this result and in view of the recent interest of composite functions, we are led to consider the corresponding issue of metric regularity for a composite of two multifunctions.

Proposition 2.1. *Let $G : X \rightrightarrows Z$ be a multifunction, $(a, \bar{z}) \in \mathrm{gph}(G)$ and let $\tau_1, \delta_1 \in (0,\ +\infty)$ be such that*

(2.2) $$d(x, G^{-1}(z)) \le \tau_1 d(z, G(x)) \quad \forall (x, z) \in B(a, \delta_1) \times B(\bar{z}, \delta_1).$$

Let $H : Z \rightrightarrows Y$ be a multifunction, $(\bar{z}, b) \in \mathrm{gph}(H)$ and let $\tau_2, \delta_2 \in (0,\ +\infty)$ be such that

$$d(z, H^{-1}(y)) \leq \tau_2 d(y, H(z)) \quad \forall (z, y) \in B(\bar{z}, \delta_2) \times B(b, \delta_2). \tag{2.3}$$

Let $\eta \in (0,\ \min\{\frac{\delta_1}{2}, \delta_2\})$, $\delta \in (0,\ \delta_1)$, $r \in (0,\ \min\{\frac{\delta_1 - 2\eta}{\tau_2}, \delta_2\})$ and $\tau \in (0,\ +\infty)$ be such that

$$d(y, H(G(x) \cap B(\bar{z}, \eta))) \leq \tau d(y, H(G(x))) \quad \forall (x, y) \in B(a, \delta) \times B(b, r). \tag{2.4}$$

Then

$$d(x, (H \circ G)^{-1}(y)) \leq \tau_1 \tau_2 \tau d(y, (H \circ G)(x)) \quad \forall (x, y) \in B(a, \delta) \times B(b, r). \tag{2.5}$$

Proof. Let $(x, y) \in B(a, \delta) \times B(b, r)$ and $\varepsilon > 0$. Then, (2.4) implies that there exists $z \in G(x) \cap B(\bar{z}, \eta)$ such that

$$d(y, H(z)) < \tau d(y, H(G(x))) + \varepsilon.$$

It follows from (2.3) that

$$d(z, H^{-1}(y)) \leq \tau_2 \tau d(y, H(G(x))) + \tau_2 \varepsilon. \tag{2.6}$$

On the other hand, (2.3) implies that

$$d(\bar{z}, H^{-1}(y)) \leq \tau_2 d(y, H(\bar{z})) \leq \tau_2 \|y - b\| < \tau_2 r,$$

and so $d(z, H^{-1}(y)) < \|z - \bar{z}\| + \tau_2 r$. Take a sequence $\{z_n\}$ in $H^{-1}(y)$ such that

$$\|z - z_n\| \to d(z, H^{-1}(y)) \tag{2.7}$$

and $\|z - z_n\| < \|z - \bar{z}\| + \tau_2 r$. Hence,

$$\|z_n - \bar{z}\| \leq \|z_n - z\| + \|z - \bar{z}\| < 2\|z - \bar{z}\| + \tau_2 r < 2\eta + \tau_2 r \leq \delta_1.$$

By (2.2), one has

$$d(x, G^{-1}(z_n)) \leq \tau_1 d(z_n, G(x)) \leq \tau_1 \|z - z_n\|.$$

Noting that $G^{-1}(z_n) \subset G^{-1}(H^{-1}(y)) = (H \circ G)^{-1}(y)$, it follows from (2.7) that

$$d(x, (H \circ G)^{-1}(y)) \leq \tau_1 d(z, H^{-1}(y)).$$

This and (2.6) imply that

$$d(x, (H \circ G)^{-1}(y)) \leq \tau_1 \tau_2 \tau d(y, (H \circ G)(x)) + \tau_1 \tau_2 \varepsilon.$$

Letting $\varepsilon \to 0$, one sees that (2.5) holds. The proof is completed.

Corollary 2.2. *Let* $g : X \to Z$ *be a single-valued function,* $a \in X$ *and let* $\tau_1, \delta_1 \in (0, +\infty)$ *be such that*

$$d(x, g^{-1}(z)) \le \tau_1 d(z, g(x)) \quad \forall (x, z) \in B(a, \delta_1) \times B(g(a), \delta_1). \tag{2.8}$$

Let $H : Z \rightrightarrows Y$ *be a multifunction,* $(g(a), b) \in \mathrm{gph}(H)$ *and let* $\tau_2, \delta_2 \in (0, +\infty)$ *be such that*

$$d(z, H^{-1}(y)) \le \tau_2 d(y, H(z)) \quad \forall (z, y) \in B(g(a), \delta_2) \times B(b, \delta_2). \tag{2.9}$$

Let $\eta \in (0, \min\{\frac{\delta_1}{2}, \delta_2\})$*,* $\delta \in (0, \delta_1)$ *and* $r \in (0, \min\{\frac{\delta_1 - 2\eta}{\tau_2}, \delta_2\})$ *be such that*

$$g(B(a, \delta)) \subset B(g(a), \eta)). \tag{2.10}$$

Then

$$d(x, (H \circ g)^{-1}(y)) \le \tau_1 \tau_2 d(y, (H \circ g)(x)) \quad \forall (x, y) \in B(a, \delta) \times B(b, r). \tag{2.11}$$

Proof. Let $x \in B(a, \delta)$. Then (2.10) implies that $H(g(x) \cap B(g(a), \eta)) = H(g(x))$ and so

$$d(y, H(g(x) \cap B(g(a), \eta)) = d(y, H(x)) \quad \forall y \in Y.$$

Thus, applying Proposition 2.1 with $G(x) = \{g(x)\}$ and $\tau = 1$, one can see that (2.11) holds.

Let g be continuous at a. Then, for any $\eta > 0$ there exists $\delta > 0$ such that $g(B(a, \delta)) \subset B(g(a), \eta)$. This and Corollary 2.2 imply the following result.

Corollary 2.3. *Let* $g : X \to Z$ *be a single-valued function and* $H : Z \rightrightarrows Y$ *be a multifunction. Let* $a \in X$ *and* $b \in H(g(a))$ *and suppose that* g *is continuous at* a*. Then*

$$\mathrm{reg}(H \circ g)(a|b) \le \mathrm{reg} g(a|g(a)) \mathrm{reg} H(g(a)|b). \tag{2.12}$$

Remark. Inequality (2.12) can be strict. Let $X = Y = Z = \mathbb{R}^2$, $g(s, t) = (\frac{1}{2}s, t)$ and $H(s, t) = \{(2s, t)\}$ for all $(s, t) \in \mathbb{R}^2$, and let $a = b = (0, 0)$. Thus $(H \circ g)(s, t) = (s, t)$ for all $(s, t) \in \mathbb{R}^2$, so $\mathrm{reg}(H \circ g)(a|b) = 1$. Let $(s_1, t_1), (s_2, t_2) \in \mathbb{R}^2$. Then,

$$d((s_1, t_1), g^{-1}(s_2, t_2)) = \|(s_1, t_1) - (2s_2, t_2)\| = ((s_1 - 2s_2)^2 + (t_1 - t_2)^2)^{\frac{1}{2}},$$

$$d((s_2, t_2), g(s_1, t_1)) = \|(s_2, t_2) - \Big(\frac{1}{2}s_1, t_1\Big)\| = \Big(\frac{1}{4}(s_1 - 2s_2)^2 + (t_1 - t_2)^2\Big)^{\frac{1}{2}},$$

$$d((s_1,t_1),H^{-1}(s_2,t_2)) = \|(s_1,t_1) - \Big(\frac{1}{2}s_2,t_2\Big)\| = \Big(\frac{1}{4}(2s_1-s_2)^2 + (t_1-t_2)^2\Big)^{\frac{1}{2}}$$

and

$$d((s_2,t_2),H(s_1,t_1)) = ((2s_1-s_2)^2 + (t_1-t_2)^2)^{\frac{1}{2}}.$$

Hence,

$$d((s_1,t_1),g^{-1}(s_2,t_2)) \le 2d(s_2,t_2),g(s_1,t_1)) \quad \forall (s_1,t_1),(s_2,t_2)\in\mathbb{R}^2,$$

$$d((s_1,t_1),H^{-1}(s_2,t_2)) \le d((s_2,t_2),H(s_1,t_1)) \quad \forall (s_1,t_1),(s_2,t_2)\in\mathbb{R}^2,$$

$$d((s_1,0),g^{-1}(s_2,0)) = 2d(s_2,0),g(s_1,0)) \quad \forall s_1,s_2\in\mathbb{R}$$

and

$$d((0,t_1),H^{-1}(0,t_2)) = d((0,t_2),H(0,t_1)) \quad \forall t_1,t_2\in\mathbb{R}.$$

It follows that $\mathrm{reg}g(a|g(a)) = 2$ and $\mathrm{reg}H(g(a)|b) = 1$. This shows that inequality (2.12) can be strict.

3. Unified Approach to the Robinson-ursescu and Lyusternik-graves Theorems

Let X, Y, Z be Banach spaces, $f : X \to Z$ a continuous function and $G : Z \rightrightarrows Y$ a closed convex multifunction. Suppose that there exist $a \in X$, $b \in G(f(a))$, $r, L, M \in (0,\ +\infty)$ and a bounded linear operator $T : X \to Z$ such that $b \in \mathrm{int}(G(Z))$, $L < M^{-1}$,

$$\|f(x_1) - f(x_2) - T(x_1 - x_2)\| \le L\|x_1 - x_2\| \quad \forall x_1, x_2 \in B(a,r)$$

and for any $y \in Y$ there exists $x \in T^{-1}(y)$ with $\|x\| \le M\|y\|$. Then, Corollary 2.3 together with both Robinson-Ursescu theorem and Lyusternik-Graves theorem implies immediately that the composite $G \circ f$ is metrically regular at (a,b). In this section, refining the original proofs of the Robinson-Ursescu theorem and Lyusternik-Graves theorem, we give an unified analysis for these two theorems. In particular, in this unification we present concrete metric regularity bounds and regions. To do this, let T be a surjective bounded linear operator between Banach spaces X and Y. Then, the open mapping theorem implies that there exists $r > 0$ such that $\bar{B}(0,r) \subset T(B(0,1))$. Hence, for any $y \in Y$ there exists $x \in X$ with $\|x\| \le \frac{1}{r}\|y\|$ such that $T(x) = y$. Let

$$\|T^{-1}\| := \inf\left\{M \in [0,\ +\infty) : \inf_{x\in T^{-1}(y)} \|x\| \le M\|y\| \quad \forall y \in Y\right\}.$$

Then, $\|T^{-1}\| \le \frac{1}{r}$.

Theorem 3.1. *Let X, Y be Banach spaces and Z be a normed sapce. Let $F : X \rightrightarrows Z$ be a multifunction such that $F(x) = G(f(x))$ for all $x \in X$, where $f : X \to Y$ is a continuous function and $G : Y \rightrightarrows Z$ is a (not necessarily closed) convex multifunction. Let $a \in X$, $b \in F(a)$ and $T : X \to Y$ be a surjective bounded linear operator. Suppose that there exist $r, \delta_1, \delta_2, L \in (0, +\infty)$ with $L < \|T^{-1}\|^{-1}$ such that*

$$B(b, r) \cap \mathrm{aff}(G(Y)) \subset G(B(f(a), \delta_1)). \tag{3.1}$$

and

$$\|f(x_1) - f(x_2) - T(x_1 - x_2)\| \le L\|x_1 - x_2\| \quad \forall x_1, x_2 \in B(a, \delta_2). \tag{3.2}$$

Let γ and δ be positive numbers such that

$$\gamma < \delta_1 \text{ and } \frac{\gamma + (\|T\| + \|T^{-1}\|^{-1})\delta}{\|T^{-1}\|^{-1} - L} \le \delta_2; \tag{3.3}$$

let $\tau := \frac{\gamma + (\|T\| + L)\delta}{\|T^{-1}\|^{-1} - L}$ and $\gamma_1 \in (0, \frac{\gamma r}{\delta_1})$. Then,

$$d(x, F^{-1}(z)) \le \frac{\tau d(z, F(x))}{\frac{\gamma r}{\delta_1} - \gamma_1 + d(z, F(x))} \tag{3.4}$$

for any $(x, z) \in B(a, \delta) \times (\mathrm{aff}(G(Y)) \cap B(b, \gamma_1))$, where $d(z, \emptyset)$ is understood as $+\infty$ and $\frac{+\infty}{+\infty}$ is understood as $+\infty$.

Remark. Inequality (3.4) is different from Robinson's inequality and stronger than usual metric inequality due to the presence of $d(z, F(x))$ in both the numerator and the denominator on the right-hand side.

We postpone the proof of Theorem 3.1 to the end of this setion. Letting $Y = Z$, $r = \delta_1$ and G be the identity mapping, the following corollary is seen to be an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let X, Y be Banach spaces and $f : X \to Y$ be a continuous function. Let $a \in X$ and $T : X \to Y$ be a surjective bounded linear operator. Suppose that there exist $\delta_2, L \in (0, +\infty)$ with $L < \|T^{-1}\|^{-1}$ such that (3.2) holds. Let γ and δ be positive numbers such that*

$$\frac{\gamma + (\|T\| + \|T^{-1}\|^{-1})\delta}{\|T^{-1}\|^{-1} - L} \le \delta_2;$$

let $\tau := \frac{\gamma + (\|T\| + L)\delta}{\|T^{-1}\|^{-1} - L}$ and $\gamma_1 \in (0, \gamma)$. Then

$$d(x, f^{-1}(y)) \le \frac{\tau\|y - f(x)\|}{\gamma - \gamma_1 + \|y - f(x)\|} \quad \forall (x, y) \in B(a, \delta) \times B(f(a), \gamma_1).$$

Note that $\frac{\|y-f(x)\|}{\gamma-\gamma_1+\|y-f(x)\|} \leq \frac{\|y-f(x)\|}{\gamma-\gamma_1}$. Corollary 3.2 implies that the Lyusternik-Graves theorem mentioned in Section 1.

Let $f : X \to Y$ be a continuous mapping, $T : X \to Y$ be a bounded linear operator and let a be a point in X. Let us introduce a constant defined by

$$L(f,T,a) := \limsup_{(x,h)\to(a,0)} \frac{\|f(x+h)-f(x)-T(h)\|}{\|h\|}.$$

Thus, for example, $L(f, f'(a), a) = 0$ if $f : X \to Y$ is strictly differentiable at $a \in X$, namely if there exists a bounded linear operator $f'(a) : X \to Y$ such that

$$\lim_{(x,h)\to(a,0)} \frac{f(x+h)-f(x)-f'(a)(h)}{\|h\|} = 0.$$

Theorem 3.3. *Let X, Y be Banach spaces and Z be a normed space. Let $F : X \rightrightarrows Z$ be a multifunction such that $F(x) = G(f(x))$ for all $x \in X$, where $f : X \to Y$ is a continuous function and $G : Y \rightrightarrows Z$ is a closed convex multifunction. Let $b \in \mathrm{ri}(F(X))$, $a \in F^{-1}(b)$ and $T : X \to Y$ be a surjective bounded linear operator. Suppose that $L(f,T,a) < \|T^{-1}\|^{-1}$ and that $\mathrm{aff}(F(X))$ is complete. Then, there exist $\tau, r \in (0, +\infty)$ such that*

(3.5) $\quad d(x, F^{-1}(z)) \leq \tau d(z, F(x)) \quad \forall (x,z) \in B(a,r) \times (\mathrm{aff}(F(X)) \cap B(b,r)).$

Proof. We claim that

$$\mathrm{aff}(F(X)) = \mathrm{aff}(G(Y)). \tag{3.6}$$

Granting this, by Lemma 1.1 (applied to $Y, Z, G, f(a), b$ in place of X, Y, F, a, b), there exist $r, \delta_1 \in (0, +\infty)$ such that (3.1) holds. This and Theorem 3.1 imply that there exist $\tau, r \in (0, +\infty)$ such that (3.5) holds. It remains to show that (3.6) holds. Since $F(X) \subset G(Y)$, we need only show that $G(Y) \subset \mathrm{aff}(F(X))$. Since $b \in \mathrm{ri}(F(X))$, there exists $r_0 > 0$ such that

$$B(b, r_0) \cap \mathrm{aff}(F(X)) \subset F(X). \tag{3.7}$$

Let $z \in G(Y)$. Then there exists $y \in Y$ such that $z \in G(y)$. Noting that $b \in F(a) = G(f(a))$, it follows from the convexity of G that $(1-\frac{1}{n})b + \frac{1}{n}z \in G((1-\frac{1}{n})f(a) + \frac{1}{n}y)$. Since $(1-\frac{1}{n})f(a) + \frac{1}{n}y \to f(a)$ as $n \to \infty$, Corollary 3.2 implies that there exists a natural number n (sufficiently large) such that $f^{-1}\left((1-\frac{1}{n})f(a) + \frac{1}{n}y\right) \neq \emptyset$. It follows that there exists $x_n \in X$ such that $(1-\frac{1}{n})f(a) + \frac{1}{n}y = f(x_n)$. Hence

$$(1-\frac{1}{n})b + \frac{1}{n}z \in G(f(x_n)) = F(x_n).$$

Since $b \in F(a)$, this implies that

$$z = b + n\left((1-\frac{1}{n})b + \frac{1}{n}z - b\right) \in \mathrm{aff}(F(X)).$$

This shows that $G(Y) \subset \mathrm{aff}(F(X))$. The proof is completed.

Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and consider the following inequality

(IE) $$\phi(x) \leq 0.$$

Let S denote the solution set of (IE), and recall that (IE) has a local error bound at $a \in S$ if there exist $\tau, \delta \in (0, +\infty)$ such that

$$d(x, S) \leq [\phi(x)]+ \quad \forall x \in B(a, \delta)$$

where $[\phi(x)]_+ = \max\{0, \phi(x)\}$. It is well-known that if ϕ is convex and (IE) satisfies the Slater condition (i.e., there exists $x_0 \in X$ such that $\phi(x_0) < 0$) then ϕ has a local error bound at each point in S. As an application of Theorem 3.3, we can extend this result to the more general "composite-convex" case.

Corollary 3.4. *Let X, Y be Banach spaces, $f : X \to Y$ be a continuous mapping, and let $\psi : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let $\phi(x) := \psi(f(x))$ for all $x \in X$ be such that the corresponding inequality (IE) satisfies the Slater condition. Let $a \in S$ and suppose that there exists a surjective bounded linear operator $T : X \to Y$ such that $L(f, T, a) < \|T^{-1}\|^{-1}$. Then (IE) has a local error bound at a.*

Proof. Let $G(y) := [\psi(y), +\infty)$ for all $y \in Y$ and $F(x) := G(f(x))$ for all $x \in X$. Then G is a convex closed multifunction. It is clear that the Slater condition means $0 \in \mathrm{int}(F(X))$. By Theorem 3.3 (applied to 0 in place of b), there exist $\tau, r \in (0, +\infty)$ such that $d(x, F^{-1}(0)) \leq \tau d(0, F(x))$ for all $x \in B(a, r)$. This implies that (IE) has a local error bound at a. The proof is completed.

Note that $\mathrm{aff}(F(X)) = Z$ if $b \in \mathrm{int}(F(X))$. The following result is a consequence of Theorem 3.3.

Corollary 3.5. *Let X, Y, Z be Banach spaces and $F : X \rightrightarrows Z$ be a multifunction such that $F(x) = G(f(x))$ for all $x \in X$, where $f : X \to Y$ is a continuous function and $G : Y \rightrightarrows Z$ is a closed convex multifunction. Let $b \in \mathrm{int}(F(X))$, $a \in F^{-1}(b)$ and $T : X \to Y$ be a surjective bounded linear mapping. Suppose that $L(f, T, a) < \|T^{-1}\|^{-1}$ (e.g., f is strictly differentiable at a with $T := f'(a)$ such that $f'(a)$ is surjective). Then, there exists $\tau, r \in (0, +\infty)$ such that*

$$d(x, F^{-1}(z)) \leq \tau d(z, F(x)) \quad \forall (x, z) \in B(a, r) \times B(b, r).$$

Corollary 3.5 generalizes Robinson's metric regularity result (let $Y = Z$, and let $f = T$ be the identify map).

We conclude this section with the proof of Theorem 3.1.

Proof of Theorem 3.1. For (3.4), let $z \in B(b, \gamma_1) \cap \operatorname{aff}(G(Y))$ and $x \in B(a, \delta) \setminus F^{-1}(z)$. Let $\varepsilon > 0$ and take $z' \in F(x) = G(f(x))$ such that

$$\|z - z'\| < d(z, F(x)) + \varepsilon.$$

Let $\gamma_2 := \frac{\gamma r}{\delta_1} - \gamma_1$. Then, $\left\| z + \frac{\gamma_2(z-z')}{\|z-z'\|} - b \right\| \le \|z - b\| + \gamma_2 < \gamma_1 + \gamma_2 = \frac{\gamma r}{\delta_1}$, and so $z + \frac{\gamma_2(z-z')}{\|z-z'\|} \in B(b, \frac{\gamma r}{\delta_1}) \cap \operatorname{aff}(G(Y))$. Note (by (3.1) and the convexity of G) that

$$\begin{aligned} B(b, \frac{\gamma r}{\delta_1}) \cap \operatorname{aff}(G(Y)) &= (1 - \frac{\gamma}{\delta_1})b + \frac{\gamma}{\delta_1} B(b, r) \cap \operatorname{aff}(G(Y)) \\ &\subset (1 - \frac{\gamma}{\delta_1}) G(f(a)) + \frac{\gamma}{\delta_1} G(B(f(a), \delta_1)) \\ &\subset G(B(f(a), \gamma)), \end{aligned}$$

and it follows that there exists $y \in B(f(a), \gamma)$ such that $z + \frac{\gamma_2(z-z')}{\|z-z'\|} \in G(y)$. Hence

$$\begin{aligned} z &= \frac{\|z - z'\|}{\gamma_2 + \|z - z'\|} \left(z + \frac{\gamma_2(z - z')}{\|z - z'\|} \right) + \frac{\gamma_2}{\gamma_2 + \|z - z'\|} z' \\ &\in \frac{\|z - z'\|}{\gamma_2 + \|z - z'\|} G(y) + \frac{\gamma_2}{\gamma_2 + \|z - z'\|} G(f(x)). \end{aligned}$$

Let

$$y' := \frac{\|z - z'\|}{\gamma_2 + \|z - z'\|} y + \frac{\gamma_2}{\gamma_2 + \|z - z'\|} f(x);$$

it follows from the convexity of G that

$$z \in G(y') \quad \text{and} \quad \|y' - f(x)\| = \frac{\|z - z'\| \|y - f(x)\|}{\gamma_2 + \|z - z'\|}. \tag{3.8}$$

On the other hand, note that $x \in B(a, \delta)$ and $\delta \le \delta_2$ (by $L < \|T^{-1}\|^{-1}$ and the second inequality in (3.3)). Thus (3.2) entails that $\|f(a) - f(x) - T(a - x)\| \le L\|a - x\|$, and so

$$\|f(a) - f(x)\| \le (\|T\| + L)\|x - a\| < (\|T\| + L)\delta.$$

It follows that

$$\|y - f(x)\| \le \|y - f(a)\| + \|f(a) - f(x)\| < \gamma + (\|T\| + L)\delta. \tag{3.9}$$

This and the equality in (3.8) imply that $\|y' - f(x)\| < \gamma + (\|T\| + L)\delta$. Take an arbitrary η in $(\|y' - f(x)\|,\ \gamma + (\|T\| + L)\delta)$ and choose τ' in $(0,\ \|T^{-1}\|^{-1} - L)$ sufficiently close to $\|T^{-1}\|^{-1} - L$ such that

$$\frac{\eta}{\tau'} < \frac{\gamma + (\|T\| + L)\delta}{\|T^{-1}\|^{-1} - L}. \tag{3.10}$$

Then, $\|T^{-1}\| < \frac{1}{\tau'+L}$, and hence, for any $v \in Y$ there exists $u \in T^{-1}(v)$ such that $\|u\| \leq \frac{1}{\tau'+L}\|v\|$. Letting $u_0 = 0$, we can construct a sequence $\{u_n\}$ in X such that for each $n \geq 1$,

$$u_n \in T^{-1}\left(y' - f\left(x + \sum_{i=0}^{n-1} u_i\right)\right) \tag{3.11}$$

and

$$\|u_n\| \leq \frac{1}{\tau' + L}\left\|y' - f\left(x + \sum_{i=0}^{n-1} u_i\right)\right\|. \tag{3.12}$$

We claim that for every nonnegative integer n,

$$\left\|y' - f\left(x + \sum_{i=0}^{n} u_i\right)\right\| \leq \frac{\eta L^n}{(\tau' + L)^n}. \tag{3.13}$$

Indeed, being true for $n = 0$, suppose that (3.13) holds for $n \leq k$. Then, (3.12) implies that

$$\sum_{i=0}^{k+1}\|u_i\| \leq \frac{1}{\tau' + L}\sum_{i=1}^{k+1}\left\|y' - f\left(x + \sum_{j=0}^{i-1} u_j\right)\right\| \leq \sum_{i=1}^{k}\frac{\eta L^{i-1}}{(\tau' + L)^i} < \frac{\eta}{\tau'} < \delta_2 - \delta$$

(the last inequality holds because of (3.3) and (3.10)). Therefore, $x + \sum_{i=0}^{n} u_i \in B(a, \delta_2)$ for all $n \leq k+1$. It follows from (3.2) that

$$\left\|f\left(x + \sum_{i=0}^{k+1} u_i\right) - f\left(x + \sum_{i=0}^{k} u_i\right) - T(u_{k+1})\right\| \leq L\|u_{k+1}\|,$$

which means that $\left\|y' - f\left(x + \sum_{i=0}^{k+1} u_i\right)\right\| \leq L\|u_{k+1}\|$ (by (3.11)). Since (3.13) holds for $n = k$, this and (3.12) imply that

$$\left\|y' - f\left(x + \sum_{i=0}^{k+1} u_i\right)\right\| \leq \frac{L}{\tau' + L}\left\|y' - f\left(x + \sum_{i=0}^{k} u_i\right)\right\| \leq \frac{\eta L^{k+1}}{(\tau' + L)^{k+1}},$$

which verifies that (3.13) holds for $n = k+1$. We have therefore shown that (3.13) holds for every nonnegative integer n. By (3.12) and (3.13), one has

$$\sum_{n=1}^{\infty} \|u_n\| \le \sum_{n=1}^{\infty} \frac{\eta L^n}{(\tau' + L)^{n+1}} \le \frac{\eta}{\tau'}$$

and so $\sum\limits_{n=0}^{\infty} u_n$ is convergent. Let $x' := x + \sum\limits_{n=0}^{\infty} u_n$. Then, $\|x' - x\| \le \frac{\eta}{\tau'}$ and $f(x') = y'$ (by (3.13)). Hence $G(y') = F(x')$, and so $z \in F(x')$. This implies that

$$d(x, F^{-1}(z)) \le \|x - x'\| \le \frac{\eta}{\tau'}.$$

Letting $\eta \to \|y' - f(x)\|$ and $\tau' \to \|T^{-1}\|^{-1} - L$, it follows from (3.8) that

$$d(x, F^{-1}(z)) \le \frac{\|y' - f(x)\|}{\|T^{-1}\|^{-1} - L} = \frac{\|z - z'\| \|y - f(x)\|}{(\gamma_2 + \|z - z'\|)(\|T^{-1}\|^{-1} - L)}$$

By (3.9) and the definition of τ, one notices that $\frac{\|y - f(x)\|}{\|T^{-1}\|^{-1} - L} < \tau$ and it follows that

$$d(x, F^{-1}(z)) \le \tau \frac{\|z - z'\|}{\gamma_2 + \|z - z'\|} \le \tau \frac{d(z, F(x)) + \varepsilon}{\gamma_2 + d(z, F(x)) + \varepsilon}.$$

Letting $\varepsilon \to 0$, we have $d(x, F^{-1}(z)) \le \tau \frac{d(z, F(x))}{\gamma_2 + d(z, F(x))}$. By the definition of γ_2, we see that inequality (3.4) holds. The proof is completed.

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Xi Yin Zheng
Department of Mathematics,
Yunnan University,
Kunming 650091,
P. R. China
E-mail: xyzheng@ynu.edu.cn

Kung Fu Ng
Department of Mathematics,
The Chinese University of Hong Kong,
Shatin, New Territory,
Hong Kong
E-mail: kfng@math.cuhk.edu.hk