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CODERIVATIVES OF EFFICIENT POINT MULTIFUNCTIONS IN PARAMETRIC VECTOR OPTIMIZATION

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Dedicated to Boris Mordukhovich on his 60th birthday

Abstract. This paper is concerned with generalized differentiation of the efficient point multifunctions of parametric vector optimization problems in Banach spaces. We give formulae for computing and/or estimating the Fréchet coderivative (precoderivative) of this multifunction in a broad class of conventional vector optimization problems with the presence of geometric, operator, (finite and infinite) functional constraints. Examples are given to illustrate the obtained results.

1. INTRODUCTION

Let $f : P \times X \to Y$ be a vector function, $C : P \rightrightarrows X$ a multifunction where P, X and Y are Banach spaces. Given a pointed (i.e., $K \cap (-K) = \{0\}$) closed convex cone $K \subset Y$, we consider the following *parametric vector optimization problem*

$$\min_K \left\{ f(p, x) \,|\, x \in C(p) \right\} \tag{1.1}$$

depending on the *parameter* $p \in P$. Here, x is a *decision variable* and the cone K induces a partial order $\preceq_K$ on Y, i.e.,

$$y \preceq_K y' \Leftrightarrow y' - y \in K, \quad y, y' \in Y. \tag{1.2}$$

The "$\min_K$" in (1.1) is understood with respect to the ordering relation $\preceq_K$ from (1.2).

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We say that $y \in A$ is an *efficient point* of a subset $A \subset Y$ with respect to K and write $y \in \text{Min}A$, if and only if $(y - K) \cap A = \{y\}$. If $A = \emptyset$, then we define $\text{Min}A = \emptyset$.

Let $F : P \rightrightarrows Y$ be a multifunction given by

$$F(p) = (f \circ C)(p) = f(p, C(p)) = \{f(p, x) \mid x \in C(p)\}. \tag{1.3}$$

We put

$$\mathcal{F}(p) = \text{Min}F(p), \quad p \in P \tag{1.4}$$

and call $\mathcal{F} : P \rightrightarrows Y$ the *efficient point multifunction* of (1.1).

Sensitivity analysis in vector optimization problems, i.e., the behavior of the efficient point multifunction $\mathcal{F}$ is analyzed by using certain concepts of *generalized derivatives* for multifunctions. The papers by Tanino [34, 35] are among the first results in this field. In those papers, the author has studied the behavior of $\mathcal{F}$ via the concept of *contingent derivative* introduced by Aubin [1]. Various sensitivity analysis results in this direction can be found in Shi [32, 33], Kuk, Tanino, and Tanaka [18, 19]. Bednarczuk and Song [2] (also see [4]) introduced the notion of *generalized contingent epiderivatives* for a multifunction and used it to study sensitivity of a family of parametric optimization problems with multifunctions. Later, Song and Wan [31] have used this concept to derive some sensitivity results on parametric vector optimization problems. Namely, the authors gave a representation of generalized contingent epiderivative of $\mathcal{F}$ in terms of the derivative of the objective function and the contingent derivative of the constraint mapping. Using the so-called *proto-differentiability* (see [29]), Lee and Huy [20] proved that, under suitable conditions, the efficient point multifunction $\mathcal{F}$ is proto-differentiable. Moreover, sufficient conditions for inner and outer approximations of the proto-derivative of $\mathcal{F}$ are given. Recently, the formulae for computing and/or estimating the *generalized Clarke epiderivative* (see [3] for the definition) of the efficient point multifunction $\mathcal{F}$ are given by the authors in [9] in terms of the Clarke tangent cone to the graph of a multifunction or the constraint mapping C and/or the Fréchet derivative of the objective function f.

To proceed further, we first emphasize that the *generalized derivatives* mentioned above for multifunctions generated by *tangent cones* to their graphs. Mordukhovich [22] introduced the very notion of *coderivatives* via a *normal* cone to the graph, independently of the normal cone used. The realization of the coderivative construction is given in [22] via the so-called *limiting normal cone* introduced by Mordukhovich in 1976 [23] which was required by applications to optimal control. The reader is referred to [25] for details on several types of coderivative and particularly the *normal/Mordukhovich coderivative* and the *Fréchet coderivative* (*precoderivative*). The *generalized derivative* approach in *primal spaces* and *coderivative* one in *dual* *spaces* are generally independent since there are normal cones that are not dual to any tangent ones, for instance the limiting normal cone (see, e.g., [25]). So it is natural to analyze the behavior of the efficient point multifunction $\mathcal{F}$ by using certain concepts of *coderivatives* instead of *generalized derivatives* for multifunctions. One part of this has been investigated in the recent paper [17]. Namely, the authors gave the formulae for computing and estimating the *Mordukhovich coderivatives* of efficient point multifunctions in parametric vector optimization problems with respect to the so-called *generalized order optimality*. There is a significant difference between dealing with the Fréchet coderivative and the Mordukhovich one is that the latter has the *exact* calculi (see [25]) while the former only has the *fuzzy*/approximate calculi (see [24, 36]).

In this paper, we make an effort to establish the formulae for computing (*precise/equality* form) and estimating the *Fréchet coderivatives* of the efficient point multifunction $\mathcal{F}$ of (1.1). The formulae for computing and/or estimating the Fréchet coderivative of this multifunction are presented in a broad class of conventional vector optimization problems with the presence of geometric, operator and (finite and infinite) functional constraints.

The rest of the paper is organized as follows. Section 2 is devoted to providing further the basic definitions and notations from vector optimization, set-valued analysis and variational analysis. In Section 3 we first establish the calculus for computing or estimating the Fréchet coderivatives of a sum of a multifunction and a cone and the *exact* rule for the Fréchet coderivative of the composition of a vector function and a multifunction. Then we derive formulae for computing (precise/equality form) and estimating the Fréchet coderivatives of the efficient point multifunction $\mathcal{F}$ in general case. The further elaboration of these formulae on the concrete/conventional classes in parametric vector optimization problems will be made in the last section. Moreover, examples are also simultaneously provided for analyzing and illustrating the obtained results.

2. Basic Definitions and Preliminaries

Throughout the paper we use the standard notations of variational analysis and generalized differentiation; see, e.g., [25, 26]. Unless otherwise stated, all spaces under consideration are Banach spaces whose norms are always denoted by $\|\cdot\|$. The canonical pairing between X and its dual X^* is denoted by $\langle\cdot\,,\cdot\rangle$. The symbol A^* denotes the adjoint operator of a linear continuous operator A. The closed ball with center x and radius ρ is denoted by $B_\rho(x)$. The *uniformly positive polar* to cone K is defined by

$$(2.1) \qquad K^*_{up} := \{y^* \in Y^* \mid \exists\beta > 0,\ \langle y^*, k\rangle \geq \beta||k|| \quad \forall k \in K\}.$$

A single-valued mapping $f\colon P \to Y$ is said to be *strictly differentiable* at $\bar{p}$ if there is a linear continuous operator $\nabla f(\bar{p})\colon P \to Y$ such that

$$\lim_{p,u\to\bar{p}} \frac{f(p)-f(u)-\langle \nabla f(\bar{p}), p-u\rangle}{\|p-u\|}=0.$$

Let $l\colon \Omega\subset X\to Y$ be a single-valued mapping and $\bar{x}\in\Omega$. l is said to be *local upper Lipschitzian* (or *calm*) at $\bar{x}$ if there are numbers $\eta>0$ and $\ell\geq 0$ such that

$$\|l(x)-l(\bar{x})\|\leq \ell\|x-\bar{x}\| \ \text{ for all } \ x\in B_\eta(\bar{x})\cap\Omega.$$

We say that a multifunction $L\colon X\rightrightarrows Y$ admits a *local upper Lipschitzian selection* at $(\bar{x},\bar{y})\in \operatorname{gph} L$ if there is a single-valued mapping $l\colon \operatorname{dom} L\to Y$ which is local upper Lipschitzian at $\bar{x}$ satisfying $l(\bar{x})=\bar{y}$ and $l(x)\in L(x)$ for all $x\in\operatorname{dom} L$ in a neighborhood of $\bar{x}$.

We say that the *domination property* holds for $F: P\rightrightarrows Y$ around $\bar{p}\in P$ if there exists a neighborhood U of $\bar{p}$ such that

$$F(p)\subset \mathrm{Min} F(p)+K \quad \forall p\in U.$$

Given a set-valued mapping $F\colon X\rightrightarrows X^*$ between a Banach space X and its dual X^*, we denote by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x):=\Big\{x^*\in X^*\Big|\ \ \exists \text{ sequences } \ x_n\to\bar{x} \ \text{ and } \ x_n^*\xrightarrow{w^*}x^* \\ \text{with } \ x_n^*\in F(x_n) \ \text{ for all } \ n\in\mathbb{N}\Big\}$$

the *sequential Painlevé-Kuratowski upper/outer limit* with respect to the norm topology of X and the weak* topology of X^* where $\mathbb{N}:=\{1,2,\ldots\}$.

Given $\Omega\subset X$ and $\varepsilon\geq 0$, define the collection of *ε-normals* to Ω at $\bar{x}\in\Omega$ by

$$\widehat{N}_\varepsilon(\bar{x};\Omega):=\Big\{x^*\in X^*\Big|\limsup_{x\xrightarrow{\Omega}\bar{x}}\frac{\langle x^*,x-\bar{x}\rangle}{\|x-\bar{x}\|}\leq\varepsilon\Big\}, \tag{2.2}$$

where $x\xrightarrow{\Omega}\bar{x}$ means that $x\to\bar{x}$ with $x\in\Omega$. When $\varepsilon=0$, the set $\widehat{N}(\bar{x};\Omega):=\widehat{N}_0(\bar{x};\Omega)$ in (2.2) is a cone called the *Fréchet normal cone* to Ω at $\bar{x}$. If $\bar{x}\notin\Omega$, we put $\widehat{N}_\varepsilon(\bar{x};\Omega):=\emptyset$ for all $\varepsilon\geq 0$.

The *limiting/Mordukhovich normal cone* $N(\bar{x};\Omega)$ is obtained from $\widehat{N}_\varepsilon(x;\Omega)$ by taking the sequential Painlevé-Kuratowski upper limit in the weak* topology of X^* as

$$N(\bar{x};\Omega):=\operatorname*{Lim\,sup}_{\substack{x\xrightarrow{\Omega}\bar{x}\\ \varepsilon\downarrow 0}}\widehat{N}_\varepsilon(x;\Omega), \tag{2.3}$$

where one can put $\varepsilon = 0$ when Ω is *closed around* $\bar{x}$, i.e., there is a neighborhood U of $\bar{x}$ such that $\Omega \cap U$ is closed and the space X is *Asplund*, i.e., a Banach space whose separable subspaces have separable duals.

For an extended real-valued function $\varphi : X \to \overline{\mathbb{R}} := [-\infty, \infty]$, we define

$$\mathrm{dom}\,\varphi = \{x \in X \mid \varphi(x) < \infty\}, \quad \text{epi } = \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\}.$$

The *limiting/Mordukhovich subdifferential* and the *Fr échet subdifferential* of φ at $\bar{x}$ with $|\varphi(\bar{x})| < \infty$ are defined, respectively, by

$$\partial\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \text{epi}\varphi)\}$$
$$\widehat{\partial}\varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}((\bar{x}, \varphi(\bar{x})); \text{epi}\varphi)\}.$$

If $|\varphi(\bar{x})| = \infty$, then one puts $\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x}) = \emptyset$. One has $\widehat{\partial}\varphi(\bar{x}) \subset \partial\varphi(\bar{x})$ for any $\bar{x}$. We say that φ is *lower regular* at $\bar{x}$ if the latter holds as equality, i.e.,

$$\widehat{\partial}\varphi(\bar{x}) = \partial\varphi(\bar{x}). \tag{2.4}$$

The collection of lower regular functions is sufficiently large including, besides *convex* and *strictly differentiable* ones, many other classes of functions important in variational analysis and optimization; see the books [25, 26, 30] for more details, discussions and applications.

In this paper we consider the *Fréchet upper subdifferential* of φ at $\bar{x}$ with $|\varphi(\bar{x})| < \infty$ which is defined by

$$\widehat{\partial}^+\varphi(\bar{x}) = -\widehat{\partial}(-\varphi)(\bar{x}). \tag{2.5}$$

It is known that (see [25, Proposition 1.87]) $\widehat{\partial}\varphi(\bar{x}) \neq \emptyset$ and $\widehat{\partial}^+\varphi(\bar{x}) \neq \emptyset$ if and only if φ is Fréchet differentiable at $\bar{x}$ in which case $\widehat{\partial}\varphi(\bar{x}) = \widehat{\partial}^+\varphi(\bar{x}) = \{\nabla\varphi(\bar{x})\}$.

Let $F\colon P \rightrightarrows Y$ be a set-valued mapping between Banach spaces with the domain and the graph

$$\mathrm{dom}\,F := \{p \in P \mid F(p) \neq \emptyset\}, \quad \text{gph}\,F := \big\{(p, y) \in P \times Y \,\big|\, y \in F(p)\big\}.$$

The *Fréchet coderivative* of F at $(\bar{p}, \bar{y}) \in \mathrm{gph}\,F$ is defined by

$$\widehat{D}^*F(\bar{p}, \bar{y})(y^*) := \{p^* \in P^* \mid (p^*, -y^*) \in \widehat{N}((\bar{p}, \bar{y}); \mathrm{gph}\,F))\}. \tag{2.6}$$

If F is a single-valued map, to simplify the notation, one writes $\widehat{D}^*F(\bar{p})(y^*)$ instead of $\widehat{D}^*F(\bar{p}, F(\bar{p}))(y^*)$. It is well known that (see [25, Theorem 1.38]) if $f : P \to Y$ is Fréchet differentiable at $\bar{p}$, then we have

$$\widehat{D}^*f(\bar{p})(y^*) = \{\nabla f(\bar{p})^* y^*\} \text{ for all } y^* \in Y^*.$$

Furthermore, for any vector function $f : P \to Y$ between Banach spaces we associate f with a scalarization function with respect to some $y^* \in Y^*$ defined by

$$\langle y^*, f\rangle(p) = \langle y^*, f(p)\rangle \text{ for all } p \in P.$$

There is a relationship between the Fréchet coderivative of Lipschitzian vector function and the Fréchet subdifferential of their scalarization which is formulated as follows (see e.g., [27, Proposition 3.5]) If f is Lipschitz continuous around $\bar{p} \in P$, then one has

$$\widehat{D}^* f(\bar{p})(y^*) = \widehat{\partial}\langle y^*, f\rangle(\bar{p}) \ \text{ for all } y^* \in Y^*. \tag{2.7}$$

In fact the equality in (2.7) remains valid if instead of f being Lipschitz continuous around $\bar{p}$ we only assume that f is local upper Lipschitzian at the corresponding point.

In what follows we also use the so-called *smooth variational description* of Fréchet subgradients in Banach spaces which is as follows.

Lemma 2.1. (See [25, Theorem 1.88 (i)]). *Let* $\varphi : X \to \overline{\mathbb{R}}$ *be finite at* $\bar{x}$. *Then* $x^* \in \widehat{\partial}\varphi(\bar{x})$ *if and only if there are a neighborhood* U *of* $\bar{x}$ *and a function* $s : U \to \mathbb{R}$ *which is Fréchet differentiable at* $\bar{x}$ *with the derivative* $\nabla s(\bar{x})$ *such that*

$$s(\bar{x}) = \varphi(\bar{x}), \nabla s(\bar{x}) = x^* \textit{ and } s(x) \le \varphi(x) \textit{ for all } x \in U.$$

3. Fréchet Coderivatives of $\mathcal{F}$ in General Vector Optimization Problems

In this section we derive formulae for computing (precise/equality form) and estimating the Fréchet coderivatives of the efficient point multifunction $\mathcal{F}$ defined in (1.4). To do this, we first need to compute or estimate the Fréchet coderivatives of a sum of a multifunction and a cone.

Proposition 3.1. *Let* $G : P \rightrightarrows Y$ *be a multifunction between Banach spaces and let* $(\bar{p}, \bar{y}) \in \mathrm{gph}G$. *One has*

$$\widehat{D}^*(G + K)(\bar{p}, \bar{y})(y^*) \subset \widehat{D}^*G(\bar{p}, \bar{y})(y^*) \quad \forall y^* \in Y^*, \tag{3.1}$$

and the converse inclusion holds if $y^* \in K^*_{up}$ *defined in* (2.1).

Proof. Since $0 \in K$, it follows that $\mathrm{gph}G \subset \mathrm{gph}(G+K)$. Taking into account the *monotonicity* property of the Fréchet normal cone (see e.g., [25, Page 5]) we have

$$\widehat{N}\big((\bar{p}, \bar{y}); \mathrm{gph}(G + K)\big) \subset \widehat{N}\big((\bar{p}, \bar{y}); \mathrm{gph}G\big)$$

and thus (3.1) is established by the definition of the Fréchet coderivative in (2.6). To justify the inverse inclusion we fix $y^* \in K^*_{up}$ and pick any $p^* \in \widehat{D}^*G(\bar{p}, \bar{y})(y^*)$.

By the Fréchet coderivative and Fréchet normal cone descriptions (2.6) and (2.2) with $\varepsilon = 0$, we have

$$\limsup_{(p,y)\xrightarrow{\text{gph}G}(\bar{p},\bar{y})} \frac{\langle (p^*,-y^*),(p,y)-(\bar{p},\bar{y})\rangle}{\|(p,y)-(\bar{p},\bar{y})\|} \le 0.$$

This means that for every $\epsilon > 0$, find $\eta_1 > 0, \eta_2 > 0$ such that

$$\langle p^*, p-\bar{p}\rangle \le \langle y^*, y-\bar{y}\rangle + \epsilon(||p-\bar{p}|| + ||y-\bar{y}||)$$

for all $p \in B_{\eta_1}(\bar{p}), y \in B_{\eta_2}(\bar{y})$ with $(p,y) \in \text{gph}\, G$. Furthermore, we claim from the choice of $y^* \in K^*_{up}$ defined in (2.1) that there exists $\beta > 0$ satisfying

$$\langle y^*, k\rangle \ge \beta||k|| \quad \forall k \in K.$$

Assume further without loss of generality that $\epsilon \le \beta$. Thus we have the inequalities

$$\begin{aligned} \langle p^*, p-\bar{p}\rangle &\le \langle y^*, y+k-\bar{y}\rangle - \beta||k|| + \epsilon(||p-\bar{p}|| + ||y-\bar{y}||) \\ &\le \langle y^*, y+k-\bar{y}\rangle + \epsilon(||p-\bar{p}|| + ||y+k-\bar{y}||) \end{aligned} \tag{3.2}$$

for all $p \in B_{\eta_1}(\bar{p}), y \in B_{\eta_2}(\bar{y})$ with $(p,y) \in \text{gph}\, G$ and all $k \in K$. Since $\epsilon > 0$ was chosen arbitrarily, it follows from (3.2) that

$$\limsup_{(p,y)\xrightarrow{\text{gph}(G+K)}(\bar{p},\bar{y})} \frac{\langle (p^*,-y^*),(p,y)-(\bar{p},\bar{y})\rangle}{\|(p,y)-(\bar{p},\bar{y})\|} \le 0.$$

Thus $p^* \in \widehat{D}^*(G+K)(\bar{p},\bar{y})(y^*)$ by (2.2) with $\varepsilon = 0$ and (2.6) which completes the proof of the theorem. ■

The following example shows that the inclusion in (3.1) may be *strict* if $y^* \notin K^*_{up}$.

Example 3.2. Let $P = Y = \mathbb{R}$ and let $K = \mathbb{R}_+ := [0, +\infty)$. Take

$$G(p) := \begin{cases} \{-\sqrt{p}, \sqrt{p}\} & \text{if } p \ge 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Observe that

$$\text{gph}G = \{(p,y) \in \mathbb{R}^2 \mid y^2 = p\} \text{ and}$$

$$\text{gph}(G+K) = \{(p,y) \in \mathbb{R}^2 \mid p \ge 0, y \ge -\sqrt{p}\}.$$

Consider $\bar{p} = \bar{y} = 0$ and $y^* = 0 \notin K^*_{up}$. By computing, we obtain

$$\widehat{D}^*(G+K)(0,0)(0) = (-\infty, 0] \text{ and } \widehat{D}^*G(0,0)(0) = \mathbb{R}.$$

This means that the inclusion in (3.1) is strict.

Next, we establish the *exact* rule for the Fréchet coderivative of the composition of a vector function and a multifunction which will be useful hereafter.

Proposition 3.3. *Let P, X and Y be Banach spaces and let $\bar{p} \in P$, $\bar{y} \in (f \circ G)(\bar{p})$ where $f : P \times X \to Y$ is a vector function and $G : P \rightrightarrows X$ is a multifunction. For $y^* \in Y^*$ suppose that for some $\bar{x} \in G(\bar{p})$ satisfying $(\bar{p}, \bar{x}) \in f^{-1}(\bar{y})$, the function f is local upper Lipschitzian at $(\bar{p}, \bar{x})$ and $\widehat{\partial}^+ \langle y^*, f\rangle(\bar{p}, \bar{x}) \neq \emptyset$. One has*

$$\widehat{D}^*(f \circ G)(\bar{p}, \bar{y})(y^*) \subset \bigcap_{(p^*,x^*)\in\widehat{\partial}^+\langle y^*,f\rangle(\bar{p},\bar{x})} \Big[p^* + \widehat{D}^* G(\bar{p}, \bar{x})(x^*)\Big]. \tag{3.3}$$

If, in addition, f is Fréchet differentiable at $(\bar{p}, \bar{x})$ with the derivative $\nabla f(\bar{p}, \bar{x}) := (\nabla_p f(\bar{p}, \bar{x}), \nabla_x f(\bar{p}, \bar{x}))$ and G admits a local upper Lipschitzian selection at $(\bar{p}, \bar{x})$, then the converse inclusion of (3.3) *is valid, i.e.,*

$$\widehat{D}^*(f \circ G)(\bar{p}, \bar{y})(y^*) = \nabla_p f(\bar{p}, \bar{x})^* y^* + \widehat{D}^* G(\bar{p}, \bar{x})\big(\nabla_x f(\bar{p}, \bar{x})^* y^*\big). \tag{3.4}$$

Proof. Take any $u^* \in \widehat{D}^*(f \circ G)(\bar{p}, \bar{y})(y^*)$. By the Fréchet coderivative and Fréchet normal cone descriptions (2.6) and (2.2) with $\varepsilon = 0$, we have

$$\limsup_{(p,y)\xrightarrow{\text{gph}(f\circ G)}(\bar{p},\bar{y})} \frac{\langle (u^*, -y^*), (p, y) - (\bar{p}, \bar{y})\rangle}{\|(p, y) - (\bar{p}, \bar{y})\|} \leq 0.$$

This means that for every $\epsilon > 0$, find $\eta_1 > 0, \eta_2 > 0$ such that

$$\langle u^*, p - \bar{p}\rangle \leq \langle y^*, y - \bar{y}\rangle + \epsilon(||p - \bar{p}|| + ||y - \bar{y}||) \tag{3.5}$$

for all $p \in B_{\eta_1}(\bar{p}), y \in B_{\eta_2}(\bar{y})$ with $y \in (f \circ G)(p) = \{f(p, x) \mid x \in G(p)\}$. Now fix any $(p^*, x^*) \in \widehat{\partial}^+\langle y^*, f\rangle(\bar{p}, \bar{x})$. Taking (2.5) into account and applying Lemma 2.1 to $(-p^*, -x^*) \in \widehat{\partial}(-\langle y^*, f\rangle)(\bar{p}, \bar{x})$ we have a function $s : U \subset P \times X \to \mathbb{R}$ that is Fréchet differentiable at $(\bar{p}, \bar{x})$ and satisfies

$$\begin{aligned} s(\bar{p}, \bar{x}) &= \langle y^*, f\rangle(\bar{p}, \bar{x}), (p^*, x^*) = \nabla s(\bar{p}, \bar{x}) : \\ &= (\nabla_p s(\bar{p}, \bar{x}), \nabla_x s(\bar{p}, \bar{x})) \text{ and} \\ s(p, x) &\geq \langle y^*, f\rangle(p, x) \text{ for all } (p, x) \in U, \end{aligned} \tag{3.6}$$

where U is some neighborhood of $(\bar{p}, \bar{x})$. Since f is local upper Lipschitzian at $(\bar{p}, \bar{x})$, there exist a neighborhood U_1 of $(\bar{p}, \bar{x})$ and $l_1 \geq 0$ such that

$$||f(p, x) - f(\bar{p}, \bar{x})|| \leq l_1(||p - \bar{p}|| + ||x - \bar{x}||) \text{ for all } (p, x) \in U_1.$$

Combining this with (3.5) and (3.6) gives us

$$
\begin{aligned}
&\langle u^*, p-\bar p\rangle \\
&\le \langle y^*, f(p,x)\rangle - \langle y^*, f(\bar p,\bar x)\rangle + \epsilon(||p-\bar p|| + ||f(p,x) - f(\bar p,\bar x)||) \\
&\le s(p,x) - s(\bar p,\bar x) + \epsilon(||p-\bar p|| + ||f(p,x) - f(\bar p,\bar x)||) \\
&= \langle \nabla_p s(\bar p,\bar x), p-\bar p\rangle + \langle \nabla_x s(\bar p,\bar x), x-\bar x\rangle + o(||p-\bar p|| + ||x-\bar x||) \\
&\quad +\epsilon(||p-\bar p|| + ||f(p,x) - f(\bar p,\bar x)||) \\
&\le \langle \nabla_p s(\bar p,\bar x), p-\bar p\rangle + \langle \nabla_x s(\bar p,\bar x), x-\bar x\rangle + o(||p-\bar p|| + ||x-\bar x||) \\
&\quad +\epsilon(1+l_1)(||p-\bar p|| + ||x-\bar x||) \\
&= \langle p^*, p-\bar p\rangle + \langle x^*, x-\bar x\rangle + o(||p-\bar p|| + ||x-\bar x||) \\
&\quad +\epsilon(1+l_1)(||p-\bar p|| + ||x-\bar x||)
\end{aligned} \tag{3.7}
$$

for all $p \in B_{\alpha_1}(\bar p), x \in B_{\alpha_2}(\bar x) \cap G(p)$ for some $\alpha_1 > 0, \alpha_2 > 0,$ where

$$\lim_{(p,x)\to(\bar p,\bar x)} \frac{o(||p-\bar p|| + ||x-\bar x||)}{||p-\bar p|| + ||x-\bar x||} = 0.$$

Since $\epsilon > 0$ was chosen arbitrarily, it follows from (3.7) that

$$\limsup_{(p,x)\xrightarrow{\text{gph}G}(\bar p,\bar x)} \frac{\langle (u^* - p^*, -x^*), (p,x) - (\bar p,\bar x)\rangle}{\|(p,x) - (\bar p,\bar x)\|} \le 0.$$

This means that $u^* - p^* \in \widehat{D}^*G(\bar p,\bar x)(x^*)$ by (2.2) with $\varepsilon = 0$ and (2.6). Thus we have the inclusion $u^* \in p^* + \widehat{D}^*G(\bar p,\bar x)(x^*)$ that justifies (3.3).

Let us now prove the opposite inclusion in (3.3), i.e., (3.4) is valid, under the additional assumptions made. Fix any $u^* \notin \widehat{D}^*(f\circ G)(\bar p,\bar y)(y^*)$. It suffices to show that

$$u^* \notin \nabla_p f(\bar p,\bar x)^* y^* + \widehat{D}^*G(\bar p,\bar x)(\nabla_x f(\bar p,\bar x)^* y^*). \tag{3.8}$$

To proceed, observe from (2.6) that $(u^*, -y^*) \notin \widehat{N}\big((\bar p,\bar y); \text{gph}(f\circ G)\big)$ whenever $u^* \notin \widehat{D}^*(f\circ G)(\bar p,\bar y)(y^*)$. Thus it follows by (2.2) with $\varepsilon = 0$ that

$$\limsup_{(p,y)\xrightarrow{\text{gph}(f\circ G)}(\bar p,\bar y)} \frac{\langle (u^*, -y^*), (p,y) - (\bar p,\bar y)\rangle}{\|(p,y) - (\bar p,\bar y)\|} > 0. \tag{3.9}$$

Since G admits a local upper Lipschitzian selection at $(\bar p,\bar x)$, there is $l : \text{dom}G \to X$, which is local upper Lipschitzian at $\bar p$ satisfying $l(\bar p) = \bar x$ and $l(x) \in G(x)$ for all $x \in \operatorname{dom} G$ sufficiently close to $\bar{p}$. It follows from (3.9) and the above properties of $l(\cdot)$ that there exist a number $\alpha > 0$ and a sequence $p_n \to \bar{p}$ as $n \to \infty$ along with

$$\langle u^*, p_n - \bar{p}\rangle \geq \langle y^*, y_n - \bar{y}\rangle + \alpha(||p_n - \bar{p}|| + ||y_n - \bar{y}||) \tag{3.10}$$

where $y_n = f(p_n, x_n), x_n := l(p_n) \in G(p_n)$ and some $\ell > 0$,

$$||x_n - \bar{x}|| \leq \ell||p_n - \bar{p}|| \tag{3.11}$$

for all $n \in \mathbb{N} := \{1, 2, ...\}$ sufficiently large. By (3.10), we have for such $n \in \mathbb{N}$ that

$$\begin{aligned}
\langle u^*, p_n - \bar{p}\rangle \geq&\langle y^*, f(p_n, x_n) - f(\bar{p}, \bar{x})\rangle + \alpha(||p_n - \bar{p}|| + ||f(p_n, x_n) - f(\bar{p}, \bar{x})||)\\
=&\langle y^*, \nabla f(\bar{p}, \bar{x})(p_n - \bar{p}, x_n - \bar{x})\rangle + o(||p_n - \bar{p}|| + ||x_n - \bar{x}||)\\
&+ \alpha(||p_n - \bar{p}|| + ||f(p_n, x_n) - f(\bar{p}, \bar{x})||)\\
=&\langle \nabla f(\bar{p}, \bar{x})^* y^*, (p_n - \bar{p}, x_n - \bar{x})\rangle + o(||p_n - \bar{p}|| + ||x_n - \bar{x}||)\\
&+ \alpha(||p_n - \bar{p}|| + ||f(p_n, x_n) - f(\bar{p}, \bar{x})||)\\
\geq&\langle \nabla f(\bar{p}, \bar{x})^* y^*, (p_n - \bar{p}, x_n - \bar{x})\rangle + o(||p_n - \bar{p}|| + ||x_n - \bar{x}||)\\
&+ \alpha||p_n - \bar{p}||.
\end{aligned}$$

This implies by (3.11) that

$$\begin{aligned}
\langle u^*, p_n - \bar{p}\rangle \geq&\langle \nabla f(\bar{p}, \bar{x})^* y^*, (p_n - \bar{p}, x_n - \bar{x})\rangle + o(||p_n - \bar{p}|| + ||x_n - \bar{x}||)\\
&+ \frac{\alpha}{2}||p_n - \bar{p}|| + \frac{\alpha}{2\ell}||x_n - \bar{x}||\\
\geq&\langle \nabla f(\bar{p}, \bar{x})^* y^*, (p_n - \bar{p}, x_n - \bar{x})\rangle + o(||p_n - \bar{p}|| + ||x_n - \bar{x}||)\\
&+ \widehat{\alpha}(||p_n - \bar{p}|| + ||x_n - \bar{x}||)
\end{aligned}$$

with $\widehat{\alpha} := \min\{\alpha/2, \alpha/(2\ell)\} > 0$. Thus

$$\limsup_{(p,x)\xrightarrow{\text{gph}G}(\bar{p},\bar{x})} \frac{\langle u^* - \nabla_p f(\bar{p}, \bar{x})^* y^*, p - \bar{p}\rangle - \langle \nabla_x f(\bar{p}, \bar{x})^* y^*, x - \bar{x}\rangle}{||p - \bar{p}|| + ||x - \bar{x}\|} \geq \widehat{\alpha},$$

which means that $(u^* - \nabla_p f(\bar{p}, \bar{x})^* y^*, -\nabla_x f(\bar{p}, \bar{x})^* y^*) \notin \widehat{N}\big((\bar{p}, \bar{x}); \text{gph}G\big)$. By (2.6), $u^* - \nabla_p f(\bar{p}, \bar{x})^* y^* \notin \widehat{D}^* G(\bar{p}, \bar{x})(\nabla_x f(\bar{p}, \bar{x})^* y^*)$. This justifies (3.8) and completes the proof of the theorem. ■

The equality in (3.4) may fail to hold if the assumption on the existence of the local upper Lipschitzian selection of G is omitted.

Example 3.4. Let $P = X = Y = \mathbb{R}$. Take $f(p, x) := x^2$ and

$$G(p) := \begin{cases} \{-\sqrt{p}, \sqrt{p}\} & \text{if } p \geq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} &\mathrm{gph}G = \{(p, x) \in \mathbb{R}^2 \mid x^2 = p\} \text{ and} \\ &\mathrm{gph}(f \circ G) = \{(p, y) \in \mathbb{R}^2 \mid p \geq 0, y = p\}. \end{aligned}$$

Consider $\bar{p} = \bar{x} = \bar{y} = 0$ and $y^* = 0$. Observe that the multifunction G does not admit any local upper Lipschitzian selections at $(0, 0)$. By computing, we obtain

$$\widehat{D}^*G(0, 0)(0) = \mathbb{R} \text{ and } \widehat{D}^*(f \circ G)(0, 0)(0) = (-\infty, 0].$$

This means that the equality in (3.4) fails to hold.

Proposition 3.3 recovers the preceding result in [24] which can be restated as follows.

Corollary 3.5. ([24, Proposition 4.6]). *Let P, X and Y be Banach spaces and let $\bar{p} \in P$, $\bar{y} \in (f \circ G)(\bar{p})$ where $f : P \times X \to Y$ is a vector function and $G : P \rightrightarrows X$ is a multifunction of closed graph. Suppose that for some $\bar{x} \in G(\bar{p})$ satisfying $(\bar{p}, \bar{x}) \in f^{-1}(\bar{y})$, the function f is Lipschitz continuous around $(\bar{p}, \bar{x})$ and Frećhet differentiable at this point. Then*

$$\widehat{D}^*(f \circ G)(\bar{p}, \bar{y})(y^*) \subset \nabla_p f(\bar{p}, \bar{x})^* y^* + \widehat{D}^*G(\bar{p}, \bar{x})\big(\nabla_x f(\bar{p}, \bar{x})^* y^*\big) \quad \forall y^* \in Y^*.$$

If, in addition, $G = g$ is single-valued and Lipschitz continuous around $\bar{p}$, then one has the equality

$$\widehat{D}^*(f \circ g)(\bar{p}, \bar{y})(y^*) = \nabla_p f(\bar{p}, \bar{x})^* y^* + \widehat{\partial}\langle \nabla_x f(\bar{p}, \bar{x})^* y^*, g\rangle(\bar{p}) \quad \forall y^* \in Y^*.$$

We are now ready to formulate and prove the main result of this section.

Theorem 3.6. *Let $\mathcal{F}$ be the efficient point multifunction of* (1.1) *defined in* (1.4) *and let $\bar{p} \in P$, $\bar{x} \in C(\bar{p})$ be such that $\bar{y} = f(\bar{p}, \bar{x}) \in \mathcal{F}(\bar{p})$. For $y^* \in K^*_{up}$ defined in* (2.1), *suppose that $\widehat{\partial}^+\langle y^*, f\rangle(\bar{p}, \bar{x}) \neq \emptyset$ and the function f is local upper*

Lipschitzian at $(\bar{p}, \bar{x})$. *Assume further the domination property holds for* F *defined in* (1.3) *around* $\bar{p}$. *One has*

$$\widehat{D}^*\mathcal{F}(\bar{p},\bar{y})(y^*) \subset \bigcap_{(p^*,x^*)\in\widehat{\partial}^+\langle y^*,f\rangle(\bar{p},\bar{x})} \Big[p^* + \widehat{D}^*C(\bar{p},\bar{x})(x^*)\Big]. \tag{3.12}$$

If, in addition, f *is Fréchet differentiable at* $(\bar{p}, \bar{x})$ *with the derivative* $\nabla f(\bar{p},\bar{x}) := (\nabla_p f(\bar{p},\bar{x}), \nabla_x f(\bar{p},\bar{x}))$ *and* C *admits a local upper Lipschitzian selection at* $(\bar{p}, \bar{x})$, *then the converse inclusion of* (3.12) *is valid, i.e.,*

$$\widehat{D}^*\mathcal{F}(\bar{p},\bar{y})(y^*) = \nabla_p f(\bar{p},\bar{x})^* y^* + \widehat{D}^*C(\bar{p},\bar{x})\big(\nabla_x f(\bar{p},\bar{x})^* y^*\big). \tag{3.13}$$

Proof. We first prove the inclusion in (3.12). Since $\mathcal{F}(p) \subset F(p)$ for all $p \in P$ and the domination property holds for F around $\bar{p} \in P$, there exists a neighborhood U of $\bar{p}$ such that

$$\mathcal{F}(p) + K = F(p) + K \quad \forall p \in U.$$

Hence

$$\widehat{D}^*(\mathcal{F}+K)(\bar{p},\bar{y})(y^*) = \widehat{D}^*(F+K)(\bar{p},\bar{y})(y^*) \quad \forall y^* \in Y^*.$$

This together with Proposition 3.1 gives

$$\begin{aligned} \widehat{D}^*\mathcal{F}(\bar{p},\bar{y})(y^*) &= \widehat{D}^*(\mathcal{F}+K)(\bar{p},\bar{y})(y^*) \\ &= \widehat{D}^*(F+K)(\bar{p},\bar{y})(y^*),\ y^* \in K^*_{up}. \end{aligned} \tag{3.14}$$

Again by Proposition 3.1, we get

$$\widehat{D}^*(F+K)(\bar{p},\bar{y})(y^*) = \widehat{D}^*F(\bar{p},\bar{y})(y^*),\ y^* \in K^*_{up}. \tag{3.15}$$

Observing further the composite form of F in (1.3) and applying Proposition 3.3 to it, we arrive at the inclusion

$$\widehat{D}^*F(\bar{p},\bar{y})(y^*) \subset \bigcap_{(p^*,x^*)\in\widehat{\partial}^+\langle y^*,f\rangle(\bar{p},\bar{x})} \Big[p^* + \widehat{D}^*C(\bar{p},\bar{x})(x^*)\Big]. \tag{3.16}$$

Combining now the relations in (3.14)–(3.16), we get (3.12).

We observe that, by using Proposition 3.3, the opposite inclusion in (3.16) is valid under the additional assumptions of the theorem. To justify (3.13) it remains to employ the relations in (3.14)–(3.16) again. ∎

It is worth mentioning here that there are examples similar to Examples 3.2 and 3.4 which show that the assumptions $y^* \in K^*_{up}$ and C admits a local upper Lipschitzian selection at the referee point in Theorem 3.6 are essential. The next example illustrates the importance of the *domination property* of F namely the inclusion in (3.12) may fail to hold if the assumption on the existence of the domination property of F around the point under consideration is dropped.

Example 3.7. Let $P = X = Y = \mathbb{R}$ and $K = \mathbb{R}_+ := [0, +\infty)$. Take $f(p, x) := x$ and

$$C(p) := \begin{cases} [0, +\infty) & \text{if } p = 0 \\ (|p|, +\infty) & \text{otherwise.} \end{cases}$$

We have

$$\text{gph}C = \{(p, x) \in \mathbb{R}^2 \mid x > 0, -x < p < x\} \cup \{(0, 0)\} \text{ and}$$
$$F(p) = C(p) \text{ for all } p \in P.$$

Consider $\bar{p} = \bar{x} = 0$ and $y^* = 1 \in K^*_{up}$ defined in (2.1). We see that

$$\mathcal{F}(p) = \begin{cases} \{0\} & \text{if } p = 0 \\ \emptyset & \text{otherwise,} \end{cases}$$

and thus $\bar{y} := f(\bar{p}, \bar{x}) = 0 \in \mathcal{F}(\bar{p})$ as well as the domination property does not hold for F around $\bar{p} = 0$. By computing, we obtain

$$\widehat{D}^*\mathcal{F}(0, 0)(1) = (-\infty, +\infty) \text{ and } \nabla_p f(0, 0)^*(1) + \widehat{D}^*C(0, 0)(1) = [-1, 1].$$

This means that the inclusion in (3.12) fails to hold.

4. Fréchet Coderivatives of $\mathcal{F}$ in Special Classes of Constrained Vector Optimization Problems

4.1. Operator constraints

We now consider the problem (1.1) with the *constraint mapping* $C : P \rightrightarrows X$ given in the form

$$C(p) := \{x \in X \mid h(p, x) \in \Theta\}, \tag{4.1}$$

where $h : P \times X \to W$ is a single-valued mapping between Banach spaces and where $\emptyset \neq \Theta \subset W$. Constraints of type (4.1) are known as *operator constraints*. They include geometric, functional, and other types of constraints under appropriate specifications of h and Θ, see [25, 26] for more discussions and examples.

The following theorem gives upper estimating and precise computing formulae to evaluate Fréchet coderivatives of $\mathcal{F}$ in (1.4) for constraints given by (4.1).

Theorem 4.1. *Let $\mathcal{F}$ be the efficient point multifunction of* (1.1) *with the constraint mapping C given by* (4.1) *and let $\bar{p} \in P$, $\bar{x} \in C(\bar{p})$ be such that $\bar{y} = f(\bar{p}, \bar{x}) \in \mathcal{F}(\bar{p})$. For $y^* \in K^*_{up}$ defined in* (2.1)*, suppose that $\widehat{\partial}^+\langle y^*, f\rangle(\bar{p}, \bar{x}) \neq \emptyset$ and the function f is local upper Lipschitzian at $(\bar{p}, \bar{x})$. Assume further the domination property holds for F defined in* (1.3) *around $\bar{p}$. The following assertions hold:*

(i) *Suppose that h in* (4.1) *is strictly differentiable at $(\bar{p}, \bar{x})$ with the surjective derivative operator $\nabla h(\bar{p}, \bar{x})$. Then one has*

$$\widehat{D}^*\mathcal{F}(\bar{p}, \bar{y})(y^*) \subset \bigcap_{(p^*,x^*)\in\widehat{\partial}^+\langle y^*,f\rangle(\bar{p},\bar{x})} \Big\{p^* + u^* \Big|\, (u^*, -x^*) \in \nabla h(\bar{p}, \bar{x})^* \widehat{N}(\bar{w}; \Theta)\Big\}, \tag{4.2}$$

where $\bar{w} := h(\bar{p}, \bar{x})$.

(ii) *Suppose in addition to* (i) *that f is Fréchet differentiable at $(\bar{p}, \bar{x})$ with the derivative $\nabla f(\bar{p}, \bar{x}) := (\nabla_p f(\bar{p}, \bar{x}), \nabla_x f(\bar{p}, \bar{x}))$ and C admits a local upper Lipschitzian selection at $(\bar{p}, \bar{x})$, then the converse inclusion of* (4.2) *is valid, i.e.,*

$$\widehat{D}^*\mathcal{F}(\bar{p}, \bar{y})(y^*) = \Big\{\nabla_p f(\bar{p}, \bar{x})^* y^* + u^* \Big|\, (u^*, -\nabla_x f(\bar{p}, \bar{x})^* y^*) \in \nabla h(\bar{p}, \bar{x})^* \widehat{N}(\bar{w}; \Theta)\Big\}. \tag{4.3}$$

Proof. Observe that the graph of the constraint mapping C in (4.1) admits the inverse image representation

$$\operatorname{gph} C = h^{-1}(\Theta) := \big\{(p, x) \in P \times X \,\big|\, h(p, x) \in \Theta\big\}. \tag{4.4}$$

By the strictly differentiability of h and the surjectivity assumption on the derivative $\nabla h(\bar{p}, \bar{x})$ we get from [25, Corollary 1.15] and (4.4) that

$$\widehat{N}\big((\bar{p}, \bar{x}); \operatorname{gph} C\big) = \widehat{N}\big((\bar{p}, \bar{x}); h^{-1}(\Theta)\big) = \nabla h(\bar{p}, \bar{x})^* \widehat{N}(\bar{w}; \Theta).$$

This together with (2.6) gives us

$$\widehat{D}^* C(\bar{p}, \bar{x})(x^*) = \big\{u^* \in P^* \mid (u^*, -x^*) \in \nabla h(\bar{p}, \bar{x})^* \widehat{N}(\bar{w}; \Theta)\big\}. \tag{4.5}$$

Substituting (4.5) into (3.12) and (3.13) of Theorem 3.6, we get (4.2) and (4.3), respectively, which complete the proof. ■

4.2. Constraints described by finitely many equalities and inequalities

Next we consider the problem (1.1) with the *functional constraints* described by finitely many equalities and inequalities given as follows

$$\begin{aligned} C(p) := \big\{x \in X \mid\ & g_i(p, x) \le 0,\ i = 1, ..., m, \\ & g_i(p, x) = 0,\ \ i = m+1, ..., m+r\big\}, \end{aligned} \tag{4.6}$$

where $g_i,\ i = 1, ..., m + r$, are real-valued functions on the Banach space $P \times X$. Constraints of this type can be treated as a particular case of the operator constraints (4.1) with $h : P \times X \to \mathbb{R}^{m+r}$ defined by

$$h(p, x) := (g_1(p, x), ..., g_{m+r}(p, x)) \tag{4.7}$$

and $\Theta \subset \mathbb{R}^{m+r}$ defined by

$$\Theta := \big\{(\alpha_1, ..., \alpha_{m+r}) \in \mathbb{R}^{m+r} \mid \alpha_i \le 0,\ i = 1, ..., m,\quad \alpha_i = 0,\ \ i = m + 1, ..., m + r\big\}. \tag{4.8}$$

However, constraints of type (4.6) is a conventional and remarkable class in parametric nonlinear programs and parametric vector optimization. Our first theorem provides an upper estimate and also a precise formula for evaluating Fréchet coderivatives of the efficient point multifunction $\mathcal{F}$ in the general Banach space setting via the classical Lagrange multipliers.

Theorem 4.2. *Let $\mathcal{F}$ be the efficient point multifunction of* (1.1) *with the constraint mapping C given by* (4.6) *and let $\bar{p} \in P$, $\bar{x} \in C(\bar{p})$ be such that $\bar{y} = f(\bar{p}, \bar{x}) \in \mathcal{F}(\bar{p})$. For $y^* \in K^*_{up}$ defined in* (2.1)*, suppose that $\widehat{\partial}^+\langle y^*, f\rangle(\bar{p}, \bar{x}) \neq \emptyset$ and the function f is local upper Lipschitzian at $(\bar{p}, \bar{x})$. Assume the domination property holds for F in* (1.3) *around $\bar{p}$ and assume further that $g_{\ i}, i = 1, ..., m + r$, in* (4.6) *are Fréchet differentiable at $(\bar{p}, \bar{x})$ and continuous around this point as well as the gradients*

$$\nabla g_1(\bar{p}, \bar{x}), ..., \nabla g_{m+r}(\bar{p}, \bar{x}) \ \ \text{are linear independent.} \tag{4.9}$$

Then one has

$$\widehat{D}^*\mathcal{F}(\bar{p}, \bar{y})(y^*) \subset \bigcap_{(p^*,x^*)\in\widehat{\partial}^+\langle y^*,f\rangle(\bar{p},\bar{x})} \ \bigcup_{\lambda\in\Lambda(\bar{p},\bar{x},x^*)} \Big[p^* + \sum_{i=1}^{m+r} \lambda_i \nabla_p g_i(\bar{p}, \bar{x})\Big], \tag{4.10}$$

where

$$\Lambda(\bar{p}, \bar{x}, x^*) := \big\{\lambda := (\lambda_1, ..., \lambda_{m+r}) \in \mathbb{R}^{m+r} \big|\, x^* + \sum_{i=1}^{m+r} \lambda_i \nabla_x g_i(\bar{p}, \bar{x}) = 0, \lambda_i \ge 0, \lambda_i g_i(\bar{p}, \bar{x}) = 0 \text{ for } i = 1, ..., m\big\} \tag{4.11}$$

denotes the set of Lagrange multipliers (see [28]). *Furthermore,* (4.10) *becomes the equality*

$$\widehat{D}^*\mathcal{F}(\bar{p}, \bar{y})(y^*) = \bigcup_{\lambda\in\Lambda(\bar{p},\bar{x},\nabla_x f(\bar{p},\bar{x})^* y^*)} \Big[\nabla_p f(\bar{p}, \bar{x})^* y^* + \sum_{i=1}^{m+r} \lambda_i \nabla_p g_i(\bar{p}, \bar{x})\Big], \tag{4.12}$$

provided that f is Fréchet differentiable at $(\bar{p},\bar{x})$ with the derivative $\nabla f(\bar{p},\bar{x}) := (\nabla_p f(\bar{p},\bar{x}), \nabla_x f(\bar{p},\bar{x}))$ and C admits a local upper Lipschitzian selection at $(\bar{p},\bar{x})$.

Proof. Observe that the graph of the constraint mapping C in (4.6) admits the inverse image representation

$$\text{gph}\, C = h^{-1}(\Theta) := \big\{(p,x) \in P \times X \big|\ h(p,x) \in \Theta\big\}, \tag{4.13}$$

where the vector function h was defined in (4.7) and the closed convex cone Θ was defined in (4.8). By our assumptions it follows from construction (4.7) that h is Fréchet differentiable at $(\bar{p},\bar{x})$ and further by (4.9) its derivative operator $\nabla h(\bar{p},\bar{x})$ is surjective (see e.g., [14, Example 13]). So we get from [25, Corollary 1.15] and (4.13) that

$$\widehat{N}\big((\bar{p},\bar{x}); \text{gph}\, C\big) = \widehat{N}\big((\bar{p},\bar{x}); h^{-1}(\Theta)\big) = \nabla h(\bar{p},\bar{x})^* \widehat{N}(h(\bar{p},\bar{x}); \Theta).$$

This together with (2.6) gives us

$$\widehat{D}^* C(\bar{p},\bar{x})(x^*) = \big\{u^* \in P^* \mid (u^*, -x^*) \in \nabla h(\bar{p},\bar{x})^* \widehat{N}(h(\bar{p},\bar{x}); \Theta)\big\}. \tag{4.14}$$

Since Θ is convex, it holds (see [25, Page 413])

$$\widehat{N}(h(\bar{p},\bar{x}); \Theta) = \big\{\lambda = (\lambda_1, ..., \lambda_{m+r}) \in \mathbb{R}^{m+r} \;\big|\; \lambda_i \geq 0,\ \lambda_i g_i(\bar{p},\bar{x}) = 0 \text{ for } i = 1, ..., m\big\}.$$

Combining this with (4.14) and taking into account the specific structure of h in (4.7), we obtain

$$\widehat{D}^* C(\bar{p},\bar{x})(x^*) = \Big\{u^* \in P^* \mid (u^*, -x^*) = \sum_{i=1}^{m+r} \lambda_i \nabla g_i(\bar{p},\bar{x}) \text{ for some } \lambda \in \mathbb{R}^{m+r} \text{ with } \lambda_i \geq 0,\ \lambda_i g_i(\bar{p},\bar{x}) = 0 \text{ as } i = 1, ..., m\Big\}. \tag{4.15}$$

Substituting now (4.15) into (3.12) and (3.13) of Theorem 3.6 and taking into account construction (4.11) of the set of Lagrange multipliers $\Lambda(\bar{p},\bar{x},x^*)$, we get (4.10) and (4.12), respectively and thus the proof is complete. ∎

Observe that in Theorem 4.2 the linear independence condition (4.9) ensures the surjectivity of $\nabla h(\bar{p},\bar{x})$ for the corresponding mapping is rather *strict*. In what follows we shall relax this condition by replacing it by the so-called *Mangasarian-Fromovitz constraint qualification* which can be formulated as follows:

(4.16) the gradients $\nabla g_{m+1}(\bar{p},\bar{x}), \ldots, \nabla g_{m+r}(\bar{p},\bar{x})$ are linearly independent, and there is $u \in P \times X$ such that $\langle \nabla g_i(\bar{p},\bar{x}), u\rangle = 0$ for $i = m+1, \ldots, m+r$ and that $\langle \nabla g_i(\bar{p},\bar{x}), u\rangle < 0$ whenever $i = 1, \ldots, m$ with $g_i(\bar{p},\bar{x}) = 0$.

However the *Asplund* structure of both spaces P and X and the *strict* differentiability of the constraint functions in (4.6) instead of their merely Fréchet differentiability at $(\bar{p},\bar{x})$ as in Theorem 4.2 held in the arbitrary Banach space setting will be presented in the next theorem.

Theorem 4.3. *In addition to all the assumptions of Theorem 4.2, suppose that the spaces P and X are Asplund, that all $g_i, i=1,...,m+r$, in* (4.6) *are strictly differentiable at $(\bar{p},\bar{x})$ and that the condition* (4.9) *is replaced by the Mangasarian-Fromovitz constraint qualification* (4.16)*. Then we have* (4.10) *and* (4.12) *respectively under the corresponding additional assumptions in Theorem 4.2.*

Proof. By [25, Corollary 4.35], we have the precise formula to compute the Fréchet coderivative under the assumptions made in the theorem as follows

$$\begin{aligned}&\widehat{D}^*C(\bar{p},\bar{x})(x^*)\\ &=\Big\{u^*\in P^*\mid (u^*,-x^*)=\sum_{i=1}^{m+r}\lambda_i\nabla g_i(\bar{p},\bar{x})\text{ for some }\lambda\in\mathbb{R}^{m+r}\\ &\quad\text{with }\lambda_i\geq 0,\ \lambda_i g_i(\bar{p},\bar{x})=0\text{ as }i=1,...,m\Big\}.\end{aligned}\tag{4.17}$$

Substituting now (4.17) into (3.12) and (3.13) of Theorem 3.6 respectively, we get the desired results. ■

4.3. Constraints described by an arbitrary (possibly infinite) number of inequalities

In this subsection we consider the problem (1.1) with the constraint mapping $C\colon P\rightrightarrows X$ defined by

$$C(p):=\{x\in X\mid g_t(p,x)\leq 0,\, t\in T\},\tag{4.18}$$

where T is an *arbitrary* (possibly *infinite*) index set and for each $t\in T$ the function $g_t:P\times X\to\overline{\mathbb{R}}$ is assumed to be *lower regular* defined in (2.4) at the reference point on the Banach space $P\times X$.

Constraints of type (4.18) are known as *semi-infinite/infinite* inequality constraints. It is well known that models of semi-infinite optimization cover, e.g., pollution control models, control of robots, engineering design, mechanical stress of materials, and the semi-definite programming. Semi-infinite optimization programming and its wide applications have attracted much attention from many researchers. We refer the reader to the book by Goberna and López [13] for more details and discussions and some recent papers [5, 6, 7, 8, 10, 11, 12] for references.

Denote by $\mathbb{R}^{(T)}$ (respectively, $\mathbb{R}^{(T)}_+$) the collection of all the functions $\lambda:T\to\mathbb{R}$ taking nonzero (respectively, nonnegative) values only at finitely many points of T, and $\text{supp } \lambda := \{t \in T \,|\, \lambda_t \neq 0\}$. Given $u \in \mathbb{R}^{(T)}$ and $\lambda \in \mathbb{R}_+^{(T)}$. We put $\langle \lambda, u \rangle = \sum_{t \in \text{supp } \lambda} \lambda_t u_t$. In connection with (4.18), we use the set of *active constraint multipliers* defined by

$$A(\bar{p}, \bar{x}) := \{\lambda \in \mathbb{R}_+^{(T)} \,|\, \lambda_t g_t(\bar{p}, \bar{x}) = 0 \ \text{ for all } t \in \text{supp } \lambda\}. \tag{4.19}$$

Definition 4.4. Let C be defined in (4.18) and let $(\bar{p}, \bar{x}) \in \text{gph}C$. We say that C satisfies the *F-regular constraint qualification* (FRCQ) at $(\bar{p}, \bar{x})$ if

$$\widehat{N}\big((\bar{p}, \bar{x}); \text{gph}\, C\big) = \bigcup_{\lambda \in A(\bar{p}, \bar{x})} \Big[\sum_{t \in \text{supp } \lambda} \lambda_t \widehat{\partial} g_t(\bar{p}, \bar{x}) \Big]. \tag{4.20}$$

Various criteria for the validity of this qualification condition can be found in [7] (also see [12, 21] for the convex case).

Our first theorem in this subsection provides an upper estimate and also a precise formula for evaluating Fréchet coderivatives of the efficient point multifunction $\mathcal{F}$ with the constraints of nondifferentiable functions via the Fréchet subdifferentials of these functions.

Theorem 4.5. *Let $\mathcal{F}$ be the efficient point multifunction of* (1.1) *with the constraint mapping C given by* (4.18) *and let $\bar{p} \in P$, $\bar{x} \in C(\bar{p})$ be such that $\bar{y} = f(\bar{p}, \bar{x}) \in \mathcal{F}(\bar{p})$. For $y^* \in K^*_{up}$ defined in* (2.1), *suppose that $\widehat{\partial}^+ \langle y^*, f \rangle(\bar{p}, \bar{x}) \neq \emptyset$ and the function f is local upper Lipschitzian at $(\bar{p}, \bar{x})$. Suppose that the domination property holds for F in* (1.3) *around $\bar{p}$ and that all g_t, $t \in T$, in* (4.18) *are lower regular at $(\bar{p}, \bar{x})$. Assume further C satisfies* (FRCQ) *in* (4.20)*. Then one has*

$$\begin{aligned} &\widehat{D}^* \mathcal{F}(\bar{p}, \bar{y})(y^*) \\ &\subset \bigcap_{(p^*, x^*) \in \widehat{\partial}^+ \langle y^*, f \rangle(\bar{p}, \bar{x})} \Big\{ p^* + u^* \Big|\ (u^*, -x^*) \in \bigcup_{\lambda \in A(\bar{p}, \bar{x})} \Big[\sum_{t \in \text{supp } \lambda} \lambda_t \widehat{\partial} g_t(\bar{p}, \bar{x}) \Big] \Big\}, \end{aligned} \tag{4.21}$$

where $A(\bar{p}, \bar{x})$ is the corresponding set of active constraint multipliers defined in (4.19)*. Furthermore,* (4.21) *becomes the equality*

$$\begin{aligned} \widehat{D}^* \mathcal{F}(\bar{p}, \bar{y})(y^*) = \Big\{ &\nabla_p f(\bar{p}, \bar{x})^* y^* + u^* \Big|\ (u^*, -\nabla_x f(\bar{p}, \bar{x})^* y^*) \\ &\in \bigcup_{\lambda \in A(\bar{p}, \bar{x})} \big[\sum_{t \in \text{supp } \lambda} \lambda_t \widehat{\partial} g_t(\bar{p}, \bar{x}) \big] \Big\}. \end{aligned} \tag{4.22}$$

provided that f is Fréchet differentiable at $(\bar{p}, \bar{x})$ with the derivative $\nabla f(\bar{p}, \bar{x}) := (\nabla_p f(\bar{p}, \bar{x}), \nabla_x f(\bar{p}, \bar{x}))$ and C admits a local upper Lipschitzian selection at $(\bar{p}, \bar{x})$.

Proof. Since C satisfies (FRCQ), it follows from (2.6) that

$$\widehat{D}^*C(\bar{p},\bar{x})(x^*)=\big\{u^*\in P^* \mid (u^*,-x^*)\in \bigcup_{\lambda\in A(\bar{p},\bar{x})}\Big[\sum_{t\in \text{supp}\,\lambda}\lambda_t\widehat{\partial}g_t(\bar{p},\bar{x})\Big]\big\}. \tag{4.23}$$

Substituting now (4.23) into (3.12) and (3.13) of Theorem 3.6, we get (4.21) and (4.22) respectively. ■

The next corollary provides an upper estimate and also a precise formula for evaluating Fréchet coderivatives of the efficient point multifunction $\mathcal{F}$ with the constraints of differentiable functions via the Lagrange multipliers.

Corollary 4.6. *In addition to all the assumptions of Theorem 4.5, suppose that all $g_t,\ t\in T$, in* (4.18) *are Fréchet differentiable at $(\bar{p},\bar{x})$. Then we have*

$$\begin{aligned}&\widehat{D}^*\mathcal{F}(\bar{p},\bar{y})(y^*)\\ &\subset \bigcap_{(p^*,x^*)\in\widehat{\partial}^+\langle y^*,f\rangle(\bar{p},\bar{x})}\ \bigcup_{\lambda\in\Lambda(\bar{p},\bar{x},x^*)}\Big[p^*+\sum_{t\in\text{supp}\,\lambda}\lambda_t\nabla_p g_t(\bar{p},\bar{x})\Big],\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}&\Lambda(\bar{p},\bar{x},x^*):=\big\{\lambda\in\mathbb{R}_+^{(T)}\big|\, x^*\\ &\quad+\sum_{t\in\text{supp}\,\lambda}\lambda_t\nabla_x g_t(\bar{p},\bar{x})=0, \lambda_t g_t(\bar{p},\bar{x})=0, \forall t\in\text{supp}\,\lambda\big\}\end{aligned} \tag{4.25}$$

denotes the set of Lagrange multipliers. Furthermore, (4.24) *becomes the equality*

$$\begin{aligned}&\widehat{D}^*\mathcal{F}(\bar{p},\bar{y})(y^*)\\ &=\bigcup_{\lambda\in\Lambda(\bar{p},\bar{x},\nabla_x f(\bar{p},\bar{x})^*y^*)}\Big[\nabla_p f(\bar{p},\bar{x})^*y^*+\sum_{t\in\text{supp}\,\lambda}\lambda_t\nabla_p g_t(\bar{p},\bar{x})\Big],\end{aligned} \tag{4.26}$$

provided that f is Fréchet differentiable at $(\bar{p},\bar{x})$ with the derivative $\nabla f(\bar{p},\bar{x}):=(\nabla_p f(\bar{p},\bar{x}),\nabla_x f(\bar{p},\bar{x}))$ and C admits a local upper Lipschitzian selection at $(\bar{p},\bar{x})$.

Proof. The proof is immediate from Theorem 4.5 so is omitted. ■

We close this section with the following example.

Example 4.7. Let $T=[0,1], P=\mathbb{R}, X=Y=\mathbb{R}^2, K=\mathbb{R}^2_+$ and let $f:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2, g_t:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}, t\in T$ be mappings which are given as follows:

$$\begin{aligned}f(p,x)&=(p+x_1+1,x_2+1)\quad \forall x=(x_1,x_2)\in\mathbb{R}^2,\ \forall p\in\mathbb{R},\\ g_t(p,x)&=tp-tx_1-(1-t)x_2,\ \forall x=(x_1,x_2)\in\mathbb{R}^2,\ \forall p\in\mathbb{R}.\end{aligned}$$

It is clear that g_t is lower regular at any $(p, x) \in P \times X$ for all $t \in T$. We consider the problem (1.1) with C defined in (4.18). By simple computation, one can find

$$C(p) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq p, x_2 \geq 0\},$$
$$F(p) = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 2p + 1, y_2 \geq 1\} \quad \forall p \in P.$$

We therefore observe that the domination property holds for F defined in (1.3) at all $p \in P$ and C admits a local upper Lipschitzian selection at any $(p, x) \in \text{gph}\, C$. Moreover, for $\bar{p} = 0$,

$$C(\bar{p}) = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 \geq 0, x_2 \geq 0\},$$
$$F(\bar{p}) = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 1, y_2 \geq 1\},$$

and thus $\bar{x} = (0, 0) \in C(\bar{p})$ as well as $\bar{y} = f(\bar{p}, \bar{x}) = (1, 1) \in \mathcal{F}(\bar{p})$. For each $t \in T$, we have

$$g_t^*(p, x) = \begin{cases} 0 & \text{if } (p, x) = (t, -t, t-1) \\ +\infty & \text{if } (p, x) \neq (t, -t, t-1), \end{cases} \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \ \forall p \in \mathbb{R},$$

$$\text{epi} g_t^* = \{t\} \times \{-t\} \times \{t - 1\} \times \mathbb{R}_+,$$

where g_t^* denotes the *conjugate function* of g_t. Thus $cone\Big(\bigcup_{t \in T} \text{epi} g_t^*\Big) = \mathbb{R}_+ \times \mathbb{R}^2_- \times \mathbb{R}_+$ which is closed in $\mathbb{R}^4$, where $cone(\Omega)$ denotes the *convex conical hull* of Ω. So it follows from [7, Theorem 3.7] that C satisfies (FRCQ). For $y^* = (y_1^*, y_2^*) \in K^*_{up}$ defined in (2.1), we have

$$\begin{aligned}
\Lambda\big(\bar{p}, \bar{x}, \nabla_x f(\bar{p}, \bar{x})^* y^*\big) &= \Big\{\lambda \in \mathbb{R}_+^{(T)} \,|\, \nabla_x f(\bar{p}, \bar{x})^* y^* \\
&\qquad + \sum_{t \in \text{supp}\lambda} \lambda_t \nabla_x g_t(\bar{p}, \bar{x}) = 0, \lambda_t g_t(\bar{p}, \bar{x}) = 0, \ \forall t \in \text{supp } \lambda\Big\} \\
&= \Big\{\lambda \in \mathbb{R}_+^{(T)} \,|\, (y_1^*, y_2^*) + \sum_{t \in \text{supp } \lambda} \lambda_t(-t, t-1) = (0, 0)\Big\} \\
&= \Big\{\lambda \in \mathbb{R}_+^{(T)} \,|\, \sum_{t \in \text{supp } \lambda} \lambda_t t = y_1^*, \sum_{t \in \text{supp } \lambda} \lambda_t = y_1^* + y_2^*\Big\}.
\end{aligned}$$

Applying now Corollary 4.6, we get

$$\widehat{D}^* \mathcal{F}(\bar{p}, \bar{y})(y^*) = \bigcup_{\lambda \in \Lambda\big(\bar{p}, \bar{x}, \nabla_x f(\bar{p}, \bar{x})^* y^*\big)} \big[y_1^* + \sum_{t \in \text{supp } \lambda} \lambda_t t\big] = \{2y_1^*\}.$$

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