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# AN INTERPOLATION THEOREM RELATED TO THE HARDY SPACE WITH NON-DOUBLING MEASURE

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**Abstract.** Let $\mu$ be a nonnegative Radon measure satisfying the growth condition that $\mu(B(x, r)) \leq Cr^n$ for any $x \in \mathbb{R}^d$ and $r > 0$ and some fixed positive constants $C$ and $n$ with $0 < n \leq d$. Let $H^{1,\infty}_{\mathrm{atb}}(\mu)$ be the Hardy space associated with $\mu$ which was introduced by Tolsa. In this paper, a new interpolation theorems related to $H^{1,\infty}_{\mathrm{atb}}(\mu)$ is established and the interpolation theorem of Tolsa is improved.

## 1. Introduction

During the last decade, considerable attention has been paid to the study of function spaces and boundedness of operators on these space (see [1-9]). Let $\mu$ be a nonnegative Radon measure on $\mathbb{R}^d$ which only satisfies the following growth condition: there exist constants $C_0>0$ and $n\in(0,d]$ such that for all $x\in\mathbb{R}^d$ and $r>0$,

$$\mu\big(B(x,r)\big) \leq C_0 r^n, \tag{1.1}$$

where $B(x,r)$ is the open ball centered at some point $x \in \mathbb{R}^d$ and having radius $r$. The measure $\mu$ in (1.1) is not assumed to satisfy the doubling condition which is a key assumption in the analysis on spaces of homogeneous type. We recall that $\mu$ is said to satisfy the doubling condition if there exists some constant $C > 0$ such that $\mu(B(x,2r)) \leq C\mu(B(x,r))$ for all $x \in \mathbb{R}^d$ and $r > 0$. Some important non-doubling measures as in (1.1) and the motivation for developing the analysis related to such measures can be found in [9], see also [4]. We only point out

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that the analysis with non-doubling measures plays an essential role in solving the long-standing open Painlevé's problem by Tolsa in [8].

In his remarkable work [6], Tolsa found a suitable substitute for the classical BMO space when the underlying measure satisfies (1.1), RBMO$(\mu)$. The space $\mathrm{RBMO}(\mu)$ enjoys the properties which are parallel to those of the space $\mathrm{BMO}(\mathbb{R}^d)$, for example, $\mathrm{RBMO}(\mu)$ is big enough so that a $L^2(\mu)$ bounded Calderón-Zygmund operator is also bounded from $L^\infty(\mu)$ to $\mathrm{RBMO}(\mu)$, and small enough to satisfy the properties (such as John-Nirenberg inequality) of the classical BMO space. Also, Tolsa established the following interpolation theorem (see [6, p. 131]).

**Theorem 1.** *Let $T$ be a linear operator which is bounded from $H^{1,\infty}_{\mathrm{atb}}(\mu)$ to $L^1(\mu)$, and bounded from $L^\infty(\mu)$ to $\mathrm{RBMO}(\mu)$. Then for any $p \in (1, \infty)$, $T$ extends boundedly to $L^p(\mu)$, where $H^{1,\infty}_{\mathrm{atb}}(\mu)$ is the atomic Hardy space with the measure $\mu$ in (1.1), see Definition 1 below.*

The main purpose of this paper is to establish a new interpolation theorem related to $H^{1,\infty}_{\mathrm{atb}}(\mu)$ which improves Tolsa's interpolation theorem above. To states our main results, we first give some definitions and notation.

By a cube $Q \subset \mathbb{R}^d$ we mean a closed cube with sides parallel to the axes and centered at some point of $\operatorname{supp}\mu$. We denote its side length by $l(Q)$. Given $\alpha > 1$ and $\beta > \alpha^n$, we say that $Q$ is $(\alpha, \beta)$-doubling if $\mu(\alpha Q) \le \beta\mu(Q)$, where $\alpha Q$ is the cube concentric with $Q$ with side length $\alpha l(Q)$. It was pointed by Tolsa in [5] that there are a lot of "big" doubling cubes. To be precise, given any point $x \in \operatorname{supp}(\mu)$ and $c > 0$, there exists some $(\alpha, \beta)$-doubling cube $Q$ centered at $x$ with $l(Q) \ge c$ due to the growth condition (1.1). On the other hand, if $\beta > \alpha^d$, then for $\mu - a.\,e.\ x \in \mathbb{R}^d$, there exists a sequence of $(\alpha, \beta)$-doubling cubes $\{Q_i\}_{i\in\mathbb{N}}$ centered at $x$ with $l(Q_i) \to 0$ as $i \to \infty$. In what follows, for definiteness, if $\alpha$ and $\beta$ are not specified, by a doubling cube we mean $(2, 2^{d+1})$-doubling cube. Given two cubes $Q_1 \subset Q_2$, set

$$K_{Q_1,Q_2} = 1 + \sum_{k=1}^{N_{Q_1,Q_2}} \frac{\mu(2^kQ_1)}{[l(2^kQ_1)]^n},$$

where $N_{Q_1,Q_2}$ is the first positive integer $k$ such that $l(2^kQ_1) \ge l(Q_2)$; see [6] for some basic properties of $K_{Q_1,Q_2}$.

Given a cube $Q \subset \mathbb{R}^d$, let $\widetilde{Q}$ be the smallest doubling cube in the sequence $\{2^kQ\}_{k\ge 0}$, and by $m_Q(f)$ the mean value of $f$ on $Q$, namely,

$$m_Q(f) = \frac{1}{\mu(Q)} \int_Q f(x)\, d\mu(x).$$

The sharp maximal operator associated with the measure $\mu$ in (1.1) is defined by

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{\mu(\frac{3}{2}Q)} \int_Q |f(y) - m_{\widetilde{Q}} f| d\mu(y) + \sup_{\substack{R \supset Q \ni x \\ Q,\, R \text{ doubling}}} \frac{|m_Q f - m_R f|}{K_{Q,R}}.$$

**Definition 1.** Let $\rho > 1$ and $1 < p \leq \infty$. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a $p$-atomic block if

(1) there exists some cube $R$ such that $\operatorname{supp} b \subset R$,

(2) $\displaystyle\int_{\mathbb{R}^d} b(x)\, d\mu(x) = 0$,

(3) for $j = 1,\, 2$, there are functions $a_j$ supported on cubes $Q_j \subset R$ and numbers $\lambda_j \in \mathbb{R}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_j\|_{L^p(\mu)} \leq \left[\mu(\rho Q_j)^{1-1/p} K_{Q_j,\, R}\right]^{-1}.$$

Then we define

$$|b|_{H^{1,\,p}_{atb}(\mu)} = |\lambda_1| + |\lambda_2|.$$

We say that $f \in H^{1,p}_{atb}(\mu)$ if there are $p$-atomic blocks $\{b_i\}_{i \in \mathbb{N}}$ such that

$$f = \sum_{i=1}^{\infty} b_i$$

with $\sum_{i=1}^{\infty} |b_i|_{H^{1,\,p}_{atb}(\mu)} < \infty$. The $H^{1,p}_{atb}(\mu)$ norm of $f$ is defined by

$$\|f\|_{H^{1,\,p}_{atb}(\mu)} = \inf \left\{ \sum_i |b_i|_{H^{1,\,p}_{atb}(\mu)} \right\},$$

where the infimum is taken over all the possible decompositions of $f$ in atomic blocks.

The space $H^{1,\infty}_{\mathrm{atb}}(\mu)$ was introduced by Tolsa in [6], and further considered by Tolsa in [7]. Moreover, it was proved by Tolsa in [6, 7] that the definition of $H^{1,p}_{atb}(\mu)$ is independent of the chosen constant $\rho > 1$. Moreover, for any $p \in (1,\, \infty)$,

$$H^{1,p}_{\mathrm{atb}}(\mu) = H^{1,\infty}_{\mathrm{atb}}(\mu)$$

with equivalent norms.

Our main result can be stated as follows.

**Theorem 2.** *Let $T$ be an operator which satisfies that*

(*i*) $|Tf_1 - Tf_2| \leq |T(f_1 - f_2)|$;

($ii$) *there is another operator $T_1$, which is bounded from $L^{p_0}(\mu)$ to $L^{q_0,\infty}(\mu)$ for some $p_0$, $q_0$ with $p_0 \le q_0$ and $p_0,\, q_0 \in (1,\, \infty]$ such that for any bounded function $f$ with compact support,*

$$M^\sharp(Tf)(x) \le CT_1f(x);$$

($iii$) *for some $q_1 \in [1,\, \infty)$, $T$ is bounded from $H^{1,\infty}_{\rm atb}(\mu)$ to $L^{q_1,\,\infty}(\mu)$, that is, there is a constant $C > 0$, such that for any $\lambda > 0$ and any $f \in H^{1,\infty}_{\rm atb}(\mu)$,*

$$\mu(\{x \in \mathbb{R}^d :\, |Tf(x)| > \lambda\}) \le C\Big(\lambda^{-1}\|f\|_{H^{1,\,\infty}_{\rm atb}(\mu)}\Big)^{q_1}.$$

*Then for any $p,\, q \in (1,\, \infty)$ with*

$$\frac{1}{p} = t + \frac{1-t}{p_0},\ \ \frac{1}{q} = \frac{t}{q_1} + \frac{1-t}{q_0},\quad t \in (0,\, 1),$$

*$T$ is bounded from $L^p(\mu)$ to $L^q(\mu)$.*

## 2. Proof of Theorem 2

We begin with the John-Strömberg sharp maximal operator with a measure in (1.1), which was introduced in [1]. For a cube $Q$ with $\mu(Q) \ne 0$, and a real-valued locally integrable function $f$, $m_f(Q)$, the median value of $f$ on the cube $Q$, is defined to be one of numbers such that

$$\mu(\{y \in Q :\, f(y) > m_f(Q)\}) \le \frac{1}{2}\mu(Q)$$

and

$$\mu(\{y \in Q :\, f(y) < m_f(Q)\}) \le \frac{1}{2}\mu(Q).$$

For the case that $\mu(Q) = 0$, we set $m_f(Q) = 0$ for any real-valued locally integrable function $f$. If $f$ is complex-valued, the median value of $f$ is defined by $m_f(Q) = m_{{\rm Re}(f)}(Q) + im_{{\rm Im}(f)}(Q)$, where $i^2 = -1$.

Let $s \in (0,\, 2^{-d-2})$. For each fixed cube $Q$ and a locally integrable function $f$, define $m_{0,\,s;\,Q}(f)$ by

$$m_{0,\,s;\,Q}(f) = \inf\Big\{t > 0 :\, \mu\big(\{y \in Q :\, |f(y)| > t\}\big) < s\mu\Big(\tfrac{3}{2}Q\Big)\Big\} \quad \text{when } \mu\big(Q\big) \ne 0, \tag{2.1}$$

and $m_{0,\,s;\,Q}(f) = 0$ when $\mu(Q) = 0$. The John-Strömberg maximal operator$M_{0,\,s}$, and the doubling John-Strömberg maximal operator$M^d_{0,\,s}$, associated with measure in (1.1) are defined by

$$M_{0,s}f(x) = \sup_{Q \ni x} m_{0,s;Q}(f), \qquad M^d_{0,s}f(x) = \sup_{\substack{Q \ni x \\ Q \text{ doubling}}} m_{0,s;Q}(f), \tag{2.2}$$

and the John-Strömberg sharp maximal operator $M^\sharp_{0,s}$ is defined by

$$M^\sharp_{0,s}f(x) = \sup_{Q \ni x} m_{0,s;Q}\big(f - m_f(\widetilde{Q})\big) + \sup_{\substack{R \supset Q \ni x \\ Q,R \text{ doubling}}} \frac{|m_f(Q) - m_f(R)|}{K_{Q,R}}. \tag{2.3}$$

We then have

**Lemma 1.** *Let $0 < s < 2^{-d-2}$. Then for any locally integrable function $f$ and any $\lambda > 0$,*

(*i*) $\{x \in \mathbb{R}^d : |f(x)| > \lambda\} \subset \{x \in \mathbb{R}^d : M^d_{0,s}f(x) \geq \lambda\} \cup \Theta$ *with* $\mu(\Theta) = 0$;

(*ii*)

$$\mu\big(\{x \in \mathbb{R}^d : M_{0,s}f(x) > \lambda\}\big) \leq Cs^{-1}\mu\big(\{x \in \mathbb{R}^d : |f(x)| > \lambda\}\big),$$

*where $C > 0$ is a constant depending on $d$.*

*Proof.* This lemma was essentially proved in [1]. For the sake of self-contained, we present the proof here. Let $M^d$ be the maximal operator defined by

$$M^d f(x) = \sup_{\substack{Q \ni x \\ Q \text{ doubling}}} \frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y).$$

By the Lebesgue differential lemma, we know that for $\mu$ almost $x \in \mathbb{R}^d$,

$$|f(x)| \leq M^d f(x)$$

and so

$$\begin{aligned}\{x \in \mathbb{R}^d : |f(x)| > \lambda\} &= \{x \in \mathbb{R}^d : \chi_{\{y \in \mathbb{R}^d : |f(y)| > \lambda\}}(x) = 1\} \\ &\subset \Big\{x \in \mathbb{R}^d : M^d\Big(\chi_{\{y \in \mathbb{R}^d : |f(y)| > \lambda\}}\Big)(x) > s2^{d+1}\Big\} \cup \Theta.\end{aligned}$$

On the other hand, a straightforward computation leads to that

$$\Big\{x \in \mathbb{R}^d : M^d\Big(\chi_{\{y \in \mathbb{R}^d : |f(y)| > \lambda\}}\Big)(x) > s2^{d+1}\Big\} \subset \{x \in \mathbb{R}^d : M^d_{0,s}f(x) \geq \lambda\}.$$

The conclusion (i) then follows directly.

To prove (ii), for each fixed $\lambda > 0$ and $r > 0$, set

$$M^r_{0,s}f(x) = \sup_{Q \ni x,\, l(Q) < r} m_{0,s;Q}(f)$$

and

$$E_{r,\lambda} = \{x \in \mathbb{R}^d : M^r_{0,s}f(x) > \lambda\}.$$

For any fixed $x \in E_{r,\lambda}$, there is a cube $Q_x$ containing $x$ and $l(Q_x) < r$, such that

$$\mu(\{y \in Q_x : |f(y)| > \lambda\}) \geq s\mu(\frac{3}{2}Q_x).$$

Applying the Besicovitch covering lemma, we can select $N$ family of cubes $\{Q_j^k\}_{1\leq j\leq N,\, k\in\Lambda_j}$ from $\{Q_x\}_{x\in E_{r,\lambda}}$, such that

(a)

$$E_{r,\lambda} \subset \bigcup_{j=1}^{N} \bigcup_{k\in\Lambda_j} \frac{3}{2}Q_j^k;$$

(b) there is a constant $C > 0$ such that for any fixed $j$, $1 \leq j \leq k$,

$$\sum_{k\in\Lambda_j} \chi_{Q_j^k} \leq C,$$

where $N$ is the Besicovitch constant. It then follows that

$$\begin{aligned} \mu(E_{r,\lambda}) &\leq \sum_{j=1}^{N} \sum_{k\in\Lambda_j} \mu(\frac{3}{2}Q_j^k) \\ &\leq s^{-1} \sum_{j=1}^{N} \sum_{k\in\Lambda_j} \mu(\{y \in Q_j^k : |f(y)| > \lambda\}) \\ &\leq Cs^{-1}\mu(\{y \in \mathbb{R}^d : |f(y)| > \lambda\}). \end{aligned}$$

Letting $r \to \infty$ then leads to our desired conclusion.

To prove Theorem 2, we also need some preliminary lemmas.

**Lemma 2.** *Let $s \in (0, 2^{-d-2})$ and $T$ be an operator which satisfies that*

$$|Tf_1(x) - Tf_2(x)| \leq |T(f_1 - f_2)(x)|.$$

*There is a constant $C > 0$ such that for any $f_1$ and $f_2$,*

$$M^\sharp_{0,s}[T(f_1 + f_2)](x) \leq CM^\sharp(Tf_1)(x) + CM_{0,s/2}(Tf_2)(x). \tag{2.4}$$

*Proof.* For any cube $Q$, a straightforward computation yields

$$\begin{aligned} m_{0,\,s;\,Q}\Big(T(f_1+f_2)-m_{T(f_1+f_2)}(\widetilde{Q})\Big) \;\le\; & m_{0,\,s/2;\,Q}\Big(Tf_1-m_{Tf_1}(\widetilde{Q})\Big) \\ & +m_{0,\,s/2;\,Q}\Big(T(f_1+f_2)-Tf_1\Big) \\ & +\Big|m_{T(f_1+f_2)}(\widetilde{Q})-m_{Tf_1}(\widetilde{Q})\Big|. \end{aligned}$$

Note that for any cube $I$, locally integrable function $h$ and constant $c$, $m_h(I)-c$ is a median value of $h-c$ on $I$, namely,

$$m_h(I)-c=m_{h-c}(I).$$

Thus,

$$\begin{aligned} \Big|m_{T(f_1+f_2)}(\widetilde{Q})-m_{Tf_1}(\widetilde{Q})\Big| \;\le\; & \Big|m_{T(f_1+f_2)-m_{Tf_1}(\widetilde{Q})}(\widetilde{Q})\Big| \\ \le\; & 2m_{0,\,s;\,\widetilde{Q}}\Big(T(f_1+f_2)-m_{Tf_1}(\widetilde{Q})\Big) \\ \le\; & 2m_{0,\,s/2;\,\widetilde{Q}}\Big(Tf_1-m_{Tf_1}(\widetilde{Q})\Big) \\ & +2m_{0,\,s/2;\,\widetilde{Q}}\Big(T(f_1+f_2)-Tf_1\Big), \end{aligned}$$

where the second inequality follows from the fact that for any doubling cube $I$, locally integrable function $h$ and $s\in(0,\,2^{-d-2})$,

$$|m_h(I)|\le 2m_{0,\,s;\,I}(h),$$

see [1, Lemma 2.5]. This in turn leads to that

$$\begin{aligned} m_{0,\,s;\,Q}\Big(T(f_1+f_2)-m_{T(f_1+f_2)}(\widetilde{Q})\Big) \le\; & 3\inf_{x\in Q} M^{\sharp}_{0,\,s/2}(Tf_1)(x) \\ & +3\inf_{x\in Q} M_{0,\,s/2}(Tf_2)(x). \end{aligned}$$

On the other hand, we can verify that for any two doubling cubes $Q\subset R$,

$$\begin{aligned} \Big|m_{T(f_1+f_2)}(Q)-m_{T(f_1+f_2)}(R)\Big| \;\le\; & \Big|m_{T(f_1+f_2)}(Q)-m_{Tf_1}(Q)\Big| \\ & +\Big|m_{T(f_1+f_2)}(R)-m_{Tf_1}(R)\Big| \\ & +|m_{Tf_1}(Q)-m_{Tf_1}(R)| \\ \le\; & 2m_{0,\,s/2;\,Q}\Big(Tf_1-m_{Tf_1}(Q)\Big) \\ & +2m_{0,\,s/2;\,Q}\Big(T(f_1+f_2)-Tf_1\Big) \end{aligned}$$

$$
\begin{aligned}
&+2m_{0,\,s/2;\,R}\Big(Tf_1 - m_{Tf_1}(R)\Big)\\
&+2m_{0,\,s/2;\,R}\Big(T(f_1+f_2) - Tf_1\Big)\\
&+|m_{Tf_1}(Q) - m_{Tf_1}(R)|\\
\le\ & 4\inf_{x\in Q} M^{\sharp}_{0,\,s/2}(Tf_1)(x) + 4\inf_{x\in Q} M_{0,\,s/2}(Tf_2)(x)\\
&+|m_{Tf_1}(Q) - m_{Tf_1}(R)|.
\end{aligned}
$$

We then get that

$$M^{\sharp}_{0,\,s}[T(f_1+f_2)](x) \le CM^{\sharp}_{0,\,s/2}(Tf_1)(x) + CM_{0,\,s/2}(Tf_2)(x).$$

Therefore, the proof of the estimate (2.4) can be reduced to proving that

$$M^{\sharp}_{0,\,s}h(x) \le CM^{\sharp}h(x). \tag{2.5}$$

Let $M^{\natural}$ be the sharp maximal operator defined by

$$
\begin{aligned}
M^{\natural}f(x) =\ & \sup_{Q\ni x} \frac{1}{\mu\big(\frac{3}{2}Q\big)} \int_Q |f(y) - m_f(\widetilde{Q})|d\mu(y)\\
&+ \sup_{\substack{R\supset Q\ni x\\ Q,\,R \text{ doubling}}} \frac{|m_f(Q) - m_f(R)|}{K_{Q,\,R}}.
\end{aligned}
$$

Observe that for any cube $Q$

$$m_{0,\,s/2;\,Q}\big(f - m_f(\widetilde{Q})\big) \le \frac{2}{s\mu\big(\frac{3}{2}Q\big)} \int_Q |f(y) - m_f(\widetilde{Q})|d\mu(y).$$

It then follows that

$$M^{\sharp}_{0,\,s/2}f(x) \le 2s^{-1}M^{\natural}f(x).$$

Recall that for any cube $I$,

$$\frac{1}{\mu(I)}\int_I |f(y) - m_f(I)|\,d\mu(y) \le \frac{1}{\mu(I)}\int_I |f(y) - m_I(f)|\,d\mu(y)$$

(see [5, p. 115]). It then follows that

$$
\begin{aligned}
\frac{1}{\mu(\frac{3}{2}Q)}\int_Q |f(y) - m_f(\widetilde{Q})|\,d\mu(y) \le\ & \frac{1}{\mu(\frac{3}{2}Q)}\int_Q |f(y) - m_{\widetilde{Q}}(f)|\,d\mu(y)\\
&+\frac{1}{\mu(\widetilde{Q})}\int_{\widetilde{Q}} |f(y) - m_f(\widetilde{Q})|\,d\mu(y)\\
\le\ & C\inf_{x\in Q} M^{\sharp}f(x).
\end{aligned}
$$

On the other hand, for any two doubling cubes $Q$ and $R$, with $Q \subset R$,

$$\begin{aligned}|m_f(Q) - m_f(R)| &\le |m_Q(f) - m_f(Q)| + |m_R(f) - m_f(R)| + |m_Q(f) - m_R(f)| \\ &\le \frac{1}{\mu(Q)} \int_Q |f(y) - m_f(Q)|\, d\mu(y) \\ &\quad + \frac{1}{\mu(R)} \int_R |f(y) - m_f(R)|\, d\mu(y) + |m_Q(f) - m_R(f)| \\ &\le CK_{Q,R} \inf_{x \in Q} M^\sharp f(x).\end{aligned}$$

Combining the last two estimates leads to that

$$M^\natural f(x) \le CM^\sharp f(x),$$

and

$$\begin{aligned}M^\sharp_{0,s}[T(f_1 + f_2)](x) &\le CM^\sharp_{0,\frac{s}{2}}(Tf_1)(x) + CM_{0,\frac{s}{2}}(Tf_2)(x) \\ &\le 2s^{-1}CM^\natural(Tf_1)(x) + CM_{0,\frac{s}{2}}(Tf_2)(x) \\ &\le CM^\sharp(Tf_1)(x) + CM_{0,\frac{s}{2}}(Tf_2)(x),\end{aligned}$$

then completes the proof of Lemma 2.

**Lemma 3.** *Let $T$, $T_1$, $T_2$ be three operators such that for any $x \in \mathbb{R}^d$,*

$$|T(f_1 + f_2)(x)| \le |T_1 f_1(x)| + |T_2 f_2(x)|.$$

*Suppose that*

(*i*) *for $p_0$, $q_0$ with $p_0 \le q_0$ and $p_0, q_0 \in (1, \infty]$, $T_1$ is bounded from $L^{p_0}(\mu)$ to $L^{q_0,\infty}(\mu)$, when $q_0 = \infty$, $L^{q_0,\infty}(\mu)$ should be replaced by $L^{q_0}(\mu)$;*

(*ii*) *for some $q_1 \in [1, \infty)$, $T_2$ is bounded from $H^{1,\infty}_{\mathrm{atb}}(\mu)$ to $L^{q_1,\infty}(\mu)$.*

*Then for any $p$, $q$ with*

$$\frac{1}{p} = t + \frac{1-t}{p_0},\ \frac{1}{q} = \frac{t}{q_1} + \frac{1-t}{q_0},\ t \in (0, 1),$$

*$T$ is bounded from $L^p(\mu)$ to $L^{q,\infty}(\mu)$.*

*Proof.* Our goal is to prove that there is a constant $C > 0$ such that for any $\lambda > 0$, and bounded function $f$ with compact support,

$$\lambda^q \mu(\{x \in \mathbb{R}^d : |Tf(x)| > \lambda\}) \le C\Big(\int_{\mathbb{R}^d} |f(x)|^p\, d\mu(x)\Big)^{q/p}. \tag{2.6}$$

By homogeneity, we may assume that $\|f\|_{L^p(\mu)} = 1$. For each fixed $\lambda > 0$ and bounded function $f$ with compact, observe that if $\|\mu\| < \infty$ and $\lambda^{q/p} \le \|f\|_{L^1(\mu)}/\|\mu\|$, the inequality (2.6) follows directly, since by the Hölder inequality,

$$\lambda^q \mu(\{x \in \mathbb{R}^d : |Tf(x)| > \lambda\}) \le C\|f\|^p_{L^1(\mu)} \le C.$$

Thus, we may assume $\|\mu\| = \infty$, or $\|\mu\| < \infty$ and $\lambda^{q/p} > \|f\|_{L^1(\mu)}/\|\mu\|$. Note that

$$\frac{\frac{1}{q} - \frac{1}{q_0}}{\frac{1}{q_1} - \frac{1}{q_0}} = \frac{\frac{1}{p} - \frac{1}{p_0}}{1 - \frac{1}{p_0}}, \quad \frac{\frac{1}{q_1} - \frac{1}{q}}{\frac{1}{q_1} - \frac{1}{q_0}} = \frac{1 - \frac{1}{p}}{1 - \frac{1}{p_0}}.$$

It then follows that

$$\frac{\frac{1}{q} - \frac{1}{q_0}}{\frac{1}{q_1} - \frac{1}{q}} = \frac{\frac{1}{p} - \frac{1}{p_0}}{1 - \frac{1}{p}}$$

and so

$$\frac{(q_0 - q)p_0}{(p_0 - p)q_0} = \frac{q - q_1}{(p-1)q_1}.$$

Let $\theta = (q - q_1)/(p-1)q_1$. Applying the Caldeón-Zygmund decomposition to $|f|^p$ at level $\lambda^{\theta p}$ (see [6, p. 131-132]), we know that there exist a sequences of cubes $\{Q_j\}_j$ such that

- $(a)$ the cubes $\{Q_j\}_j$ have bounded overlaps, that is, there is a constant $C$ such that $\sum_j \chi_{Q_j}(x) \le C$;
- $(b)$ $\frac{1}{\mu(2Q_j)} \int_{Q_j} |f(x)|^p \, d\mu(x) > \frac{\lambda^{\theta p}}{2^{d+1}}$;
- $(c)$ for any $\eta > 0$, $\frac{1}{\mu(2\eta Q_j)} \int_{\eta Q_j} |f(x)|^p \, d\mu(x) \le \frac{\lambda^{\theta p}}{2^{d+1}}$;
- $(d)$ $|f(x)| \le \lambda^\theta$, $\mu$-a. e. $x \in \mathbb{R}^d \setminus \cup_j Q_j$;
- $(e)$ for each fixed $j$, let $R_j$ be the smallest $(6, 6^{n+1})$-doubling cube of the form $6^k Q_j$ for $k \in \mathbb{N}$. Set $w_j = \chi_{Q_j}/\sum_{k\ge1} \chi_{Q_k}(x)$. Then there is a function $\phi_j$ with $\operatorname{supp}\phi_j \subset R_j$ and some positive constant $C$ satisfying

$$\int_{\mathbb{R}^d} \phi_j(x) \, d\mu(x) = \int_{Q_j} f(x) w_j(x) \, d\mu(x), \ \sum_j |\phi_j(x)| \le C\lambda^\theta,$$

  and

$$\Big( \int_{R_j} |\phi_j(x)|^p \, d\mu(x) \Big)^{1/p} [\mu(R_j)]^{1/p'} \le \frac{C}{\lambda^{\theta(p-1)}} \int_{Q_j} |f(x)|^p \, d\mu(x).$$

We can decompose $f$ as

$$f(x) = g(x) + b(x).$$

where

$$g(x) = f(x)\chi_{\mathbb{R}^d \setminus \cup_j Q_j}(x) + \sum_j \phi_j(x)$$

and

$$b(x) = \sum_j \left(f(x)w_j(x) - \phi_j(x)\right).$$

It is easy to verify that

$$\|g\|_{L^{p_0}}^{p_0} \le \|g\|_{L^\infty(\mu)}^{p_0-p} \|g\|_{L^p(\mu)}^p \le C\lambda^{\theta(p_0-p)}$$

and

$$\|b\|_{H^{1,\,p}_{\mathrm{atb}}(\mu)} \le C\lambda^{-\theta(p-1)}.$$

This in turn leads to that

$$\mu(\{x \in \mathbb{R}^d :\ |T_1 g(x)| > \lambda/2\}) \le C\lambda^{-q_0}\|g\|_{L^{p_0}(\mu)}^{q_0} \le C\lambda^{-q_0}\lambda^{\theta(p_0-p)q_0/p_0} \le C\lambda^{-q}.$$

and

$$\mu(\{x \in \mathbb{R}^d :\ |T_2 b(x)| > \lambda/2\}) \le C\lambda^{-q_1}\|b\|_{H^{1,\,p}_{\mathrm{atb}}(\mu)}^{q_1} \le C\lambda^{-q}.$$

then

$$\mu(\{x \in \mathbb{R}^d :\ |Tf(x)| > \lambda\}) \le C\lambda^{-q}.$$

This completes the proof of Lemma 3.

**Lemma 4.** (see [1]). *Let* $s_1 \in (0,\, 2^{-d-2})$ *and* $p \in (0, \infty)$. *There is a constant* $C_1 \in (0,1)$ *depending on* $s_1$ *such that for any* $s_2 \in (0,\, C_1 s_1)$,

($i$) *if* $\|\mu\| = \infty$, $f \in L^{p_0,\,\infty}(\mu)$ *with* $p_0 \in [1,\, \infty)$ *and*

$$\sup_{0<\lambda<R} \lambda^p \mu(\{x \in \mathbb{R}^d :\ |f(x)| > \lambda\}) < \infty$$

*for any* $R > 0$, *then*

$$\sup_{\lambda>0} \lambda^p \mu(\{x \in \mathbb{R}^d :\ M^d_{0,\,s_1} f(x) > \lambda\}) \le C \sup_{\lambda>0} \lambda^p \mu(\{x \in \mathbb{R}^d :\ M^\sharp_{0,\,s_2} f(x) > \lambda\});$$

($ii$) *if* $\|\mu\| < \infty$ *and* $f \in L^{p_0,\,\infty}(\mu)$ *with* $p_0 \in [1,\, \infty)$, *then*

$$\begin{aligned} &\sup\nolimits_{\lambda>0} \lambda^p \mu(\{x \in \mathbb{R}^d :\ M^d_{0,\,s_1} f(x) > \lambda\}) \\ &\le C \sup\nolimits_{\lambda>0} \lambda^p \mu(\{x \in \mathbb{R}^d :\ M^\sharp_{0,\,s_2} f(x) > \lambda\}) \\ &\quad + \|\mu\|(s_1\|\mu\|))^{-p/p_0} \|f\|_{L^{p_0,\,\infty}(\mu)}^p. \end{aligned}$$

*Proof of Theorem 2.* At first, for any $s \in (0,\, 2^{-d-2})$, our assumption (ii) along with the estimate (2.5) tells us that the operator $M^{\sharp}_{0,\,s} \circ T$ is bounded from $L^{p_0}(\mu)$ to $L^{q_0,\,\infty}(\mu)$. On the other hand, the assumption (iii) in Theorem 2 via Lemma 1 (ii) states that $M_{0,\,s} \circ T$ is bounded from $H^{1,\,\infty}_{\rm atb}(\mu)$ to $L^{q_1,\,\infty}(\mu)$. Therefore, it follows from Lemma 2 and Lemma 3 that $M^{\sharp}_{0,\,s} \circ T$ is bounded from $L^{p}(\mu)$ to $L^{q,\,\infty}(\mu)$, that is, for any fixed $\lambda > 0$ and bounded function $f$ with compact support,

$$\lambda^q \mu(\{x \in \mathbb{R}^d : M^{\sharp}_{0,\,s} Tf(x) > \lambda\}) \leq C\|f\|^q_{L^p(\mu)} \tag{2.7}$$

provided that $p$ and $q$ satisfies

$$\frac{1}{p} = t + \frac{1-t}{p_0},\ \ \frac{1}{q} = \frac{t}{q_1} + \frac{1-t}{q_0},\quad t \in (0,\,1).$$

We can now conclude the proof of Theorem 2. Set

$$L^{\infty}_{0,\,0}(\mu) = \{f :\ f \text{ is bounded, has compact support, } \int_{\mathbb{R}^d} f(x)d\mu(x) = 0\}.$$

It is well known that $L^{\infty}_{0,\,0}(\mu)$ is a density subset of $L^{p}(\mu)$ for any $p \in [1,\,\infty)$. For each fixed $f \in L^{\infty}_{0,\,0}(\mu)$, which implies $f \in H^{1,\infty}_{atb}(\mu)$, our hypothesis guarantee that $Tf \in L^{q_1,\,\infty}(\mu)$ and so for any $R > 0$, $q > q_1$,

$$\sup_{0<\lambda<R} \lambda^q \mu(\{x\in\mathbb{R}^d :\ |Tf(x)| > \lambda\}) \leq R^{q-q_1} \sup_{\lambda>0} \lambda^{q_1} \mu(\{x\in\mathbb{R}^d :\ |Tf(x)| > \lambda\}) < \infty.$$

By the standard density argument, we need only to prove Theorem 2 for $f \in L^{\infty}_{0,0}(\mu)$ in the following two cases:

**Case 1.** $\|\mu\| = \infty$. By Lemma 1, Lemma 4 (i) and (2.7), we have

$$\begin{aligned}
\sup_{\lambda>0}\lambda^q \mu(\{x \in \mathbb{R}^d :\ |Tf(x)| > \lambda\}) &\leq \sup_{\lambda>0}\lambda^q \mu(\{x \in \mathbb{R}^d :\ M^{d}_{0,\,s_1} Tf(x) \geq \lambda\}) \\
&\leq C\sup_{\lambda>0}\lambda^q \mu(\{x \in \mathbb{R}^d :\ M^{\sharp}_{0,\,s_2} Tf(x) \geq \lambda\}) \\
&\leq C\|f\|^q_{L^p(\mu)}.
\end{aligned}$$

This implies $T$ is bounded from $L^p(\mu)$ to $L^{q,\infty}(\mu)$. On the other hand, the assumption (i) implies that $T$ is sublinear. Thus, by Marcinkiewicz's interpolation theorem, we know that $T$ is also bounded from $L^p(\mu)$ to $L^q(\mu)$.

**Case 2.** $\|\mu\| < \infty$. By a trivial computation, we see that for each fixed $p \in (1,\,\infty)$,

$$\|f\|_{H^{1,\,p}_{\rm atb}(\mu)} \leq C\|\mu\|^{1-1/p}\|f\|_{L^p(\mu)}.$$

By Lemma 1 (i), Lemma 4 (ii) and (2.7), we have

$$\begin{aligned}
&\sup_{\lambda>0}\lambda^q\mu(\{x\in\mathbb{R}^d:\,|Tf(x)|>\lambda\})\\
&\le \sup_{\lambda>0}\lambda^q\mu(\{x\in\mathbb{R}^d:\,M^d_{0,\,s_1}Tf(x)\ge\lambda\})\\
&\le C\sup_{\lambda>0}\lambda^q\mu(\{x\in\mathbb{R}^d:\,M^\sharp_{0,\,s_2}Tf(x)\ge\lambda\})+\|\mu\|(s_1\|\mu\|)^{-\frac{q}{q_1}}\|Tf\|^q_{L^{q_1,\infty}(\mu)}\\
&\le C\|f\|^q_{L^p(\mu)}+Cs_1^{-\frac{q}{q_1}}\|\mu\|^{1-\frac{q}{q_1}}\|f\|^q_{H^{1,\infty}_{\rm atb}(\mu)}\\
&\le C\|f\|^q_{L^p(\mu)},
\end{aligned}$$

which together with the same arguments as in Case 1 implies that $T$ is bounded from $L^p(\mu)$ to $L^q(\mu)$. This completes the proof of Theorem 2.

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