

GENERALIZED CONSTRAINTS QUALIFICATION CONDITIONS AND INFINITE DIMENSIONAL DUALITY

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Abstract. Following the papers [7, 8, 9] and [19], the strong duality between a convex optimization problem with both cone and equality constraints is further studied by means of the concepts of normal cone and of quasi relative interior. In such a way the difficulty that often the ordering cone has empty interior is overcome.

1. INTRODUCTION

In the papers [7, 8, 9] and [19] the authors present an infinite dimensional duality theory which, with the aid of generalized constraints qualification assumptions related to the notion of quasi relative interior called *Assumption S* ([7, 8]) and *Assumption N* ([9, 19]), guarantees the existence of strong duality between a convex optimization problem and its Lagrange dual. The use of quasi relative interior, introduced by Borwein and Lewis [2], and the notions of tangent and normal cone, allows to overcome the difficulty that in many cases the interior of the set involved in the regularity condition is empty. This is the case of all the Optimization Problems or Variational Inequalities connected with network equilibrium problems, the obstacle problem, the elastic-plastic torsion problem (see [1, 5, 6, 10, 12, 13, 14, 15, 17, 18, 20]) which use positive cones of $L^p(\Omega)$ or of Sobolev spaces. Then the usual interior conditions, as the core, the intrinsic core or the strong-quasi relative interior condition (see [22]) are not suitable for our problem because, for example, the strong-quasi relative interior of the positive cone of $L^p(\Omega)$, namely the most general condition among the above mentioned ones, is empty. Also the result of Jeyakumar and Wolkowicz [16] which uses the notion of quasi relative interior, however requires that the cone defining the constraints has a non empty interior.

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The convex optimization problem and its Lagrange dual we are concerning with are the following. Given $f : S \rightarrow \mathbb{R}$, $g : S \rightarrow Y$, $h : S \rightarrow Z$, where S is a convex subset of a real linear topological space X , Y is a real normed space ordered by a convex cone C , Z is a real normed space and h is an affine-linear mapping, we consider the optimization problem:

$$f(x_0) = \min_{x \in \mathbb{K}} f(x), \quad \text{Problem 1}$$

with

$$\mathbb{K} = \{x \in S : g(x) \in -C, h(x) = \theta_Z\}$$

and the Lagrange dual problem:

$$\max_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in S} [f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle], \quad \text{Problem 2}$$

where

$$C^* = \{u \in Y^* : \langle u, y \rangle \geq 0, \forall y \in C\}.$$

Now let us recall the main concepts which we need in order to present the duality results.

Given a point $x \in X$ and a subset C of X , the set

$$T_C(x) = \left\{ h \in X : h = \lim_{n \rightarrow \infty} \lambda_n(x_n - x), \lambda_n \in \mathbb{R} \text{ and } \lambda_n > 0 \forall n \in \mathbb{N}, \right. \\ \left. x_n \in C, \forall n \in \mathbb{N} \text{ and } \lim_n x_n = x \right\}$$

is called the *tangent cone* to C at x . Of course, if $T_C(x) \neq \emptyset$, then $x \in \text{cl } C$. If $x \in \text{cl } C$ and C is convex, then we have:

$$T_C(x) = \text{cl cone } (C - \{x\}),$$

where

$$\text{cone } (C) = \{\lambda x : x \in C, \lambda \in \mathbb{R}, \lambda \geq 0\}.$$

Following Borwein and Lewis (see [2]), we give the following definition of quasi-relative interior for a convex set.

Definiton 1.1. *Let C be a convex subset of X . The quasi-relative interior of C , denoted by $\text{qri } C$, is the set of those $x \in C$ for which $T_C(x)$ is a linear subspace of X .*

If we define the normal cone to C at x by

$$N_C(x) = \{\xi \in X^* : \langle \xi, y - x \rangle \leq 0, \forall y \in C\},$$

where X^* is a topological dual space of X , the following result holds true.

Proposition 1.1. *Let C be a convex subset of X and $x \in C$. Then $x \in \text{qri } C$ if and only if $N_C(x)$ is a linear subspace of X^* .*

The first theorem on duality is the following (see [9]).

Theorem 1.1. *Assume that the functions $f : S \rightarrow \mathbb{R}$, $g : S \rightarrow Y$ are convex and that $h : S \rightarrow Z$ is an affine-linear mapping. Assume that the following Assumption S is fulfilled at the extremal solution $x_0 \in \mathbb{K}$ to Problem 1, namely*

$$T_{\widetilde{M}}(f(x_0), \theta_Y, \theta_Z) \cap]-\infty, 0[\times \theta_Y \times \theta_Z = \emptyset, \quad \text{Assumption } S$$

where

$$\widetilde{M} = \{(f(x) + \alpha, g(x) + y, h(x)) : x \in S \setminus \mathbb{K}, \alpha \geq 0, y \in C\}.$$

Then also Problem 2 is solvable and if $\bar{u} \in C^*$, $\bar{v} \in Z^*$ are the optimal points to Problem 2, we have:

$$\langle \bar{u}, g(x_0) \rangle = 0$$

and the optimal values of the two problems are equal, namely

$$(1) \quad \begin{aligned} f(x_0) &= f(x_0) + \langle \bar{u}, g(x_0) \rangle + \langle \bar{v}, h(x_0) \rangle \\ &= \max_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in S} [f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle]. \end{aligned}$$

Next, in [9], it is also proved the following result based on the notion of normal cone.

Theorem 1.2. *Let $f : S \rightarrow \mathbb{R}$, $g : S \rightarrow Y$, $h : S \rightarrow Z$ be three functions such that the following assumption holds:*

$$\exists \bar{x} \in \mathbb{K}, \exists (\hat{\xi}, \hat{y}^*, \hat{z}^*) \in N_M(0, \theta_Y, \theta_Z) \text{ such that}$$

$$\hat{\xi}(f(\bar{x}) - f(x_0)) + \langle \hat{y}^*, g(\bar{x}) \rangle + \langle \hat{z}^*, h(\bar{x}) \rangle < 0 \quad \text{Assumption } N$$

where

$$M = \{(f(x) - f(x_0) + \alpha, g(x) + y, h(x)), x \in S, \alpha \geq 0, y \in C\}$$

and $x_0 \in \mathbb{K}$ is the optimal solution to Problem 1. Then Problem 2 is solvable and the optimal values of both problems are equal.

Assumption S, besides the strong duality result, guarantees that the point $(0, \theta_Y, \theta_Z)$ does not belong to *qri M*, namely that $T_M(0, \theta_Y, \theta_Z)$ or $N_M(0, \theta_Y, \theta_Z)$ are not linear subspaces.

In turns, also *Assumption N* is strictly connected with the *qri M* notion. In fact, if *Assumption N* holds, then there exists a point $(\hat{\xi}, \hat{y}^*, \hat{z}^*) \in N_M(0, \theta_Y, \theta_Z)$ with $\hat{\xi} \neq 0$ and $N_M(0, \theta_Y, \theta_Z)$ is not a linear subspace (see Section 3). Then, if M is convex, namely for instance if f and g are convex and h is affine-linear, also the tangent cone $T_M(0, \theta_Y, \theta_Z)$ (see Proposition 1.1) is not a linear subspace and $(0, \theta_Y, \theta_Z) \notin \text{qri } M$. Vice versa, if $(0, \theta_Y, \theta_Z) \notin \text{qri } M$, $N_M(0, \theta_Y, \theta_Z)$ cannot be a linear subspace and there exists $(\hat{\xi}, \hat{y}^*, \hat{z}^*) \neq (0, \theta_{Y^*}, \theta_{Z^*})$ such that $(\hat{\xi}, \hat{y}^*, \hat{z}^*) \in N_M(0, \theta_Y, \theta_Z)$. However, in order to obtain the strong duality result we need that $\hat{\xi} \neq 0$, therefore it is necessary to make some additional assumptions. In Section 2, we will prove the following result, which is slightly different from Theorem 1.2.

Theorem 1.3. *Let $f : S \rightarrow \mathbb{R}$, $g : S \rightarrow Y$, $h : S \rightarrow Z$ be three functions with h affine-linear. Let us assume that S is a linear subspace of X , $cl(C - C) = Y$, $cl(h(S - S)) = Z$ and there exists $\hat{x} \in S$ with $g(\hat{x}) \in -\text{qri } C$ and $h(\hat{x}) = \theta_Z$. If there exists $(\hat{\xi}, y^*, z^*) \in N_M(0, \theta_{Y^*}, \theta_{Z^*})$ with $(\hat{\xi}, y^*, z^*) \neq (0, \theta_{Y^*}, \theta_{Z^*})$ and Problem 1 is solvable, also Problem 2 is solvable and the optimal values of both problems are equal.*

In Section 3, we point out the reason for which the strong duality result is connected with the normal cone and hence with the tangent cone and the quasi relative interior of M . In fact, let us denote by $\bar{u} \in C^*$ and $\bar{z} \in Z^*$ the optimal points of Problem 2. We will show the following result.

Theorem 1.4. *The strong duality between Problem 1 and Problem 2 (Lagrange dual) holds if and only if $(-1, -\bar{u}, -\bar{v})$ belongs to the normal cone $N_M(0, \theta_Y, \theta_Z)$.*

In Section 4, we study the duality for various infinite-dimensional equilibrium problems and it is worth mentioning that *Assumption S* is the most effective for the applications, whereas the other assumptions have only a theoretical interest.

2. PROOF OF THEOREM 1.3

First we report, for the reader's convenience, the proof of the sign property of the points of $N_M(0, \theta_Y, \theta_Z)$ (see [9], Sec. 4). The points $(\xi, \tilde{y}, \tilde{z}) \in \mathbb{R} \times Y^* \times Z^*$ of the cone $N_M(0, \theta_Y, \theta_Z)$ are such that:

$$(2) \quad \begin{aligned} \xi(f(x) - f(x_0) + \alpha) + \langle \tilde{y}, g(x) + y \rangle + \langle \tilde{z}, h(x) \rangle &\leq 0 \\ \forall x \in S, \forall \alpha \geq 0, \forall y \in C. \end{aligned}$$

Assuming in (2) $x \in \mathbb{K}$, $\alpha = 0$ and $y = -g(x) \in C$, we get:

$$\xi(f(x) - f(x_0)) \leq 0, \quad \forall x \in \mathbb{K},$$

and hence we obtain:

$$(3) \quad \xi \leq 0.$$

Now, choosing in (2) $x = x_0$, $\alpha = 0$ and $y = -g(x_0) + z \in C \quad \forall z \in C$, we get:

$$\langle \tilde{y}, z \rangle \leq 0, \quad \forall z \in C,$$

namely

$$(4) \quad \tilde{y} \in C^-.$$

Then, each point $(\xi, \tilde{y}, \tilde{z})$ of $N_M(0, \theta_Y, \theta_Z)$ is such that $\xi \leq 0$ and $\tilde{y} \in C^-$ and, if $N_M(0, \theta_Y, \theta_Z)$ contains some points $(\xi, \tilde{y}, \tilde{z})$, with $\xi \neq 0$ or $\tilde{y} \neq \theta_{Y^*}$, it cannot be a linear subspace.

Now, let us pass to the proof of Theorem 1.3. From the assumptions there exists $(\hat{\xi}, y^*, z^*) \in N_M(0, \theta_{Y^*}, \theta_{Z^*})$ with $(\hat{\xi}, y^*, z^*) \neq (0, \theta_{Y^*}, \theta_{Z^*})$. Then $(\hat{\xi}, y^*, z^*)$ verifies (2), (3) and (4). We recall that $\langle -y^*, g(x_0) \rangle = 0$. In fact, choosing $x = x_0$, $y = \theta_Y$ and $\alpha = 0$, from (2) we get

$$\langle y^*, g(x_0) \rangle \leq 0.$$

Because $g(x_0) \in -C$, it follows

$$\langle y^*, g(x_0) \rangle \geq 0$$

and hence the claim.

Then, from (2) written for $\xi = \hat{\xi}$, $\tilde{y} = y^*$, $\tilde{z} = z^*$, we get

$$(5) \quad \begin{aligned} &\hat{\xi}(f(x) - f(x_0) + \alpha) + \langle y^*, g(x) + y \rangle + \langle z^*, h(x) \rangle \leq 0, \\ &\forall x \in S, \forall \alpha \geq 0, \forall y \in C. \end{aligned}$$

Let us prove that $\hat{\xi} < 0$. Assume that $\hat{\xi} = 0$. From (5), when $y = \theta_Y$, we get

$$(6) \quad \langle y^*, g(x) \rangle + \langle z^*, h(x) \rangle \geq 0 \quad \forall x \in S.$$

By assumption, $\exists \hat{x} \in S$ such that

$$-g(\hat{x}) \in \text{qri } C, \quad h(\hat{x}) = \theta_Z$$

and, this fact, taking into account that in virtue of (6) it results that $\langle y^*, g(\hat{x}) \rangle \leq 0$, leads to $\langle y^*, g(\hat{x}) \rangle = 0$. Therefore

$$\langle y^*, y + g(\hat{x}) \rangle \leq 0 \quad \forall y \in C$$

and then $-y^* \in N_C(-g(\hat{x}))$.

In virtue of Proposition 1.1, also $-y^* \in N_C(-g(\hat{x}))$ and $\langle y^*, y \rangle \geq 0 \quad \forall y \in C$. This implies $\langle y^*, C \rangle = 0$ and $\langle y^*, \text{cl}(C - C) \rangle = 0$, namely $y^* = \theta_{Y^*}$. Now let us prove that also z^* must be θ_{Z^*} . From (5) we get, being $\mu = 0$, $y^* = \theta_{Y^*}$

$$(7) \quad \langle z^*, h(x) \rangle \leq 0 \quad \forall x \in S$$

and, taking into account that $h(\hat{x}) = 0$ and that S is a linear subspace, we obtain

$$-h(x) = h(-x + 2\hat{x}) \quad \forall x \in S,$$

and hence

$$\langle z^*, h(-x + 2\hat{x}) \rangle = -\langle z^*, h(x) \rangle \geq 0, \quad \forall x \in S.$$

Then it results that

$$\langle z^*, h(x) \rangle = 0 \quad \forall x \in S$$

and also

$$\langle z^*, h(S) \rangle = 0, \quad \langle z^*, -h(S) \rangle = 0, \quad \langle z^*, h(S) - h(S) \rangle = 0, \quad \langle z^*, \text{cl } h(S - S) \rangle = 0,$$

namely, since it is $\text{cl } h(S - S) = Z$ by assumption, it turns out to be $z^* = \theta_{Z^*}$. In such a way we obtain $(\hat{\xi}, y^*, z^*) = (0, \theta_{Y^*}, \theta_{Z^*})$ that is an absurdity. Consequently, it is $\hat{\xi} > 0$ and from (5) for $\alpha = 0$ and $y = \theta_Y$ we get:

$$(8) \quad f(x_0) \leq f(x) - \frac{1}{\hat{\xi}} \langle y^*, g(x) \rangle - \frac{1}{\hat{\xi}} \langle z^*, h(x) \rangle \quad \forall x \in S.$$

Setting $\bar{u} = -\frac{1}{\hat{\xi}} y^* \in C^*$, $\bar{v} = -\frac{1}{\hat{\xi}} z^* \in Z^*$ and having in mind that $\langle \bar{u}, g(x_0) \rangle = 0$ and $\langle \bar{v}, h(x_0) \rangle = 0$, we obtain

$$\inf_{x \in S} [f(x) + \langle \bar{u}, g(x) \rangle + \langle \bar{v}, h(x) \rangle] \geq f(x_0) + \langle \bar{u}, g(x_0) \rangle + \langle \bar{v}, h(x_0) \rangle.$$

But, taking into account that $\langle u, g(x_0) \rangle \leq 0 \quad \forall u \in C^*$ and $\langle v, h(x_0) \rangle = 0 \quad \forall v \in Z^*$, it results

$$\inf_{x \in S} [f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle] \leq f(x_0) + \langle u, g(x_0) \rangle + \langle v, h(x_0) \rangle \leq f(x_0) \\ \forall u \in C^*, \forall v \in Z^*.$$

Then

$$\sup_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in S} [f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle] \leq f(x_0) + \langle \bar{u}, g(x_0) \rangle + \langle \bar{v}, h(x_0) \rangle$$

from which we deduce

$$\begin{aligned} f(x_0) + \langle \bar{u}, g(x_0) \rangle + \langle \bar{v}, h(x_0) \rangle &\leq \sup_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in S} [f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle] \\ &\leq f(x_0) + \langle \bar{u}, g(x_0) \rangle + \langle \bar{v}, h(x_0) \rangle, \end{aligned}$$

namely

$$\begin{aligned} f(x_0) &= f(x_0) + \langle \bar{u}, g(x_0) \rangle + \langle \bar{v}, h(x_0) \rangle \\ &= \max_{\substack{u \in C^* \\ v \in Z^*}} \inf_{x \in S} [f(x) + \langle u, g(x) \rangle + \langle v, h(x) \rangle] \end{aligned}$$

and the assert is proved.

3. PROOF OF THEOREM 1.4

Let us assume that the strong duality (1) holds. Then, we have the following inequality:

$$(9) \quad f(x) + \langle \bar{u}, g(x) \rangle + \langle \bar{v}, h(x) \rangle \geq f(x_0) + \langle \bar{u}, g(x_0) \rangle + \langle \bar{v}, h(x_0) \rangle, \quad \forall x \in S,$$

form which we get

$$(10) \quad \begin{aligned} -(f(x) - f(x_0) + \alpha) + \langle -\bar{u}, g(x) + y \rangle + \langle -\bar{v}, h(x) \rangle &\leq 0, \\ \forall \alpha \geq 0, x \in S, y \in C, \end{aligned}$$

because $\langle \bar{u}, g(x_0) \rangle = 0$, $\langle \bar{v}, h(x_0) \rangle = 0$ and $\langle \bar{u}, y \rangle \geq 0, \forall y \in C$. But (10) means that $(-1, -\bar{u}, -\bar{v}) \in N_M(0, \theta_Y, \theta_Z)$.

Vice versa, if $(-1, -\bar{u}, -\bar{v}) \in N_M(0, \theta_Y, \theta_Z)$ then we have (10). Assuming $\alpha = 0, x = x_0$ and $y = -g(x_0) + z, \forall z \in C$, being $-g(x_0) \in C$ and C a convex cone, we get $y \in C$ and we have

$$\langle -\bar{u}, z \rangle \leq 0, \quad \forall z \in C,$$

namely

$$\bar{u} \in C^*.$$

Moreover, from (10) assuming $\alpha = 0, x = x_0$ and $y = 0$, using the usual technique, we get

$$\langle \bar{u}, g(x_0) \rangle = 0.$$

Finally, from (10) we have

$$f(x) + \langle \bar{u}, g(x) \rangle + \langle \bar{v}, h(x) \rangle \geq f(x_0), \quad \forall x \in S,$$

from which (see Proof of Theorem 1.1 in [9]), the strong duality result follows.

4. APPLICATIONS TO DYNAMIC EQUILIBRIUM PROBLEMS

The aim of this section is to show the effectiveness of *Assumption S* in the applications. In paper [9] the validity of *Assumption S* has been shown for the archetype problem which models all the equilibrium problems (see [5, 6, 10, 11, 12, 13, 14, 15, 17, 18, 20]), that is the variational inequality

$$(11) \quad \int_0^T \langle C(x_0(t)), x(t) - x_0(t) \rangle dt \geq 0 \quad \forall x \in \mathbb{K},$$

where

$$\mathbb{K} = \{x \in L^2([0, T], \mathbb{R}^m) : x(t) \geq 0, \Phi x(t) = \rho(t) \text{ a.e. in } [0, T]\},$$

with $\rho \in L^2([0, T], \mathbb{R}^l)$, $\rho(t) > 0$ a.e. in $[0, T]$, $\Phi = \{\Phi_{ij}\}_{\substack{i=1, \dots, l \\ j=1, \dots, m}}$, $\Phi_{ij} \in \{0, 1\}$, and in each column there is one entry different from zero and $C : \mathbb{K} \rightarrow L^2([0, T], \mathbb{R}^m)$ is the cost trajectory.

Here, we would like to show that *Assumption S* is verified by the variational inequality which expresses the dynamic Cournot-Nash equilibrium, namely the dynamic oligopolistic market equilibrium problem (see [1]):

$$(12) \quad \text{Find } x^* \in \mathbb{K} : \ll -\nabla v(t, x^*(t)), x - x^* \gg \geq 0 \quad \forall x \in \mathbb{K}$$

where

$$\mathbb{K} = \{x \in L^2([0, T], \mathbb{R}^m) : 0 \leq \lambda(t) \leq x(t) \leq \mu(t) \text{ a.e. in } [0, T]\}.$$

The function $v_i(t, x(t)) \in C^1([0, T], \mathbb{R}^m)$, $i = 1, \dots, m$, and such that

$$\|v_i(t, x(t))\| \leq A(t)\|x(t)\| + B(t),$$

is the profit of the firm P_i at time $t \in [0, T]$. In the paper [1] the following Lemma is proved.

Lemma 4.1. *Let $x^* \in \mathbb{K}$ be a solution to the variational inequality (12). Then, setting for $i = 1, \dots, m$*

$$\begin{aligned}
 E_-^i &= \{t \in [0, T] : x_i^*(t) = \lambda_i(t) \text{ a.e. in } [0, T]\}, \\
 E_0^i &= \{t \in [0, T] : \lambda_i(t) < x_i^*(t) < \mu_i(t) \text{ a.e. in } [0, T]\}, \\
 E_+^i &= \{t \in [0, T] : x_i^*(t) = \mu_i(t) \text{ a.e. in } [0, T]\},
 \end{aligned}$$

we have:

$$\begin{aligned}
 -\frac{\partial v(t, x^*(t))}{\partial x_i} &\geq 0 \text{ a.e. in } E_-^i, \\
 \frac{\partial v(t, x^*(t))}{\partial x_i} &= 0 \text{ a.e. in } E_0^i, \\
 -\frac{\partial v(t, x^*(t))}{\partial x_i} &\leq 0 \text{ a.e. in } E_+^i.
 \end{aligned}$$

Now, let us rewrite Problem (12) in a more suitable form. If $x^* \in \mathbb{K}$ is a solution to variational inequality (12) and we set

$$\psi(x) = \ll \nabla v(x^*), x - x^* \gg, \quad \forall x \in \mathbb{K},$$

we get

$$\psi(x) \geq 0 \quad \forall x \in \mathbb{K}$$

and x^* is a minimal solution of the problem

$$(13) \quad \min_{x \in \mathbb{K}} \psi(x) = \psi(x^*) = 0.$$

With the aid of this Lemma we can show the following theorem:

Theorem 4.1. *Problem (13) verifies Assumption S.*

Proof. Let us recall that

$$\begin{aligned}
 &T_{\widetilde{M}}(\psi(x^*), \theta_{L^2([0, T], \mathbb{R}^m)}, \theta_{L^2([0, T], \mathbb{R}^m)}) \\
 &= \left\{ y : y = \lim_{n \rightarrow +\infty} \lambda_n \left[\left(\psi(x_n) + \alpha_n, -x_n + \lambda + y_n, \right. \right. \right. \\
 &\quad \left. \left. \left. -\mu + x_n + z_n \right) - \left(\psi(x^*), \theta_{L^2([0, T], \mathbb{R}^m)}, \theta_{L^2([0, T], \mathbb{R}^m)} \right) \right], \text{ with } \lambda_n > 0, \right. \\
 &\quad \theta_{L^2([0, T], \mathbb{R}^m)} = \lim_{n \rightarrow +\infty} \lambda_n (\lambda - x_n + y_n), \\
 &\quad \theta_{L^2([0, T], \mathbb{R}^m)} = \lim_{n \rightarrow +\infty} \lambda_n (x_n - \mu + z_n), \\
 &\quad \theta_{L^2([0, T], \mathbb{R}^m)} = \lim_{n \rightarrow +\infty} (\psi(x_n) + \alpha_n), \\
 &\quad \theta_{L^2([0, T], \mathbb{R}^m)} = \lim_{n \rightarrow +\infty} (\lambda - x_n + y_n), \\
 &\quad \theta_{L^2([0, T], \mathbb{R}^m)} = \lim_{n \rightarrow +\infty} (x_n - \mu + z_n), \\
 &\quad \left. x_n \in L^2([0, T], \mathbb{R}^m) \setminus \mathbb{K}, \alpha_n \geq 0, y_n, z_n \in C \right\},
 \end{aligned}$$

where C is the ordering cone in $L^2([0, T], \mathbb{R}^m)$. In order to achieve *Assumption S*, we prove that, if $(l, \theta_{L^2([0, T], \mathbb{R}^m)}, \theta_{L^2([0, T], \mathbb{R}^m)})$ belongs to $T_{\widetilde{M}}(\psi(x^*), \theta_{L^2([0, T], \mathbb{R}^m)}, \theta_{L^2([0, T], \mathbb{R}^m)})$, then $l \geq 0$. It results:

$$\begin{aligned} l &= \lim_{n \rightarrow +\infty} \lambda_n \left(\sum_{i=1}^m \int_0^T \frac{\partial v(t, x^*(t))}{\partial x_i} (x_i^n(t) - x_i^*(t)) dt + \alpha_n \right) \\ &\geq \lim_{n \rightarrow +\infty} \lambda_n \left(\sum_{i=1}^m \int_{E_-^i} \frac{\partial v(t, x^*(t))}{\partial x_i} (x_i^n(t) - \lambda_i(t)) dt \right. \\ &\quad \left. + \sum_{i=1}^m \int_{E_0^i} \frac{\partial v(t, x^*(t))}{\partial x_i} (x_i^n(t) - x_i^*(t)) dt \right. \\ &\quad \left. + \sum_{i=1}^m \int_{E_+^i} \frac{\partial v(t, x^*(t))}{\partial x_i} (x_i^n(t) - \mu_i(t)) dt \right). \end{aligned}$$

Furthermore we remark that

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \lambda_n \sum_{i=1}^m \int_{E_-^i} \frac{\partial v(t, x^*(t))}{\partial x_i} (x_i^n(t) - \lambda_i(t)) dt \\ &= \lim_{n \rightarrow +\infty} \lambda_n \left[\sum_{i=1}^m \left(\int_{E_-^i} \frac{\partial v(t, x^*(t))}{\partial x_i} (x_i^n(t) - \lambda_i(t) - y_i^n(t)) dt \right. \right. \\ &\quad \left. \left. + \int_{E_-^i} \frac{\partial v(t, x^*(t))}{\partial x_i} y_i^n(t) dt \right) \right] \geq 0 \end{aligned}$$

because

$$\lim_{n \rightarrow +\infty} (x_i^n(t) - \lambda_i(t) - y_i^n(t)) = \theta_L, \quad y_i^n(t) \geq 0, \quad \lambda_n \geq 0.$$

Moreover we have

$$\lim_{n \rightarrow +\infty} \lambda_n \sum_{i=1}^m \int_{E_0^i} \frac{\partial v(t, x^*(t))}{\partial x_i} (x_i^n(t) - x_i^*(t)) dt = 0$$

and, finally,

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \lambda_n \sum_{i=1}^m \int_{E_+^i} \frac{\partial v(t, x^*(t))}{\partial x_i} (x_i^n(t) - \mu_i(t)) dt \\ &= \lim_{n \rightarrow +\infty} \lambda_n \sum_{i=1}^m \int_{E_+^i} \frac{\partial v(t, x^*(t))}{\partial x_i} (x_i^n(t) + z_i^n(t) - \mu_i(t) - z_i^n(t)) dt \\ &= \lim_{n \rightarrow +\infty} \sum_{i=1}^m \int_{E_+^i} \frac{\partial v(t, x^*(t))}{\partial x_i} \lambda_n (x_i^n(t) + z_i^n(t) - \mu_i(t)) dt \\ &\quad + \lim_{n \rightarrow +\infty} \lambda_n \sum_{i=1}^m \int_{E_+^i} \frac{\partial v(t, x^*(t))}{\partial x_i} (-z_i^n(t)) dt \geq 0 \end{aligned}$$

because

$$\lim_{n \rightarrow +\infty} \lambda_n (x_i^n(t) + z_i^n(t) - \mu_i(t)) = 0, \quad \lambda_n \geq 0, \quad z_i^n \geq 0.$$

Hence the *Assumption S* holds. ■

Finally we recall that *Assumption S* guarantees the existence of the Lagrange multiplier associated to the elastic-plastic torsion problem (see [3, 4, 8, 21]).

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