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GOOD SOLUTIONS FOR A CLASS OF INFINITE HORIZON DISCRETE-TIME OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper we establish the existence of good solutions for a large class of infinite horizon discrete-time optimal control problems. This class contains optimal control problems arising in economic dynamics which describe a model proposed by Robinson, Solow and Srinivasan with nonconcave utility functions representing the preferences of the planner.

1. INTRODUCTION

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [4, 7-9, 11, 14, 19-23, 31-33] and the references mentioned therein. These problems arise in engineering [1, 12], in models of economic growth [2, 6, 10, 15, 17, 18, 26, 29, 34-36], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3, 30] and in the theory of thermodynamical equilibrium for materials [5, 13, 16]. In this paper we study a large class of infinite horizon discrete-time optimal control problems. This class contains optimal control problems arising in economic dynamics which describe a model proposed by Robinson, Solow and Srinivasan [24, 25, 27, 28] with nonconcave utility functions representing the preferences of the planner.

We begin with some preliminary notation. Let R (R_+) be the set of real (non-negative) numbers and let R^n be a finite-dimensional Euclidean space with non-negative orthant $R^n_+ = \{x \in R^n : \ x_i \geq 0, \ i = 1, \ldots, n\}$. For any $x, y \in R^n$, let the inner product $xy = \sum_{i=1}^n x_i y_i$, and $x >> y$, $x > y$, $x \geq y$ have their usual meaning. Let $e(i)$, $i = 1, \ldots, n$, be the ith unit vector in R^n, and e be an element of R^n_+ all of whose coordinates are unity. For any $x \in R^n$, let $||x||_2$ denote the Euclidean norm of x.

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For each mapping $a : X \to 2^Y \setminus \{\emptyset\}$, where X, Y are nonempty sets, put graph$(a) = \{(x, y) \in X \times Y : \ y \in a(x)\}$.

Let K be a nonempty compact subset of R^n. Denote by $\mathcal{P}(K)$ the set of all nonempty closed subsets of K. We assume that $||\cdot||$ is a norm on R^n.

For each nonempty $A, B \subset R^n$ set

$$H(A,B) = \sup\{\sup_{x\in A}\inf_{y\in B}||x-y||, \ \sup_{y\in B}\inf_{x\in A}||x-y||\}. \tag{1.1}$$

For any integer $t \geq 0$ let $a_t : K \to \mathcal{P}(K)$ be such that graph(a_t) is a closed subset of $R^n \times R^n$.

Suppose that there exists $\kappa \in (0, 1)$ such that for each $x, y \in K$ and each integer $t \geq 0$,

$$H(a_t(x), a_t(y)) \leq \kappa||x - y|| \tag{1.2}$$

and that for each integer $t \geq 0$ the upper semicontinuous function

$$u_t : \{(x, x') \in K \times K, \ x' \in a_t(x)\} \to [0, \infty)$$

satisfies

$$\sup\{\sup\{u_t(x, x') : \ (x, x') \in \text{graph}(a_t)\} : \ t = 0, 1, \dots\} < \infty. \tag{1.3}$$

A sequence $\{x(t)\}_{t=0}^{\infty} \subset K$ is called a program if $x(t + 1) \in a(x(t))$ for all integers $t \geq 0$.

Let T_1, T_2 be integers such that $T_1 < T_2$. A sequence $\{x(t)\}_{t=T_1}^{T_2} \subset K$ is called a program if $x(t + 1) \in a_t(x(t))$ for all integers t satisfying $T_1 \leq t < T_2$.

We suppose that the following assumptions hold:

$(A1)$ for each $\delta > 0$ there exists $\lambda > 0$ such that if an integer $t \geq 0$ and if $(x, x') \in$ graph(a_t) satisfies $u_t(x, x') \geq \delta$, then there is $z \in a_t(x)$ for which $z \geq x' + \lambda e$;

$(A2)$ there exist a program $\{\widehat{x}(t)\}_{t=0}^{\infty}$ and $\widehat{\Delta} > 0$ such that $u_t(\widehat{x}(t), \widehat{x}(t+1)) \geq \widehat{\Delta}$ for all integers $t \geq 0$;

$(A3)$ for each integer $t \geq 0$, each $(x, y) \in$ graph(a_t) and each $\tilde{x} \in K$ satisfying $\tilde{x} \geq x$ there is $\tilde{y} \in a_t(\tilde{x})$ such that

$$\tilde{y} \geq y, \ u_t(\tilde{x}, \tilde{y}) \geq u_t(x, y).$$

In the sequel we assume that supremum of empty set is $-\infty$.

For each $x_0 \in K$ and each integer $T > 0$ set

$$U(x_0, T) = \sup\left\{\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) : \{x(t)\}_{t=0}^{T-1} \text{ is a program and } x(0) = x_0\right\}. \tag{1.4}$$

Let $x_0, \tilde{x}_0 \in K$ and let T be a natural number. Set

$$U(x_0, \tilde{x}_0, T) = \sup \left\{ \sum_{t=0}^{T-1} u_t(x(t), x(t+1)) : \{x(t)\}_{t=0}^{T} \text{ is a program such that } x(0) = x_0,\ x(T) \geq \tilde{x}_0 \right\}. \tag{1.5}$$

Let T be a natural number. Set

$$\widehat{U}(T) = \sup\{ \sum_{t=0}^{T-1} u_t(x(t), x(t+1)) : \{x(t)\}_{t=0}^{T} \text{ is a program} \}. \tag{1.6}$$

Upper semicontinuity of u_t, $t = 0, 1, \dots$ implies the following two results.

Proposition 1.1. *For each $x_0 \in K$ and each natural number T there exists a program $\{x(t)\}_{t=0}^{T}$ such that $x(0) = x_0$ and*

$$\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) = U(x_0, T).$$

Proposition 1.2. *For each natural number T there exists a program $\{x(t)\}_{t=0}^{T}$ such that $\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) = \widehat{U}(T)$.*

For each $x_0 \in K$ and each pair of integers $T_1 < T_2$ set

$$U(x_0, T_1, T_2) = \sup \left\{ \sum_{t=T_1}^{T_2} u_t(x(t), x(t+1)) : \{x(t)\}_{t=T_1}^{T_2} \text{ is a program and } x(T_1) = x_0 \right\}. \tag{1.7}$$

Upper semicontinuity of u_t, $t = 0, 1, \dots$ implies the following result.

Proposition 1.3. *For each $x_0 \in K$ and each pair of integers $T_1 < T_2$ there exists a program $\{x(t)\}_{t=T_1}^{T_2}$ such that $x(T_1) = x_0$ and*

$$\sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) = U(x_0, T_1, T_2).$$

Let $x_0, \tilde{x}_0 \in K$ and let $T_1 < T_2$ be integers. Set

$$U(x_0, \tilde{x}_0, T_1, T_2) = \sup\{ \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) : \{x(t)\}_{t=T_1}^{T_2} \text{ is a program and}$$

$$x(T_1) = x_0, \ \{x(T_2) \geq \tilde{x}_0\} \tag{1.8}$$

Let T_1, T_2 be integers such that $T_1 < T_2$. Set

$$\widehat{U}(T_1, T_2) = \sup \left\{ \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) : \ \{x(t)\}_{t=T_1}^{T_2} \text{ is a program} \right\}. \tag{1.9}$$

We will establish the following theorem which is our main result.

Theorem 1.1. *There is $M > 0$ such that for each $x_0 \in K$ there exists a program $\{\bar{x}(t)\}_{t=0}^{\infty}$ such that $\bar{x}(0) = x_0$ and that for each pair of integers $T_1, T_2 \geq 0$ satisfying $T_1 < T_2$,*

$$\left| \sum_{t=T_1}^{T_2-1} u_t(\bar{x}(t), \bar{x}(t+1)) - \widehat{U}(T_1, T_2) \right| \leq M.$$

Moreover, for each integer $T > 0$,

$$\sum_{t=0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1)) = U(\bar{x}(0), \bar{x}(T), 0, T),$$

if the following properties hold:

for each integer $t \geq 0$ and each $(z, z') \in$ graph(a_t) satisfying $u_t(z, z') > 0$ the function u_t is continuous at (z, z'); for each integer $t \geq 0$ and each $z, \ z_1, \ z_2, \ z_3 \in K$ satisfying $z_1 \leq z_2 \leq z_3$ and $z_i \in a_t(z)$, $i = 1, 3$ the inclusion $z_2 \in a_t(z)$ holds.

The program $\{\bar{x}(t)\}_{t=0}^{\infty}$ whose existence is guaranteed by Theorem 1.1 in infinite horizon optimal control is considered as an (approximately) optimal program [3, 5, 11, 13, 16, 35, 36].

We will also establish the following result.

Theorem 1.2. *Assume that $\{x(t)\}_{t=0}^{\infty}$ is a program, there exists $M_0 > 0$ such that for each integer $T > 0$,*

$$\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) \geq U(0, T, x(0), x(T)) - M_0$$

and that

$$\limsup_{t \to \infty} u_t(x(t), x(t+1)) > 0.$$

Then there exists $M_1 > 0$ *such that for each pair of integers* $T_1 \geq 0$, $T_2 > T_1$,

$$\left|\sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) - \widehat{U}(T_1, T_2)\right| \leq M_1.$$

Theorem 1.1 is proved in Section 6 while Theorem 1.2 is obtained in Section 7.

Let $M > 0$ be as guaranteed by Theorem 1.1.

Proposition 1.4. *Let* $x_0 \in K$ *and let a program* $\{\bar{x}(t)\}_{t=0}^{\infty}$ *be as guaranteed by Theorem 1.1. Assume that* $\{x(t)\}_{t=0}^{\infty}$ *is a program. Then either the sequence*

$$\{\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) - \sum_{t=0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1))\}_{T=1}^{\infty}$$

is bounded or

$$\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) - \sum_{t=0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1)) \to -\infty \text{ as } T \to \infty. \tag{1.10}$$

Proof. Assume that the sequence $\{\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) - \sum_{t=0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1))\}_{T=1}^{\infty}$ is not bounded. Then by Theorem 1.1,

$$\liminf_{T\to\infty}[\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) - \sum_{t=0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1))] = -\infty.$$

Let $Q > 0$. Then there exists an integer $T_0 > 0$ such that

$$\sum_{t=0}^{T_0-1} u_t(x(t), x(t+1)) - \sum_{t=0}^{T_0-1} u_t(\bar{x}(t), \bar{x}(t+1)) < -Q - M. \tag{1.11}$$

By (1.11), the choice of $\{\bar{x}(t)\}_{t=0}^{\infty}$ and Theorem 1.1 for each integer $T > T_0$,

$$\begin{aligned}
&\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) - \sum_{t=0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1)) = \sum_{t=0}^{T_0-1} u_t(x(t), x(t+1)) \\
&- \sum_{t=0}^{T_0-1} u_t(\bar{x}(t), \bar{x}(t+1)) + \sum_{t=T_0}^{T-1} u_t(x(t), x(t+1)) - \sum_{t=T_0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&< -Q - M + \widehat{U}(T_0, T) - \sum_{t=T_0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1)) < -Q.
\end{aligned}$$

Since Q is any positive number we conclude that (1.10) is true. Proposition 1.4 is proved.

Note that if the program $\{x(t)\}_{t=0}^{\infty}$ satisfies (1.10), the it is called bad; otherwise it is called good [6, 11, 34-36]. Thus in view of Theorem 1.1 for any initial state there exists a good program. This is a difficult result because we study the infinite horizon optimal control problem with constraints and the cost functions u_t are not assumed to be concave. The existence of good programs is established for a large class of infinite horizon problems. We show in Section 3 that this class contains optimal control problems arising in economic dynamics which describe a nonstationary model proposed by Robinson, Solow and Srinivasan [24, 25, 27, 28] with nonconcave utility functions representing the preferences of the planner. Existence of good programs for the stationary Robinson-Solow-Srinivasan model with a nonconcave utility function was obtained in [35].

Now assume that $u_t = u_0$ and $a_t = a_0$, $t = 0, 1, \ldots$. Let $M > 0$ be as guaranteed by Theorem 1.1 and set $u = u_0$, $a = a_0$. The following result which will be proved in Section 8 is a generalization of one of the main results of [35].

Theorem 1.3. *There exists* $\mu = \lim_{p\to\infty} \widehat{U}(0, p)/p$ *and*

$$|p^{-1}\widehat{U}(0, p) - \mu| \le 2M/p \text{ for all natural numbers } p.$$

2. Upper Semicontinuity of Cost Functions

We use the notation from Section 1. For each integer $t \ge 0$ let $a_t : K \to \mathcal{P}(K)$ be such that graph(a_t) is a closed set and assume that for each integer $t \ge 0$ an upper semicontinuous function $\phi_t : R^n_+ \to [0, \infty)$ be such that

$$\sup\{\sup\{\phi_t(z) : \ z \in (K - R^n_+) \cap R^n_+\} : \ t = 0, 1, \ldots\} < \infty. \tag{2.1}$$

For each integer $t \ge 0$ and each $(x, x') \in$ graph(a_t) define

$$u_t(x, x') = \sup\{\phi_t(z) : \ z \in R^n_+, \ x' + z \in a(x)\}. \tag{2.2}$$

In view of (2.1) and (2.2) u_t, $t = 0, 1, \ldots$ satisfy (1.3). Note that in many models of economic dynamics cost functions u_t, $t = 0, 1, \ldots$ are defined by (2.2).

Lemma 2.1. *For each integer* $t \ge 0$ *the function* $u_t :$ *graph*$(a_t) \to [0, \infty)$ *is upper semicontinuous.*

Proof. Let $t \ge 0$ be an integer and let $\{(x^{(j)}, y^{(j)})\}_{j=1}^{\infty} \subset$ graph(a_t) satisfy

$$\lim_{j\to\infty} (x^{(j)}, y^{(j)}) = (x, y). \tag{2.3}$$

We show that $u_t(x,y) \geq \limsup_{j\to\infty} u(x^{(j)}, y^{(j)})$. Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that there exists $\lim_{j\to\infty} u(x^{(j)}, y^{(j)})$. By (2.2), for each integer $j \geq 1$ there exists $z^{(j)} \in R^n_+$ such that

$$y^{(j)} + z^{(j)} \in a_t(x^{(j)}),\ \phi_t(z^{(j)}) \geq u_t(x^{(j)}, y^{(j)}) - 1/j. \tag{2.4}$$

Clearly, the sequence $\{z^{(j)}\}_{j=1}^{\infty}$ is bounded. Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that there exists

$$z = \lim_{j\to\infty} z^{(j)}. \tag{2.5}$$

By (2.3), (2.4) and (2.5), $z \geq 0$ and $(x, y+z) = \lim_{j\to\infty}(x^{(j)}, y^{(j)} + z^{(j)}) \in$ graph(a_t). Combined with (2.2), (2.4) and (2.5) this implies that

$$\begin{aligned} u_t(x,y) \geq \phi_t(z) \geq \limsup_{j\to\infty} \phi_t(z^{(j)}) &\geq \limsup_{j\to\infty} [u_t(x^{(j)}, y^{(j)}) - 1/j] \\ &= \lim_{j\to\infty} u_t(x^{(j)}, y^{(j)}). \end{aligned}$$

Lemma 2.1 is proved.

3. The Nonstationary Robinson-solow-srinivasan Model

In this section we consider a subclass of the class of infinite horizon optimal control problems considered in Section 1. Infinite horizon problems of this subclass correspond to the nonstationary Robinson-Solow-Srinivasan models [24, 25, 27, 28].

For each integer $t \geq 0$ let

$$\begin{aligned} \alpha^{(t)} &= (\alpha_1^{(t)}, \ldots, \alpha_n^{(t)}) >> 0, \\ b^{(t)} &= (b_1^{(t)}, \ldots, b_n^{(t)}) >> 0, \\ d^{(t)} &= (d_1^{(t)}, \ldots, d_n^{(t)}) \in ((0,1])^n \end{aligned} \tag{3.1}$$

and for each integer $t \geq 0$ let $w_t : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function such that

$$w_t(0) = 0,\ \inf\{w_t(z) :\ t = 0,1,\ldots\} > 0 \text{ for all } z > 0 \tag{3.2}$$

and such that the following assumption holds:

(A4) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each integer $t \geq 0$ and each $z \in [0,\delta]$ the inequality $w_t(z) \leq \epsilon$ is true.

Let $t \geq 0$ be an integer. For each $x \in R^n_+$ set

$$a_t(x) = \{y \in R^n_+ : \ y_i \ge (1-d_i^{(t)})x_i, \ i=1,\dots,n, \sum_{i=1}^n \alpha_i^{(t)}(y_i-(1-d_i^{(t)})x_i) \le 1\}. \tag{3.3}$$

It is clear that for each $x \in R^n$, $a_t(x)$ is a nonempty closed bounded subset of R^n_+ and graph(a_t) is a closed subset of $R^n_+ \times R^n_+$. Suppose that

$$\inf\{d_i^{(t)} : \ i=1,\dots,n, \ t=0,1,\dots\} > 0, \tag{3.4}$$

$$\inf\{eb^{(t)} : \ t=0,1,\dots\} > 0, \tag{3.5}$$

$$\inf\{\alpha_i^{(t)} : \ i=1,\dots,n, \ t=0,1,\dots\} > 0, \tag{3.6}$$

$$\sup\{b_i^{(t)} : \ i=1,\dots,n, \ t=0,1,\dots\} < \infty, \tag{3.7}$$

$$\sup\{\alpha_i^{(t)} : \ i=1,\dots,n, \ t=0,1,\dots\} < \infty \tag{3.8}$$

and that for each $M > 0$

$$\sup\{w_t(M) : \ t=0,1,\dots\} < \infty, \ \inf\{w_t(M) : \ t=0,1,\dots\} > 0. \tag{3.9}$$

The constraint mappings a_t, $t=0,1,\dots$ have already been defined. Let us now define the cost functions u_t, $t=0,1,\dots$.

For each integer $t \ge 0$ and each $(x,x') \in$ graph(a_t) set

$$u_t(x,x') = sup\{w_t(b^{(t)}y) : \ 0 \le y \le x, ey + \sum_{i=1}^n \alpha_i^{(t)}(x'_i-(1-d_i^{(t)})x_i) \le 1\}. \tag{3.10}$$

Choose α^*, $\alpha_* > 0$, $d_* > 0$ such that

$$\alpha_* < \alpha_i^{(t)} < \alpha^*, \ d_* < d_i^{(t)}, \ i=1,\dots,n, \ t=0,1,\dots. \tag{3.11}$$

Lemma 3.1. *Let a number $M_0 > (\alpha_* d_*)^{-1}$, an integer $t \ge 0$ and let $(x,x') \in$ graph(a_t) satisfy $x \le M_0 e$. Then $x' \le M_0 e$.*

Proof. By (3.3), $\sum_{i=1}^n \alpha_i^{(t)}(x'_i-(1-d_i^{(t)})x_i) \le 1$ and in view of (3.11) for each $i=1,\dots,n$,

$$\begin{aligned} x'_i &\le (\alpha_i^{(t)})^{-1} + (1-d_i^{(t)})x_i \le \alpha_*^{-1} + (1-d_*)x_i \le \alpha_*^{-1} + (1-d_*)M_0 \\ &\le d_*(\alpha_* d_*)^{-1} + (1-d_*)M_0 \le d_* M_0 + (1-d_*)M_0 = M_0. \end{aligned}$$

Lemma 3.1 is proved.

Lemma 3.2. *Let $t \geq 0$ be an integer. Then the function u_t : graph$(a_t) \to [0, \infty)$ is upper semicontinuous. Moreover, if $(x, y) \in$ graph(a_t) and $u_t(x, y) > 0$, then u_t is continuous at (x, y).*

Proof. Let

$$(x, y) \in \text{graph}(a_t),\ \{(x^{(j)}, y^{(j)})\}_{j=1}^{\infty} \subset \text{graph}(a_t),\ \lim_{j\to\infty}(x^{(j)}, y^{(j)}) = (x, y). \tag{3.12}$$

We show that $u_t(x, y) \geq \limsup_{j\to\infty} u_t(x^{(j)}, y^{(j)})$. Extracting a subsequence and re-indexing we may assume that there exists $\lim_{j\to\infty} u_t(x^{(j)}, y^{(j)})$. By (3.9) and (3.10) for each integer $j \geq 1$ there exists $z^{(j)} \in R^n_+$ such that

$$z^{(j)} \leq x^{(j)},\ ez^{(j)} + \sum_{i=1}^{n} \alpha_i^{(t)}(y_i^{(j)} - (1 - d_i^{(t)})x_i^{(j)}) \leq 1, \tag{3.13}$$

$$w_t(b^{(t)}z^{(j)}) \geq u_t(x^{(j)}, y^{(j)}) - 1/j. \tag{3.14}$$

Extracting a subsequence and re-indexing we may assume without loss of generality that there exists

$$z = \lim_{j\to\infty} z^{(j)}. \tag{3.15}$$

In view of (3.12) and (3.15)

$$0 \leq z \leq x. \tag{3.16}$$

By (3.10), (3.12) and (3.15),

$$\begin{aligned} &ez + \sum_{i=1}^{n} \alpha_i^{(t)}(y_i - (1 - d_i^{(t)})x_i) \\ &= \lim_{j\to\infty}[ez^{(j)} + \sum_{i=1}^{n} \alpha_i^{(t)}(y_i^{(j)} - (1 - d_i^{(t)})x_i^{(j)})] \leq 1. \end{aligned}$$

Together with (3.10), (3.14) and (3.16) this implies that

$$u_t(x, y) \geq w_t(b^{(t)}z) = \lim_{j\to\infty} w_t(b^{(t)}z^{(j)}) = \lim_{j\to\infty} u_t(x^{(j)}, y^{(j)}).$$

Thus u_t is upper lower semicontinuous.

Assume now that $(x, y) \in \mathrm{graph}(a_t)$ satisfies

$$u_t(x, y) > 0 \tag{3.17}$$

and show that u_t is continuous at (x, y). Clearly, it is sufficient to show that u_t is lower semicontinuous at (x, y). Assume that

$$(x^{(j)}, y^{(j)}) \in \mathrm{graph}(a_t) \text{ for all integers } j \geq 1, \ \lim_{j\to\infty}(x^{(j)}, y^{(j)}) = (x, y). \tag{3.18}$$

Let $\epsilon > 0$. It is sufficient to show that $\liminf_{j\to\infty} u_t(x^{(j)}, y^{(j)}) \geq u_t(x, y) - \epsilon$. By (3.10) and (3.17) there is $z \in R^n_+$ such that

$$z \leq x, \ ez + \sum_{i=1}^{n} \alpha_i^{(t)}(y_i - (1 - d_i^{(t)})x_i) \leq 1, \tag{3.19}$$

$$w_t(b^{(t)}z) > 0, \ w_t(b^{(t)}z) > u_t(x, y) - \epsilon/4. \tag{3.20}$$

In view of (3.2) and (3.20) there is $q \in \{1, \dots, n\}$ such that

$$b_q^{(t)} z_q > 0. \tag{3.21}$$

It follows from (3.2) and (3.21) that there is $\gamma \in (0, 1)$ such that

$$w_t(b^{(t)}\gamma z) \geq w_t(b^{(t)}z) - \epsilon/4. \tag{3.22}$$

By (3.18), (3.19) and (3.21) there exists a natural number j_0 such that for each integer $j \geq j_0$,

$$\gamma z \leq x^{(j)}, \ e(\gamma z) + \sum_{i=1}^{n} \alpha_i^{(t)}(y_i^{(j)} - (1 - d_i^{(t)})x_i^{(j)}) \leq 1. \tag{3.23}$$

Relations (3.10), (3.20), (3.22) and (3.23) imply that for all integers $j \geq j_0$,

$$u_t(x^{(j)}, y^{(j)}) \geq w_t(b^{(t)}\gamma z) \geq w_t(b^{(t)}z) - \epsilon/4 > u_t(x, y) - \epsilon/2.$$

This implies that u_t is lower semicontinuous at (x, y). Lemma 3.2 is proved.

For each $x = (x_1, \dots, x_n) \in R^n$ set

$$||x||_1 = \sum_{i=1}^{n} |x_i|, \ ||x||_\infty = \max\{|x_i| : \ i = 1, \dots, n\}. \tag{3.24}$$

By (3.3) and (3.11) for each integer $t \geq 0$, each $x, y \in K$ and for $|| \cdot || = || \cdot ||_p$, where $p = 1, 2, \infty$,

$$H(a_t(x), a_t(y)) \leq ||((1-d_i^{(t)})x_i)_{i=1}^n - ((1-d_i^{(t)})y_i)_{i=1}^n|| \leq (1-d_*)||x-y|| \tag{3.25}$$

(see (1.2)).

Proposition 3.1. *Let $\delta > 0$. Then there exists $\lambda > 0$ such that for each integer $t \geq 0$ and each $(x, y) \in$ graph(a_t) which satisfies $u_t(x, y) \geq \delta$ the inclusion $y + \lambda e \in a_t(x)$ holds.*

Proof. By (A4) there is $\delta_0 > 0$ such that for each integer $t \geq 0$ and each $\xi \in R_+$ satisfying $w_t(\xi) \geq \delta/2$ the following inequality holds:

$$\xi \geq \delta_0. \tag{3.26}$$

Set

$$b_* = \sup\{b_i^{(t)} :\ t = 0, 1, \ldots,\ i = 1, \ldots, n\} \tag{3.27}$$

(see (3.7)). Choose a positive number λ such that

$$\lambda n \alpha^* < 2^{-1} b_*^{-1} \delta_0. \tag{3.28}$$

Assume that an integer $t \geq 0$,

$$(x, y) \in \text{graph}(a_t),\ u_t(x, y) \geq \delta. \tag{3.29}$$

By (3.10) and (3.29) there exists $z \in R^n_+$ such that

$$0 \leq z \leq x,\ ez + \sum_{i=1}^{n} \alpha_i^{(t)}(y_i - (1 - d_i^{(t)})x_i) \leq 1,\ w_t(b^{(t)}z) \geq \delta/2. \tag{3.30}$$

In view of (3.30) and the choice of δ_0,

$$b^{(t)} z \geq \delta_0. \tag{3.31}$$

It follows from (3.27) and (3.31) that

$$ez = \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} (b_i^{(t)})^{-1} b_i^{(t)} z_i \geq b_*^{-1} bz \geq b_*^{-1} \delta_0. \tag{3.32}$$

We show that $y + \lambda e \in a_t(x)$. It is clear (see (3.3) and (3.29)) that for any $i = 1, \ldots, n$

$$y_i + \lambda \geq y_i \geq (1 - d_i^{(t)})x_i. \tag{3.33}$$

It follows from (3.11), (3.28), (3.30) and (3.32) that

$$\sum_{i=1}^{n} \alpha_i^{(t)}((y + \lambda e)_i - (1 - d_i^{(t)})x_i) = \sum_{i=1}^{n} \alpha_i^{(t)}(y_i - (1 - d_i^{(t)})x_i) + \lambda \sum_{i=1}^{n} \alpha_i^{(t)}$$
$$\leq 1 - ez + \lambda \sum_{i=1}^{n} \alpha_i^{(t)} \leq 1 - b_*^{-1}\delta_0 + \lambda n \alpha^* < 1$$

and together with (3.33) this implies that $y + \lambda e \in a_t(x)$. Proposition 3.1 is proved.

Proposition 3.2. *There exist a program* $\{\widehat{x}(t)\}_{t=0}^{\infty}$ *and* $\widehat{\Delta} > 0$ *such that*

$$u_t(\widehat{x}(t), \widehat{x}(t+1)) \geq \widehat{\Delta} \text{ for all integers } t \geq 0.$$

Proof. Choose $\lambda_0 > 0$, $\lambda_1 > 0$ such that

$$\lambda_0 n \alpha^* < 1/2,\ \lambda_1 < \lambda_0,\ \lambda_1 n < 1/4. \tag{3.34}$$

By (3.5), there is $\epsilon_0 > 0$ such that

$$eb^{(t)} \geq \epsilon_0,\ t = 0, 1, \ldots. \tag{3.35}$$

Put

$$\widehat{\Delta} = \inf\{w_t(\lambda_1 \epsilon_0) :\ t = 0, 1, \ldots\}. \tag{3.36}$$

By (3.9), $\widehat{\Delta} > 0$. Set

$$\widehat{x}(t) = \lambda_0 e,\ t = 0, 1, \ldots,\ \widehat{y}(t) = \lambda_1 e,\ t = 0, 1, \ldots \tag{3.37}$$

By (3.11), (3.34) and (3.37) for $i = 1, \ldots, n$, $t = 0, 1, \ldots$,

$$\widehat{x}_i(t+1) - (1 - d_i^{(t)})\widehat{x}_i(t) = \lambda_0 d_i^{(t)} > 0, \tag{3.38}$$

$$\begin{aligned} &\sum_{i=1}^{n} \alpha_i^{(t)} [\widehat{x}_i(t+1) - (1 - d_i^{(t)})\widehat{x}_i(t)] \\ &= \left(\sum_{i=1}^{n} \alpha_i^{(t)} d_i^{(t)} \right) \lambda_0 \leq \lambda_0 \sum_{i=1}^{n} \alpha_i^{(t)} \leq \lambda_0 n \alpha^* < 1/2 \end{aligned} \tag{3.39}$$

and for $t = 0, 1, \ldots$,

$$e\widehat{y}(t) + \sum_{i=1}^{n} \alpha_i^{(t)} [\widehat{x}_i(t+1) - (1 - d_i^{(t)})\widehat{x}_i(t)] \leq \lambda_1 n + 1/2 < 1. \tag{3.40}$$

Therefore $\{\widehat{x}(t)\}_{t=0}^{\infty}$ is a program. By (3.10), (3.34), (3.35), (3.37), (3.36) and (3.40) for all integers $t \geq 0$,

$$u_t(\widehat{x}(t), \widehat{x}(t+1)) \geq w_t(b^{(t)}\widehat{y}(t)) \geq w_t(\lambda_1 e b^{(t)}) \geq w_t(\lambda_1 \epsilon_0) \geq \widehat{\Delta}.$$

Proposition 3.2 is proved.

Proposition 3.3. *Let $t \geq 0$ be an integer, $(x, y) \in graph(a_t)$ and let $\tilde{x} \in R^n_+$ satisfy $\tilde{x} \geq x$. Then there is $\tilde{y} \in a_t(\tilde{x})$ such that $\tilde{y} \geq y$ and $u_t(\tilde{x}, \tilde{y}) \geq u_t(x, y)$.*

Proof. By (3.10), there is $z \in R^n_+$ such that

$$0 \leq z \leq x,\ ez + \sum_{i=1}^{n} \alpha_i^{(t)}(y_i - (1 - d_i^{(t)})x_i) \leq 1,\ w_t(b^{(t)}z) = u_t(x, y). \tag{3.41}$$

For any $i = 1, \dots, n$ set

$$\tilde{y}_i = \tilde{x}_i(1 - d_i^{(t)}) + y_i - (1 - d_i^{(t)})x_i. \tag{3.42}$$

By (3.3), (3.41) and (3.42), for $i = 1, \dots, n$, $\tilde{y}_i \geq (1 - d_i^{(t)})\tilde{x}_i$,

$$\sum_{i=1}^{n} \alpha_i^{(t)}(\tilde{y}_i - (1 - d_i^{(t)})\tilde{x}_i) = \sum_{i=1}^{n} \alpha_i^{(t)}(y_i - (1 - d_i^{(t)})x_i) \leq 1 - ez.$$

Therefore $\tilde{y} \in a_t(\tilde{x})$. In view of the inequality $\tilde{x} \geq x$ and (3.42) we have $\tilde{y} \geq y$. It is easy to see that

$$u_t(\tilde{x}, \tilde{y}) \geq w_t(b^{(t)}z) = u_t(x, y).$$

This completes the proof of Proposition 3.3 is proved.

It is easy to see that the following result is true.

Proposition 3.4. *Let an integer $t \geq 0$, $x,\ x_1,\ x_2,\ x_3 \in R^n_+$, $x_i \in a_t(x)$, $i = 1, 3$, $x_1 \leq x_2 \leq x_3$. Then $x_2 \in a(x_t)$.*

Thus we have defined the mappings a_t and the cost functions u_t, $t = 0, 1, \dots$. The control system considered in this section is a special case of the control system studied in Section 1. As we have already mentioned before this control system corresponds to the nonstationary Robinson-Solow-Srinivasan model [24, 25, 27, 28]. Note that this control system satisfies the assumptions posed in Section 1 and therefore all the results stated there hold for this system. Indeed, choose $M_0 > (\alpha_* d_*)^{-1}$ and put $K = \{z \in R^n_+ : z \leq M_0 e\}$. By Lemma 3.1, $a_t(K) \subset K$, $t = 0, 1, \dots$. Relation (1.2) follows from (3.25). Clearly, (1.3) holds. In view of Lemma 3.2, u_t is upper semicontinuous for all integers $t \geq 0$. Proposition 3.1 implies (A1). (A2) follows from Proposition 3.2 and (A3) follows from Proposition 3.3.

4. Auxiliary Results for Theorems 1.1-1.3

In this section we use the notation and the assumptions of Section 1.

Lemma 4.1. *Let $\delta > 0$. Then there exists a natural number $T_0 \geq 4$ such that for each integer $\tau_1 \geq 0$, each integer $\tau_2 \geq T_0 + \tau_1$, each program $\{x(t)\}_{t=\tau_1}^{\tau_2}$ which satisfies*

$$u_{\tau_2-1}(x(\tau_2 - 1), x(\tau_2)) \geq \delta \tag{4.1}$$

and each $\tilde{x}_0 \in K$ there exists a program $\{\tilde{x}(t)\}_{t=\tau_1}^{\tau_2}$ such that

$$\tilde{x}(\tau_1) = \tilde{x}_0,\ \tilde{x}(\tau_2) \geq x(\tau_2).$$

Proof. By (A1) there exists $\lambda \in (0, 1)$ such that the following property holds:

(P1) For each integer $t \geq 0$ and each $(x, x') \in$ graph(a_t) satisfying $u_t(x, x') \geq \delta$ there is $z \in a_t(x)$ such that $z \geq x' + \lambda e$.

Choose $D_0 > 0$ such that

$$||z|| \leq D_0 \text{ for all } z \in K. \tag{4.2}$$

There is $c_0 > 0$ such that

$$||z||_2 \leq c_0||z|| \text{ for all } z \in K. \tag{4.3}$$

Choose a natural number $T_0 \geq 4$ such that

$$2D_0c_0\kappa^{T_0} < \lambda \tag{4.4}$$

(see (1.2)).

Assume that integers $\tau_1 \geq 0$, $\tau_2 \geq T_0 + \tau_1$, a program $\{x(t)\}_{t=\tau_1}^{\tau_2}$ satisfies (4.1) and that $\tilde{x}_0 \in K$. By (4.1) and (P1) there exists $z \in R^n_+$ such that

$$z \in a_{\tau_2-1}(x(\tau_2 - 1)),\ z \geq x(\tau_2) + \lambda e. \tag{4.5}$$

By (1.2) there exists a program $\{\tilde{x}(t)\}_{t=\tau_1}^{\tau_2-1}$ such that

$$\begin{aligned} &\tilde{x}(\tau_1) = \tilde{x}_0, \\ &||\tilde{x}(t + 1) - x(t + 1)|| \leq \kappa||\tilde{x}(t) - x(t)||,\ t = \tau_1, \dots, \tau_2 - 2. \end{aligned} \tag{4.6}$$

In view of (1.2) and (4.5) there is $\tilde{x}(\tau_2) \in a_{\tau_2-1}(\tilde{x}(\tau_2 - 1))$ such that

$$||\tilde{x}(\tau_2) - z|| \leq \kappa||x(\tau_2 - 1) - \tilde{x}(\tau_2 - 1)||. \tag{4.7}$$

Clearly, $\{\tilde{x}(t)\}_{t=\tau_1}^{\tau_2}$ is a program. Relations (4.2), (4.6) and (4.7) imply that

$$||\tilde{x}(\tau_2) - z|| \leq \kappa^{\tau_2-\tau_1}||\tilde{x}(\tau_1) - x(\tau_1)|| \leq \kappa^{\tau_2-\tau_1}(2D_0) \leq \kappa^{T_0}(2D_0)$$

and in view of (4.3) $||\tilde{x}(\tau_2) - z||_2 \leq 2D_0c_0\kappa^{T_0}$. This implies that for each integer $i = 1, \dots, n$, $|\tilde{x}_i(\tau_2) - z_i| \leq 2D_0c_0\kappa^{T_0}$ and in view of (4.4) and (4.5)

$$\tilde{x}(\tau_2) \geq z - 2D_0c_0\kappa^{T_0}e \geq x(\tau_2) + [\lambda - 2D_0c_0\kappa^{T_0}]e \geq x(\tau_2).$$

Lemma 4.1 is proved.

Choose a positive number γ such that

$$\gamma < 1/2 \text{ and } \gamma < 4^{-1}\widehat{\Delta}. \tag{4.8}$$

Lemma 4.2. *Let $M_1 > 0$. Then there exist natural numbers $L_1, L_2 \geq 4$ such that for each pair of integers $T_1 \geq 0$, $T_2 \geq L_1 + L_2 + T_1$, each program $\{x(t)\}_{t=T_1}^{T_2}$ which satisfies*

$$\sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) \geq U(x(T_1), T_1, T_2) - M_1 \tag{4.9}$$

and each integer $\tau \in [T_1 + L_1, T_2 - L_2]$ the following inequality holds:

$$\max\{u_t(x(t), x(t+1)) : \ t = \tau, \dots, \tau + L_2 - 1\} \geq \gamma. \tag{4.10}$$

Proof. By Lemma 4.1 there exists a natural number $L_1 \geq 4$ such that the following property holds:

(P2) If integers $S_1 \geq 0$, $S_2 \geq S_1 + L_1$, if a program $\{v(t)\}_{t=S_1}^{S_2}$ satisfies

$$u_{S_2-1}(v(S_2 - 1), v(S_2)) \geq \gamma$$

and if $\tilde{v}_0 \in K$, then there exists a program $\{\tilde{v}(t)\}_{t=S_1}^{S_2}$ such that $\tilde{v}(S_1) = \tilde{v}_0$, $\tilde{v}(S_2) \geq v(S_2)$.

Choose a number M_2 such that

$$M_2 > u_t(z, z') \text{ for each integer } t \geq 0 \text{ and each } (z, z') \in \text{graph}(a_t) \tag{4.11}$$

and a natural number L_2 such that

$$L_2 > 4(L_1 + 1) + 16\widehat{\Delta}^{-1}(M_1 + L_1\gamma + 1) + 16\widehat{\Delta}^{-1}(M_1 + M_2 + L_1 + 2). \tag{4.12}$$

Assume that integers $T_1 \geq 0$, $T_2 \geq L_1 + L_2 + T_1$, a program $\{x(t)\}_{t=T_1}^{T_2-1}$ satisfies (4.9) and an integer τ satisfies

$$T_1 + L_1 \leq \tau \leq T_2 - L_2. \tag{4.13}$$

We show that (4.10) holds. Let us assume the contrary. Then

$$u_t(x(t), x(t+1)) < \gamma,\ t = \tau, \dots, \tau + L_2 - 1. \tag{4.14}$$

There are two cases:

$$u_t(x(t), x(t+1)) < \gamma,\ t = \tau, \dots, T_2 - 1; \tag{4.15}$$

$$\max\{u_t(x(t), x(t+1)) :\ t = \tau, \dots, T_2 - 1\} \geq \gamma. \tag{4.16}$$

Now we define a natural number τ_0 as follows. If (4.15) is true, then we set $\tau_0 = T_2$. If (4.16) is true, then by (4.14) there is a natural number τ_0 such that

$$\tau + L_2 \leq \tau_0 \leq T_2 - 1, \tag{4.17}$$

$$u_{\tau_0}(x(\tau_0), x(\tau_0 + 1)) \geq \gamma, \tag{4.18}$$

$$u_t(x(t), x(t+1)) < \gamma,\ t = \tau, \dots, \tau_0 - 1. \tag{4.19}$$

It is clear that in the both cases (4.19) holds and that in the both cases

$$\tau_0 - \tau \geq L_2. \tag{4.20}$$

Assume that (4.15) is true. It follows from the choice of L_1, (A2), property (P2), (4.8), (4.12) and (4.13) that there exists a program $\{\tilde{x}(t)\}_{t=\tau}^{\tau+L_1}$ such that

$$\tilde{x}(\tau) = x(\tau),\ \tilde{x}(\tau + L_1) \geq \widehat{x}(\tau + L_1). \tag{4.21}$$

Set

$$\tilde{x}(t) = x(t),\ t = T_1, \dots, \tau. \tag{4.22}$$

By (4.21), (4.22), (A3) and (A2) there exists $\tilde{x}(t) \in K$, $t = \tau + L_1 + 1, \dots, T_2$ such that $\{\tilde{x}(t)\}_{t=T_1}^{T_2}$ is a program,

$$\tilde{x}(t) \geq \widehat{x}(t) \text{ for all integers } t = \tau + L_1, \dots, T_2, \tag{4.23}$$

$$u_t(\tilde{x}(t), \tilde{x}(t+1)) \geq u_t(\widehat{x}(t), \widehat{x}(t+1)),\ t = \tau + L_1, \dots, T_2 - 1. \tag{4.24}$$

It follows from (4.9), (4.13), (4.22), (4.24), (A2), (4.15), (4.8) and (4.12) that

$$\begin{aligned}
M_1 &\geq U(x(T_1), T_1, T_2) - \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) \\
&\geq \sum_{t=T_1}^{T_2-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) \\
&= \sum_{t=\tau}^{T_2-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=\tau}^{T_2-1} u_t(x(t), x(t+1)) \\
&\geq \sum_{t=\tau}^{\tau+L_1-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) + \sum_{t=\tau+L_1}^{T_2-1} u_t(\widehat{x}(t), \widehat{x}(t+1)) - \sum_{t=\tau}^{T_2-1} u_t(x(t), x(t+1)) \\
&\geq \sum_{t=\tau+L_1}^{T_2-1} u_t(\widehat{x}(t), \widehat{x}(t+1)) - \sum_{t=\tau}^{T_2-1} u_t(x(t), x(t+1)) \\
&\geq (T_2 - \tau - L_1)\widehat{\Delta} - (T_2 - \tau)\gamma \\
&= (T_2 - \tau - L_1)(\widehat{\Delta} - \gamma) - L_1\gamma \geq \widehat{\Delta}2^{-1}(T_2 - \tau - L_1) - L_1\gamma \\
&\geq 2^{-1}\widehat{\Delta}(L_2 - L_1) - L_1\gamma \geq 4^{-1}\widehat{\Delta}L_2 - L_1\gamma
\end{aligned}$$

and

$$L_2 \leq 8\widehat{\Delta}^{-1}(M_1 + L_1\gamma).$$

This inequality contradicts (4.12). The contradiction we have reached proves that (4.15) does not hold. Therefore (4.16) is true and there is a natural number τ_0 which satisfies (4.17)-(4.19). It follows from the choice of L_1, property (P2), (A2) and (4.8) that there exists a program $\{\tilde{x}(t)\}_{t=\tau}^{\tau+L_1}$ such that

$$\tilde{x}(\tau) = x(\tau),\ \tilde{x}(\tau + L_1) \geq \widehat{x}(\tau + L_1). \tag{4.25}$$

Set

$$\tilde{x}(t) = x(t),\ t = T_1, \ldots, \tau. \tag{4.26}$$

In view of (A2), (A3), (4.25), (4.17) and (4.12) there exist $\tilde{x}(t) \in K$, $t = \tau + 1 + L_1, \ldots, \tau_0 - L_1$ such that $\{\tilde{x}(t)\}_{t=\tau+L_1}^{\tau_0-L_1}$ is a program,

$$\tilde{x}(t) \geq \widehat{x}(t),\ t = \tau + L_1, \ldots, \tau_0 - L_1, \tag{4.27}$$

$$u_t(\tilde{x}(t), \tilde{x}(t+1)) \geq u_t(\widehat{x}(t), \widehat{x}(t+1)),\ t = \tau + L_1, \ldots, \tau_0 - L_1 - 1. \tag{4.28}$$

Clearly, $\{\tilde{x}(t)\}_{t=T_1}^{\tau_0-L_1}$ is a program. By the choice of L_1, property (P2) and (4.18) there exist $\tilde{x}(t) \in K$, $t = \tau_0 - L_1 + 1, \ldots, \tau_0 + 1$ such that $\{\tilde{x}(t)\}_{t=\tau_0-L_1}^{\tau_0+1}$ is a program,

$$\tilde{x}(\tau_0 + 1) \geq x(\tau_0 + 1). \tag{4.29}$$

Clearly, $\{\tilde{x}(t)\}_{t=T_1}^{\tau_0+1}$ is a program. If $T_2 > \tau_0 + 1$, then it follows from (4.29) and (A3) that there exist $\tilde{x}(t) \in K$, $t = \tau_0 + 2, \dots, T_2$ such that $\{\tilde{x}(t)\}_{t=\tau_0+1}^{T_2}$ is a program,

$$\tilde{x}(t) \geq x(t),\ t = \tau_0 + 1, \dots, T_2, \tag{4.30}$$

$$u_t(\tilde{x}(t), \tilde{x}(t+1)) \geq u_t(x(t), x(t+1)),\ t = \tau_0 + 1, \dots, T_2 - 1. \tag{4.31}$$

By (4.9), (4.26), (4.13), (4.17), (4.31), (4.12), (4.19), (4.28), (4.8), (4.11) and (A2),

$$\begin{aligned}
M_1 &\geq U(x(T_1), T_1, T_2) - \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) \\
&\geq \sum_{t=T_1}^{T_2-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) \\
&= \sum_{t=\tau}^{T_2-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=\tau}^{T_2-1} u_t(x(t), x(t+1)) \\
&\geq \sum_{t=\tau}^{\tau_0} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=\tau}^{\tau_0} u_t(x(t), x(t+1)) \\
&\geq \sum_{t=\tau+L_1}^{\tau_0-L_1-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - (\tau_0 - \tau)\gamma - u_{\tau_0}(x(\tau_0), x(\tau_0+1)) \\
&\geq \sum_{t=\tau+L_1}^{\tau_0-L_1-1} u_t(\widehat{x}(t), \widehat{x}(t+1)) - (\tau_0 - \tau)\gamma - u_{\tau_0}(x(\tau_0), x(\tau_0+1)) \\
&\geq \widehat{\Delta}(\tau_0 - \tau - 2L_1) - (\tau_0 - \tau)\gamma - M_2 = (\widehat{\Delta} - \gamma)(\tau_0 - \tau - 2L_1) - 2L_1\gamma - M_2 \\
&\geq (\widehat{\Delta}/2)(\tau_0 - \tau - 2L_1) - 2L_1 - M_2 \\
&\geq (\widehat{\Delta}/2)(L_2 - 2L_1) - 2L_1 - M_2 \geq 4^{-1}L_2\widehat{\Delta} - 2L_1 - M_2
\end{aligned}$$

and

$$L_2 \leq 4(\widehat{\Delta})^{-1}(M_1 + M_2 + 2L_1).$$

This inequality contradicts (4.12). The contradiction we have reached proves (4.10). Lemma 4.2 is proved.

Lemma 4.3. *Let $M_1 > 0$. Then there exist natural numbers $\bar{L}_1, \bar{L}_2$ and $M_2 > 0$ such that for each pair of integers $\tau_1 \geq 0$, $\tau_2 \geq \bar{L}_1 + \bar{L}_2 + \tau_1$ and each program $\{x(t)\}_{t=\tau_1}^{\tau_2}$ which satisfies*

$$\sum_{t=\tau_1}^{\tau_2-1} u_t(x(t), x(t+1)) \geq U(x(\tau_1), \tau_1, \tau_2) - M_1 \tag{4.32}$$

the following assertion holds.

If integers $T_1, T_2 \in [\tau_1, \tau_2 - \bar{L}_2]$ *satisfy* $\bar{L}_1 \le T_2 - T_1$*, then*

$$\sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) \ge U(x(T_1), T_1, T_2) - M_2. \tag{4.33}$$

Proof. Let natural numbers $L_1, L_2 \ge 4$ be as guaranteed by Lemma 4.2. By Lemma 4.1 there exists a natural number $L_3 \ge 4$ such that the following property holds:

(P3) If integers $S_1 \ge 0$, $S_2 \ge L_3 + S_1$, if a program $\{v(t)\}_{t=S_1}^{S_2}$ satisfies

$$u_{S_2-1}(v(S_2-1), v(S_2)) \ge \gamma$$

and if $\tilde{v}_0 \in K$, then there exists a program $\{\tilde{v}(t)\}_{t=S_1}^{S_2}$ such that $\tilde{v}(S_1) = \tilde{v}_0$, $\tilde{v}(S_2) \ge v(S_2)$.

Choose a number M_0 such that

$$M_0 > u_t(z, z') \text{ for each integer } t \ge 0 \text{ and each } (z, z') \in \text{graph}(a_t), \tag{4.34}$$

natural numbers $\bar{L}_1, \bar{L}_2$ and positive number M_2 such that

$$\bar{L}_1 \ge L_1,\ \bar{L}_2 > 2(L_1 + L_2 + L_3 + 1), \tag{4.35}$$

$$M_2 > M_1 + M_0(L_3 + L_2). \tag{4.36}$$

Assume that integers $\tau_1 \ge 0$, $\tau_2 \ge \bar{L}_1 + \bar{L}_2 + \tau_1$, a program $\{x(t)\}_{t=\tau_1}^{\tau_2}$ which satisfies (4.32) and integers T_1, T_2 satisfy

$$T_1, T_2 \in [\tau_1, \tau_2 - \bar{L}_2],\ \bar{L}_1 \le T_2 - T_1. \tag{4.37}$$

We show that (4.33) is true. By Proposition 1.3 there exists a program $\{x^{(1)}(t)\}_{t=T_1}^{T_2}$ such that

$$x^{(1)}(T_1) = x(T_1),\ \sum_{t=T_1}^{T_2-1} u_t(x^{(1)}(t), x^{(1)}(t+1)) = U(x(T_1), T_1, T_2). \tag{4.38}$$

Relations (4.35) and (4.37) imply that

$$T_1 + L_1 \le T_1 + \bar{L}_1 + L_3 \le T_2 + L_3 \le \tau_2 - \bar{L}_2 + L_3 \le \tau_2 - 2L_2 - L_3. \tag{4.39}$$

It follows from the choice of L_1, L_2, Lemma 4.2, (4.32), (4.35) and (4.39) that

$$\max\{u_t(x(t), x(t+1)) :\ t = T_2 + L_3, \dots, T_2 + L_2 + L_3 - 1\} \ge \gamma.$$

Thus there exists an integer $\tau \in [T_2 + L_3, \dots, T_2 + L_3 + L_2 - 1]$ such that

$$u_\tau(x(\tau), x(\tau + 1)) \geq \gamma. \tag{4.41}$$

It follows from property (P3) and (4.41) that there exists a program $\{x^{(2)}(t)\}_{t=T_2}^{\tau+1}$ such that

$$x^{(2)}(T_2) = x^{(1)}(T_2),\ x^{(2)}(\tau + 1) \geq x(\tau + 1). \tag{4.42}$$

Set

$$\tilde{x}(t) = x(t),\ t = \tau_1, \dots, T_1,\ \tilde{x}(t) = x^{(1)}(t),\ t = T_1 + 1, \dots, T_2,$$

$$\tilde{x}(t) = x^{(2)}(t),\ t = T_2 + 1, \dots, \tau + 1. \tag{4.43}$$

It is clear that $\{\tilde{x}(t)\}_{t=\tau_1}^{\tau+1}$ is a program. In view of (4.42) and (4.43)

$$\tilde{x}(\tau + 1) \geq x(\tau + 1). \tag{4.44}$$

It follows from (4.44) and (A3) that there exist $\tilde{x}(t) \in K,\ t = \tau + 2, \dots, \tau_2$ such that $(\tilde{x}(t)\}_{t=\tau_1}^{\tau_2}$ is a program,

$$\tilde{x}(t) \geq x(t),\ t = \tau + 1, \dots, \tau_2, \tag{4.45}$$

$$u_t(\tilde{x}(t), \tilde{x}(t + 1)) \geq u_t(x(t), x(t + 1)),\ t = \tau + 1, \dots, \tau_2 - 1. \tag{4.46}$$

It follows from (4.32), (4.43), (4.46), (4.38). (4.34), (4.36) and the choice of $\bar{L}$ that

$$\begin{aligned}
&M_1 \geq U(x(\tau_1), \tau_1, \tau_2) - \sum_{t=\tau_1}^{\tau_2-1} u_t(x(t), x(t+1)) \\
&\geq \sum_{t=\tau_1}^{\tau_2-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=\tau_1}^{\tau_2-1} u_t(x(t), x(t+1)) \\
&= \sum_{t=T_1}^{\tau_2-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=T_1}^{\tau_2-1} u_t(x(t), x(t+1)) \\
&\geq \sum_{t=T_1}^{\tau} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=T_1}^{\tau} u_t(x(t), x(t+1)) \\
&\geq \sum_{t=T_1}^{T_2-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) - \sum_{t=T_2}^{\tau} u_t(x(t), x(t+1)) \\
&\geq U(x(T_1), T_1, T_2) - \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) - (\tau - T_2 + 1)M_0
\end{aligned}$$

and

$$\sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) \geq U(x(T_1), T_1, T_2) - M_1 - M_0(L_3 + L_2) >$$

$$U(x(T_1), T_1, T_2) - M_2.$$

Lemma 4.3 is proved.

5. Properties of the Function U

It is not difficult to see that the following proposition is true.

Proposition 5.1. *Let $\tau_1 \geq 0$, $\tau_1 > \tau_1$ be integers, $\Delta \geq 0$, T_1, T_2 be integers such that $\tau_1 \leq T_1 < T_2 \leq \tau_2$ and let $\{x(t)\}_{t=\tau_1}^{\tau_2}$ be a program satisfying*

$$\sum_{t=\tau_1}^{\tau_2-1} u_t(x(t), x(t+1)) \geq U(x(\tau_1), x(\tau_2), \tau_1, \tau_2) - \Delta.$$

Then

$$\sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) \geq U(x(T_1), x(T_2), T_1, T_2) - \Delta.$$

Lemma 5.1. *There exist a natural number L and $M_1 > 0$ such that for each $x_0, \tilde{x}_0 \in K$ and each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + L$ the following inequality holds:*

$$|U(x_0, T_1, T_2) - U(\tilde{x}_0, T_1, T_2)| \leq M_1.$$

Proof. Let natural numbers $L_1, L_2 \geq 4$ be as guaranteed by Lemma 4.2 with $M_1 = 1$. By Lemma 4.1 there exists an integer $L_3 \geq 4$ such that the following property holds:

(P4) If integers $S_1 \geq 0$, $S_2 \geq S_1 + L_3$, a program $\{v(t)\}_{t=S_1}^{S_2}$ satisfies

$$u_{S_2-1}(v(S_2 - 1), v(S_2)) \geq \gamma$$

and if $\tilde{v}_0 \in K$, then there exists a program $\{\tilde{v}(t)\}_{t=S_1}^{S_2}$ such that $\tilde{v}(S_1) = \tilde{v}_0$, $\tilde{v}(S_2) \geq v(S_2)$.

Choose a natural number

$$L > 2(L_1 + L_2 + L_3 + 1), \tag{5.1}$$

a number

$$M_0 > u_t(z,z'),\ t=0,1,\dots,\ (z,z')\in \text{graph}(a_t) \tag{5.2}$$

and put

$$M_1 = M_0(L_1+L_2+L_3). \tag{5.3}$$

Assume that $x_0, \tilde{x}_0 \in K$ and that integers $T_1 \geq 0$, $T_2 \geq T_1 + L$. By Proposition 1.3 there exists a program $\{x(t)\}_{t=T_1}^{T_2}$ such that

$$x(T_1) = x_0,\ \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) = U(x_0, T_1, T_2). \tag{5.4}$$

In view of (5.1)

$$T_1 + L_1 + L_3 < T_1 + L - L_2 \leq T_2 - L_2. \tag{5.5}$$

It follows from the choice of L_1, L_2, Lemma 4.2, (5.1) and (5.4) that

$$\max\{u_t(x(t), x(t+1)) :\ t = L_3 + L_1 + T_1, \dots, L_3 + L_1 + L_2 + T_1 - 1\} \geq \gamma.$$

Hence there is an integer

$$\tau \in \{T_1 + L_1 + L_3, \dots, T_1 + L_3 + L_1 + L_2 - 1\} \tag{5.6}$$

such that

$$u_\tau(x(\tau), x(\tau+1)) \geq \gamma. \tag{5.7}$$

It follows from the property (P4), the choice of L_3, (5.6) and (5.7) that there exists a program $\{\tilde{x}(t)\}_{t=T_1}^{\tau+1}$ such that

$$\tilde{x}(T_1) = \tilde{x}_0,\ \tilde{x}(\tau+1) \geq x(\tau+1). \tag{5.8}$$

By (5.8) and (A3) there exist $\tilde{x}(t) \in K$, $t = \tau+2, \dots, T_2$ such that $\{\tilde{x}(t)\}_{t=\tau+1}^{T_2}$ is a program,

$$\tilde{x}(t) \geq x(t),\ t = \tau+1, \dots, T_2, \tag{5.9}$$

$$u_t(\tilde{x}(t), \tilde{x}(t+1)) \geq u_t(x(t), x(t+1)),\ t = \tau+1, \dots, T_2 - 1. \tag{5.10}$$

Clearly, $\{\tilde{x}(t)\}_{t=T_1}^{T_2}$ is a program. By (5.2), (5.3), (5.4), (5.6) and (5.8),

$$
\begin{aligned}
&U(\tilde{x}_0, T_1, T_2) \geq \sum_{t=T_1}^{T_2-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) \\
&= \sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) - \left[\sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) - \sum_{t=T_1}^{T_2-1} u_t(\tilde{x}(t), \tilde{x}(t+1))\right] \\
&\geq U(x_0, T_1, T_2) - [\sum_{t=T_1}^{\tau} u_t(x(t), x(t+1)) - \sum_{t=T_1}^{\tau} u_t(\tilde{x}(t), \tilde{x}(t+1))] \\
&\geq U(x_0, T_1, T_2) - \sum_{t=T_1}^{\tau} u_t(x(t), x(t+1)) \geq U(x_0, T_1, T_2) - (\tau - T_1)M_0 \\
&\geq U(x_0, T_1, T_2) - (L_1 + L_2 + L_3)M_0 = U(x_0, T_1, T_2) - M_1.
\end{aligned}
$$

Thus we have shown that for each $x_0, \tilde{x}_0 \in K$ and each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + L$, $U(\tilde{x}_0, T_1, T_2) \geq U(x_0, T_1, T_2) - M_1$. This completes the proof of Lemma 5.1.

Corollary 5.1. *There exists $M_1 > 0$ and a natural number L such that for each pair of integers $T_1 \geq 0$, $T_2 \geq T_1 + L$ and each $x_0 \in K$, $|U(x_0, T_1, T_2) - \widehat{U}(T_1, T_2)| \leq M_1$.*

Lemmas 4.2 and 4.3 and Corollary 5.1 imply the following result.

Lemma 5.2. *Let $M_1 > 0$. Then thee exist natural numbers $\bar{L}_1, \bar{L}_2$ and $M_2 > 0$ such that for each pair of integers $\tau_1 \geq 0$, $\tau_2 \geq \tau_1 + \bar{L}_1 + \bar{L}_2$ and each program $\{x(t)\}_{t=\tau_1}^{\tau_2}$ which satisfies $\sum_{t=\tau_1}^{\tau_2-1} u_t(x(t), x(t+1)) \geq U(x(\tau_1), \tau_1, \tau_2) - M_1$ the following assertion holds:*

If integers $T_1, T_2 \in [\tau_1, \tau_2 - \bar{L}_2]$ satisfy $\bar{L}_1 \leq T_2 - T_1$, then

$$\sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) \geq \widehat{U}(T_1, T_2) - M_2.$$

6. Proof of Theorem 1.1

Let $M_1 = 1$ and let natural numbers $\bar{L}_1, \bar{L}_2$ and $M_2 > 0$ be as guaranteed by Lemma 5.2.

Let $x_0 \in K$. By Proposition 1.3 for each natural number k there exists a program $\{x^{(k)}(t)\}_{t=0}^{k}$ such that

$$x^{(k)}(0) = x_0, \sum_{t=0}^{k-1} u_t(x^{(k)}(t), x^{(k)}(t+1)) = U(x_0, 0, k). \tag{6.1}$$

It follows from the choice of $\bar{L}_1, \bar{L}_2, M_2$ and Lemma 5.2 that the following property holds:

(i) For each integer $k \geq \bar{L}_1 + \bar{L}_2$ and each pair of integers $T_1, T_2 \in [0, k - \bar{L}_2]$ satisfying $\bar{L}_1 \leq T_2 - T_1$, $\sum_{t=T_1}^{T_2-1} u_t(x^{(k)}(t), x^{(k)}(t+1)) \geq \widehat{U}(T_1, T_2) - M_2$.

Clearly, there exists a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ such that for each integer $t \geq 0$ there exists

$$\bar{x}(t) = \lim_{j \to \infty} x^{(k_j)}(t). \tag{6.2}$$

Evidently, $\{\bar{x}(t)\}_{t=0}^{\infty}$ is a program. In view of (6.1) and (6.2),

$$\bar{x}(0) = x_0. \tag{6.3}$$

It follows from (6.2), the property (i) and upper semicontinuity of the functions u_t, $t - 0, 1, \ldots$ that the following property holds:

(ii) for each pair of integers $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \geq \bar{L}_1$,

$$|\sum_{t=T_1}^{T_2-1} u_t(\bar{x}(t), \bar{x}(t+1)) - \widehat{U}(T_1, T_2)| \leq M_2. \tag{6.4}$$

Choose a positive number M_0 such that

$$M_0 > u_t(z, z') \text{ for each integer } t \geq 0 \text{ and each } (z, z') \in \text{graph}(a_t). \tag{6.5}$$

Set

$$M = M_2 + M_0 \bar{L}_1. \tag{6.6}$$

Assume that nonnegative integers T_1, T_2 satisfy $T_1 < T_2$. If $T_2 - T_1 \geq \bar{L}_1$, then by property (ii), (6.4) and (6.6),

$$|\sum_{t=T_1}^{T_2-1} u_t(\bar{x}(t), \bar{x}(t+1)) - \widehat{U}(T_1, T_2)| \leq M_2 \leq M.$$

If $T_2 - T_1 \leq \bar{L}_1$, then by (6.5) and (6.6)

$$|\sum_{t=T_1}^{T_2-1} u_t(\bar{x}(t), \bar{x}(t+1)) - \widehat{U}(T_1, T_2)| \leq (T_2 - T_1) M_0 \leq M_0 \bar{L}_1 < M.$$

Thus in the both cases

$$|\sum_{t=T_1}^{T_2-1} u_t(\bar{x}(t), \bar{x}(t+1)) - \widehat{U}(T_1, T_2)| \leq M. \tag{6.7}$$

Assume now that the following properties hold:

(iii) for each integer $t \geq 0$ and each $(z, z') \in \text{graph}(a_t)$ satisfying $u_t(z, z') > 0$ the function u_t is continuous at (z, z');

(iv) if an integer $t \geq 0$ and $z,\ z_1,\ z_2,\ z_3 \in K$ satisfy $z_i \in a_t(z),\ i = 1, 3$ and $z_1 \leq z_2 \leq z_3$, then $z_2 \in a_t(z)$.

In order to complete the proof of the theorem it is sufficient to show that for each integer $T > 0$,

$$\sum_{t=0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1)) = U(x(0), x(T), 0, T). \tag{6.8}$$

Denote by E the set of all natural numbers τ such that

$$u_{\tau-1}(\bar{x}(\tau-1), \bar{x}(\tau)) > 0. \tag{6.9}$$

By (A2) and (6.7) the set E is infinite. In view of Proposition 5.1 it is sufficient to show that (6.8) holds for all $T = \tau - 1$, where $\tau \in E$.

Let $\tau \in E$ and $T = \tau - 1$. We show that (6.8) is valid. Let us assume the contrary. Then there exist a program $\{x(t)\}_{t=0}^{T}$ and a positive number Δ such that

$$x(0) = \bar{x}(0),\ x(T) \geq \bar{x}(T), \tag{6.10}$$

$$\sum_{t=0}^{T-1} u_t(x(t), x(t+1)) \geq \sum_{t=0}^{T-1} u_t(\bar{x}(t), \bar{x}(t+1)) + 2\Delta. \tag{6.11}$$

By the inclusion $\tau \in E$ and the definition of E,

$$u_T(\bar{x}(T), \bar{x}(T+1)) = u_{\tau-1}(\bar{x}(\tau-1), \bar{x}(\tau)) > 0. \tag{6.12}$$

In view of (6.12) and (A1) there is a number $\lambda_0 \in (0, 1)$,

$$z_0 \in a_{\tau-1}(\bar{x}(\tau-1)) = a_T(\bar{x}(T)) \tag{6.13}$$

such that

$$z_0 \geq \bar{x}(\tau) + \lambda_0 e = \bar{x}(T+1) + \lambda_0 e. \tag{6.14}$$

There is $c_0 > 1$ such that

$$||y|| \leq c_0||y||_2 \leq c_0^2||y|| \text{ for all } y \in R^n. \tag{6.15}$$

By (6.12), (6.14) and properties (iii) and (iv) we may assume without loss of generality that

$$|u_{\tau-1}(\bar{x}(\tau-1), z_0) - u_{\tau-1}(\bar{x}(\tau-1), \bar{x}(\tau))| \leq \Delta/4. \tag{6.16}$$

It follows from (A3), (6.10) and (6.13) that there is $z_1 \in a_T(x(T))$ such that

$$z_1 \geq z_0, \; u_T(x(T), z_1) \geq u_T(\bar{x}(T), z_0). \tag{6.17}$$

Choose a positive number

$$\delta < \min\{1, \lambda_0, \Delta\tau^{-1}\}. \tag{6.18}$$

By the construction of the program $\{\bar{x}(t)\}_{t=0}^{\infty}$ (see (6.2)) and upper semicontinuity of u_t, $t = 0, 1, \dots$ there is a natural number $k > \tau + 4$ such that

$$||x^{(k)}(t) - \bar{x}(t)||_2 \leq \delta, \; t = 0, \dots, \tau + 2, \tag{6.19}$$

$$u_t(x^{(k)}(t), x^{(k)}(t+1)) \leq u_t(\bar{x}(t), \bar{x}(t+1)) + \delta, \; t = 0, \dots, \tau + 2. \tag{6.20}$$

Set

$$\tilde{x}(t) = x(t), \; t = 0, \dots, \tau - 1. \tag{6.21}$$

We show that $z_1 \geq x^{(k)}(\tau)$. By (6.19),

$$||x^{(k)}(\tau) - \bar{x}(\tau)||_2 \leq \delta. \tag{6.22}$$

In view of (6.18), (6.22), (6.14) and (6.17),

$$x^{(k)}(\tau) \leq \bar{x}(\tau) + \delta e \leq \bar{x}(\tau) + \lambda_0 e \leq z_0 \leq z_1. \tag{6.23}$$

Set

$$\tilde{x}(\tau) = z_1. \tag{6.24}$$

Since $z_1 \in a_T(x_T) = a_{\tau-1}(\tilde{x}_{\tau-1})$, $\{\tilde{x}(t)\}_{t=0}^{\tau}$ is a program. By (6.21), (6.10), (6.3), (6.23), and (6.24),

$$\tilde{x}(0) = \bar{x}(0) = x_0, \; \tilde{x}(\tau) \geq x^{(k)}(\tau). \tag{6.25}$$

In view of (6.21), (6.11), equality $T = \tau - 1$, (6.24), (6.17) and (6.16),

$$\begin{aligned}
&\sum_{t=0}^{\tau-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=0}^{\tau-1} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&\geq \sum_{t=0}^{\tau-2} u_t(x(t), x(t+1)) + u_{\tau-1}(\tilde{x}(\tau-1), \tilde{x}(\tau)) - \sum_{t=0}^{\tau-1} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&\geq \sum_{t=0}^{\tau-2} u_t(\bar{x}(t), \bar{x}(t+1)) + 2\Delta + u_{\tau-1}(x(\tau-1), z_1) - \sum_{t=0}^{\tau-1} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&\geq 2\Delta + \sum_{t=0}^{\tau-2} u_t(\bar{x}(t), \bar{x}(t+1)) + u_{\tau-1}(\bar{x}(\tau-1), z_0) - \sum_{t=0}^{\tau-1} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&\geq 2\Delta + u_{\tau-1}(\bar{x}(\tau-1), z_0) - u_{\tau-1}(\bar{x}(\tau-1), \bar{x}(\tau)) \geq (3/2)\Delta.
\end{aligned} \tag{6.26}$$

Relations (6.18), (6.20) and (6.26) imply that

$$\begin{aligned}
&\sum_{t=0}^{\tau-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=0}^{\tau-1} u_t(x^{(k)}(t), x^{(k)}(t+1)) \\
&= \sum_{t=0}^{\tau-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - \sum_{t=0}^{\tau-1} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&\quad + \sum_{t=0}^{\tau-1} u_t(\bar{x}(t), \bar{x}(t+1)) - \sum_{t=0}^{\tau-1} u_t(x^{(k)}(t), x^{(k)}(t+1)) \\
&\geq (3/2)\Delta - \delta\tau \geq \Delta/2.
\end{aligned} \tag{6.27}$$

By (6.25) and (6.27),

$$U(x_0, x^{(k)}(\tau), 0, \tau) \geq \sum_{t=0}^{\tau-1} u_t(x^{(k)}(t), x^{(k)}(t+1)) + \Delta/2.$$

This inequality contradicts (6.1). The contradiction we have reached proves that (6.8) is valid for all $T = \tau - 1$ where $\tau \in E$. This completes the proof of Theorem 1.1.

7. Proof of Theorem 1.2

In the sequel we assume that the sum over empty set is zero. There exist $\Delta > 0$ and a strictly increasing sequence of natural numbers $\{\tau_i\}_{i=1}^{\infty}$ such that $\tau_1 \geq 4$ and

$$u_{\tau_i - 1}(x(\tau_{i-1}), x(\tau_i)) \geq \Delta \text{ for all integers } i \geq 1. \tag{7.1}$$

Let $M > 0$ be as guaranteed by Theorem 1.1. By Lemma 4.1 there exists a natural number $L_0 \geq 4$ such that the following property holds:

(P5) For each integer $S_1 \geq 0$, each integer $S_2 \geq S_1 + L_0$, each program $\{v(t)\}_{t=S_1}^{S_2}$ which satisfies $u_{S_2-1}(v(S_2-1), v(S_2)) \geq \Delta$ and each $\tilde{v}_0 \in K$ there exists a program $\{\tilde{v}(t)\}_{t=S_1}^{S_2}$ such that $\tilde{v}(S_1) = \tilde{v}_0,\ \tilde{v}(S_2) \geq v(S_2)$.

By Corollary 5.1 and (1.3) there exists $M_* > 0$ such that

$$|U(v_0, T_1, T_2) - \widehat{U}(T_1, T_2)| \leq M_* \text{ for each } v_0 \in K \text{ and each pair of integers } T_1 < T_2, \tag{7.2}$$

$$u_t(z, z') \leq M_* \text{ for each integer } t \geq 0, \text{ and each } (z, z') \in \mathrm{graph}(a_t). \tag{7.3}$$

Choose a positive number

$$M_1 > L_0 M_* + M_0 + 3M. \tag{7.4}$$

By Theorem 1.1 there exists a program $\{\bar{x}(t)\}_{t=0}^{\infty}$ such that

$$\bar{x}(0) = x(0) \tag{7.5}$$

and that for each pair of integers S_1, S_2 satisfying $S_1 < S_2$,

$$|\sum_{t=S_1}^{S_2-1} u_t(\bar{x}(t), \bar{x}(t+1)) - \widehat{U}(S_1, S_2)| \le M. \tag{7.6}$$

Assume that T_1, T_2 are integers such that $0 \le T_1 < T_2$. We show that

$$|\sum_{t=T_1}^{T_2-1} u_t(x(t), x(t+1)) - \widehat{U}(T_1, T_2)| \le M_1. \tag{7.7}$$

If $T_2 \le T_1 + L_0$, then this inequality follows from (7.3) and (7.4).

Assume that $T_2 > T_1 + L_0$. There exists an integer $i \ge 1$ such that

$$\tau_i > T_2 + 2L_0. \tag{7.8}$$

It follows from (7.1), (7.8) and (P5) that there exists a program $\{\tilde{x}(t)\}_{t=\tau_i-L_0}^{\tau_i}$ such that

$$\tilde{x}(\tau_i - L_0) = \bar{x}(\tau_i - L_0),\ \tilde{x}(\tau_i) \ge x(\tau_i). \tag{7.9}$$

Set

$$\tilde{x}(t) = \bar{x}(t),\ t = 0, \dots, \tau_i - L_0 - 1. \tag{7.10}$$

Clearly, $\{\tilde{x}(t)\}_{t=0}^{\tau_i}$ is a program and in view of (7.9),

$$\sum_{t=0}^{\tau_i-1} u_t(x(t), x(t+1)) \ge \sum_{t=0}^{\tau_i-1} u_t(\tilde{x}(t), \tilde{x}(t+1)) - M_0. \tag{7.11}$$

It follows from (7.11) and (7.3) that

$$\sum_{t=0}^{\tau_i-1} u_t(x(t), x(t+1)) \ge \sum_{t=0}^{\tau_i-L_0-1} u_t(\bar{x}(t), \bar{x}(t+1)) - M_0$$
$$\ge \sum_{t=0}^{\tau_i-1} u_t(\bar{x}(t), \bar{x}(t+1)) - M_0 - L_0 M_*.$$

Combined with (7.6) this implies that

$$
\begin{aligned}
&-(M_0 + L_0 M_*) \le \sum_{t=0}^{\tau_i - 1} u_t(x(t), x(t+1)) - \sum_{t=0}^{\tau_i - 1} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&\le \sum \{u_t(x(t), x(t+1)) :\ 0 \le t < T_1\} - \sum \{u_t(\bar{x}(t), \bar{x}(t+1)) :\ 0 \le t < T_1\} \\
&\quad + \sum_{t=T_1}^{T_2 - 1} u_t(x(t), x(t+1)) - \sum_{t=T_1}^{T_2 - 1} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&\quad + \sum_{t=T_2}^{\tau_i - 1} u_t(x(t), x(t+1)) - \sum_{t=T_2}^{\tau_i - 1} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&\le M + \sum_{t=T_1}^{T_2 - 1} u_t(x(t), x(t+1)) - (\widehat{U}(T_1, T_2) - M) + \widehat{U}(T_2, \tau_i) \\
&\quad - \sum_{t=T_2}^{\tau_i} u_t(\bar{x}(t), \bar{x}(t+1)) \\
&\le \sum_{t=T_1}^{T_2 - 1} u_t(x(t), x(t+1)) - \widehat{U}(T_1, T_2) + 3M
\end{aligned}
$$

and together with (7.4) this implies that

$$
\sum_{t=T_1}^{T_2 - 1} u_t(x(t), x(t+1)) - \widehat{U}(T_1, T_2) \ge -3M - (M_0 + L_0 M_*) > -M_1.
$$

Theorem 1.2 is proved.

8. Proof of Theorem 1.3

Let $x_0 \in K$ and let $\{\bar{x}(t)\}_{t=0}^{\infty}$ be as guaranteed by Theorem 1.1. Then for each pair of integers $T_1, T_2 \ge 0$ satisfying $T_1 < T_2$,

$$
|\sum_{t=T_1}^{T_2 - 1} u_t(\bar{x}(t), \bar{x}(t+1)) - \widehat{U}(T_1, T_2)| \le M. \tag{8.1}
$$

Choose $\Delta > 0$ such that

$$
\Delta > u(z, z') \text{ for each } (z, z') \in \text{graph}(a). \tag{8.2}
$$

Let p be a natural number. We show that for all sufficiently large natural numbers T,

$$
|p^{-1} \widehat{U}(0, p) - T^{-1} \sum_{t=0}^{T-1} u(\bar{x}(t), \bar{x}(t+1))| \le 2M/p. \tag{8.3}
$$

Assume that $T \geq p$ is a natural number. Then there exist integers q, s such that

$$q \geq 1,\ 0 \leq s < p,\ T = pq + s. \tag{8.4}$$

It follows from (8.4) that

$$\begin{aligned}
&T^{-1}\sum_{t=0}^{T-1} u(\bar{x}(t),\bar{x}(t+1)) - p^{-1}\widehat{U}(0,p) = T^{-1}(\sum_{t=0}^{pq-1} u(\bar{x}(t),\bar{x}(t+1)) \\
&+\sum\{u(\bar{x}(t),\bar{x}(t+1)) :\ t \text{ is an integer such that} \\
&pq \leq t \leq T-1\}) - p^{-1}\widehat{U}(0,p) \\
=\ &T^{-1}\sum\{u(\bar{x}(t),\bar{x}(t+1)) :\ t \text{ is an integer such that} \\
&\quad pq \leq t \leq T-1\} \\
&+(T^{-1}pq)(pq)^{-1}\sum_{i=0}^{q-1}\sum_{t=ip}^{(i+1)p-1} u(\bar{x}(t),\bar{x}(t+1)) - p^{-1}\widehat{U}(0,p) \\
=\ &(T^{-1}pq)(pq)^{-1}[\sum_{i=0}^{q-1}(\sum_{t=ip}^{(i+1)p-1} u(\bar{x}(t),\bar{x}(t+1)) \\
&-\widehat{U}(0,p)) + q\widehat{U}(0,p)] - p^{-1}\widehat{U}(0,p) \\
&+T^{-1}\{\sum u(\bar{x}(t),\bar{x}(t+1)) :\ t \text{ is an integer such that} \\
&\quad pq \leq t \leq T-1\}.
\end{aligned} \tag{8.5}$$

By (8.1), (8.2), (8.4) and (8.5),

$$\begin{aligned}
&|T^{-1}\sum_{t=0}^{T-1} u(\bar{x}(t),\bar{x}(t+1)) - p^{-1}\widehat{U}(0,p)| \\
&\leq T^{-1}p\Delta + (pq)^{-1}qM + \widehat{U}(0,p)|q/T - 1/p| \\
&\leq T^{-1}p\Delta + M/p + \widehat{U}(0,p)s(pT)^{-1} \to M/p \text{ as } T \to \infty.
\end{aligned}$$

Since p is any natural number we conclude that $T^{-1}\sum_{t=0}^{T-1} u(\bar{x}(t),\bar{x}(t+1))\}_{T=1}^{\infty}$ is a Cauchy sequence. Clearly, there exists $\lim_{T\to\infty} T^{-1}\sum_{t=0}^{T-1} u(\bar{x}(t),\bar{x}(t+1))$ and that for each natural number p,

$$|p^{-1}\widehat{U}(0,p) - \lim_{T\to\infty} T^{-1}\sum_{t=0}^{T-1} u(\bar{x}(t),\bar{x}(t+1))| \leq 2M/p. \tag{8.6}$$

Since (8.6) is true for any natural number p we obtain that

$$\lim_{T\to\infty} T^{-1}\sum_{t=0}^{T-1} u(\bar{x}(t),\bar{x}(t+1)) = \lim_{p\to\infty} \widehat{U}(0,p)/p. \tag{8.7}$$

Set

$$\mu = \lim_{p \to \infty} \widehat{U}(0,p)/p. \tag{8.8}$$

By (8.6)-(8.8), for all natural numbers p, $|p^{-1}\widehat{U}(0,p) - \mu| \leq 2M/p$. Theorem 1.3 is proved.

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