TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 5, pp. 1583-1608, October 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

INTEGRAL PRODUCTS, BOCHNER-MARTINELLI TRANSFORMS AND APPLICATIONS

Chia-Chi Tung

Abstract. A generalized Bochner-Martinelli formula for the push-forward of a Lipschitz function over a weak Stokes domain is proved. By means of integral products, the $\bar{\partial}$-Euler and the $\bar{\partial}$-Neumann vector fields, local and global characterizations of the holomorphicity of functions on a Riemann domain are given. As further applications characterizations of isogeneity and Liouville properties for the push-forward of semi-harmonic functions on an analytic covering space are obtained.

1. INTRODUCTION

This paper is a sequel to [19], in which semi-harmonic functions on a complex space are introduced and characterizations of semi-harmonicity are given, on a Riemann domain, in terms of the local properties of the function such as the solid, spherical, as well as the near, resp. weak, harmonicity.

On a semi-Riemann domain the connection between semi-harmonicity and holomorphicity of functions seems to lie in the $\bar{\partial}$-Euler vector fields (see [19], §2). Alternatively, one may consider the Dirichlet product of functions, which offers a natural link, in a sense, between the Cauchy-Riemann and the Laplace operators. Consequently, on a normal semi-Riemann domain local characterizations of the holomorphicity of functions are obtained in terms of the $\bar{\partial}$-Euler vector fields (Corollary 5.1) as well as *isogeneity* (Theorem 3.1 and Corollary 3.1) and weak-harmonicity (Corollary 3.2). Also, by means of the Dirichlet product conditions for a complex-valued function to be isogenic to a semi-harmonic function are given (Proposition 3.3).

Received May 19, 2007, accepted January 24, 2008.
Communicated by Der-Chen Chang.
2000 *Mathematics Subject Classification*: Primary: 31C05; Secondary: 32C30, 31B10.
Key words and phrases: Semi-Riemann domain, $\bar{\partial}$-Euler vector field, $\bar{\partial}$-Neumann vector field, Semi-harmonicity, Dirichlet product, Push-forward, Bochner-Martinelli transform.
Supports by Minnesota State University, Mankato and the Grant "Globale Methoden in der komplexen Geometrie" of the German research society DFG are gratefully acknowledged.

To consider global characterizations of holomorphicity, a generalized Bochner-Martinelli representation, in the form of a push-forward formula, needs to be proved: (Theorem 4.1) *Let* X *be a complex space of dimension* $m > 0$ *and* $f : X \to \mathbb{C}^m$ *a holomorphic map. Let* $G \subset X$ *be a weak Stokes domain with* $dG \neq \emptyset$, *and* $\phi \in C^\lambda(\overline{G})$. (1) *Assume that :* (i) $w \in \mathbb{C}^m \backslash f(\mathrm{Spt}_{\partial G}(\phi))$, *and* (ii) $\overline{G} \cap f^{-1}(w)$ *is discrete. Then* $\forall \xi \in G \cap f^{-1}(w)$,

$$\langle\!\langle \phi \rangle\!\rangle_{f,G}(w) = \mathfrak{B}^+_{\partial G}(\xi) - \int_G \bar\partial \phi \wedge \mathcal{K}_\xi. \tag{1.1}$$

(2) *For every* $\xi \in X \backslash \widehat{\overline{G}}$,

$$\mathfrak{B}^-_{\partial G}(\xi) = \int_G \bar\partial \phi \wedge \mathcal{K}_\xi.$$

Here $\mathfrak{B}^+_{\partial G}(\xi)$, resp. $\mathfrak{B}^-_{\partial G}(\xi)$, denotes the *interior*, resp. *exterior, Bochner-Martinelli transform* of ϕ (relative to G). On a Riemann domain an alternative form of the representation (1.1) is given in (5.11). The formula (1.1) implies that the push-forward of a locally Lipschitz function with a vanishing $\bar\partial$-Euler derivative admits a universal Bochner-Martinelli representation ((4.15)). By means of the Euler product and the $\bar\partial$-Neumann vector field, global characterizations of holomorphicity are given in Proposition 3.1, resp. Theorems 5.1 and 5.3. The latter gives a generalization of the Aronov-Kytmanov theorem ([3], Theorem 1) in the following form: *Let* $p : X \to \mathbb{C}^m$ *be a normal Riemann domain. Assume that* $G \subseteq X$ *is a weak Stokes domain with* $dG \neq \emptyset$, *and* $\phi \in \mathfrak{H}_w(G) \cap C^{1,1}(\overline{G})$. *Then the push-forward of* ϕ *admits the Bochner-Martinelli representation*

$$p^* \langle\!\langle \phi \rangle\!\rangle_{p,G}(\xi) = \mathfrak{B}^+_{\partial G}(\xi), \quad \forall \xi \in G \backslash \widehat{\partial G},$$

iff $\phi \in \mathcal{O}(G)$. Finally, Liouville properties for the push-forward of semi-harmonic functions on an analytic covering space are proved.

2. Preliminaries

In what follows every complex space is assumed to be reduced and has a countable topology. The notations and terminology of [19] shall be used throughout this paper. Some of these are recalled here for the convenience of the reader. Let $\|z\|$ denote the Euclidean norm of $z = (z_1, \cdots, z_m) \in \mathbb{C}^m$, where $z_j = x_j + i\, y_j$. The space $\mathbb{C}^m$ shall be oriented so that the form $\upsilon^m := (dd^c \|z\|^2)^m$ is positive. Let X be a complex space of dimension $m > 0$ and $p : X \to \mathbb{C}^m$ a holomorphic map. Set $a' := p(a)$, $p^{[a]} := p - a'$, $\forall a \in X$. Clearly the form

$$\upsilon_p := dd^c \|p^{[a]}\|^2 = (\frac{i}{2\pi})\partial\bar\partial\,\|p^{[a]}\|^2 \tag{2.1}$$

is non-negative and independent of a. Let $\mathfrak{z}_j := (p^{[a]})^* z_j,\ j = 1, \cdots, m,\ d\mathfrak{z} := d\mathfrak{z}_1 \wedge \cdots \wedge d\mathfrak{z}_m$, and $d\bar{\mathfrak{z}}_{[j]} := d\bar{\mathfrak{z}}_1 \wedge \cdots \wedge d\bar{\mathfrak{z}}_{j-1} \wedge d\bar{\mathfrak{z}}_{j+1} \wedge \cdots \wedge d\bar{\mathfrak{z}}_m$. The *Bochner-Martinelli form* on $X \times X$ is defined by

$$\mathcal{K}(z,a) := (-1)^{\frac{m(m-1)}{2}} \frac{(m-1)!}{(2\pi i)^m} \sum_{j=1}^{m} (-1)^{j-1} (\frac{\bar{\mathfrak{z}}_j}{\|p(z) - p(a)\|^{2m}})\, d\bar{\mathfrak{z}}_{[j]} \wedge d\mathfrak{z}, \tag{2.2}$$

where $z \notin p^{-1}(a')$. The form $\mathcal{K}_a = \mathcal{K}(\ ,a)$ can be written more compactly as

$$\mathcal{K}_a = \frac{1}{2\pi i}\frac{1}{\|p^{[a]}\|^{2m}}\,\partial\|p^{[a]}\|^2 \wedge \upsilon_p^{m-1}. \tag{2.3}$$

The real part of $\mathcal{K}_a$,

$$\sigma_a := \frac{1}{2}(\mathcal{K}_a + \overline{\mathcal{K}_a}), \tag{2.4}$$

is d-closed:

$$d\,\sigma_a = (dd^c \log\|p^{[a]}\|^2)^m \;=\; 0. \tag{2.5}$$

Define the *Newtonian functions*

$$\mathfrak{g}_a(z) := \begin{cases} -\dfrac{1}{2\pi i}\log\dfrac{1}{\|z' - a'\|^2}, & \text{if } m = 1, \\ -\dfrac{(m-2)!}{(2\pi i)^m}\,\dfrac{1}{\|z' - a'\|^{2m-2}}, & \text{if } m > 1, \end{cases} \tag{2.6}$$

where $z \in X\backslash p^{-1}(a')$. The Bochner-Martinelli form admits a useful alternative representation in terms of the Newtonian functions:

$$\mathcal{K}_a = (-1)^{\frac{m(m-1)}{2}} \sum_{k=1}^{m} (-1)^{k-1}\frac{\partial \mathfrak{g}_a}{\partial p_k} dp \wedge d\bar{p}_{[k]}. \tag{2.7}$$

If $f : X \to Y$ is a holomorphic map between complex spaces X, Y, and f is light at a point $z \in f^{-1}(y)$, denote by $\nu_f^y(z)$ the *multiplicity* of f at z ([17], p. 22); set $\nu_f^y(z') = 0$, if $z' \notin f^{-1}(y)$. Let $\phi : X \to \mathbb{C}^N$ be a continuous map. A notion central to value distribution theory is the *push-forward* (or *fiber integration*) of ϕ with respect to the restriction $f\rfloor D$. This is defined, in the case $f\rfloor D$ has discrete fiber over $y \in Y$, by

$$\langle\!\langle \phi \rangle\!\rangle_{f,D}(y) \;:=\; \sum_{\xi \in D} \nu_f^y(\xi)\,\phi(\xi),$$

provided the sum on the right-side exists (see Stoll [17]; for a topological treatment see Radò and Reichelderfer [14] and Martio, Rickman and Väisälä [13]). Here $\langle\!\langle\phi\rangle\!\rangle_{f,D}(y) := 0,$ if $D \cap f^{-1}(y) = \emptyset,$ and D is omitted if $D = X.$

A complex space X together with a holomorphic map $p : X \to \Omega,$ where Ω is a domain in $\mathbb{C}^m$, is called a *semi-Riemann domain* iff there exists a thin analytic subset Σ of Ω such that the inverse image $\Sigma_p := p^{-1}(\Sigma)$ is thin in X and the restriction $p : X^0 := X \setminus \Sigma_p \to \Omega_0 := \Omega \setminus \Sigma$ has discrete fibers. If $\Sigma = \emptyset,$ then (X, p) is a *Riemann domain* ([5], p. 19, or [8], p.135; also [7], p. 116, where X is assumed a normal space). If $p : X \to \Omega$ is in addition a local homeomorphism, then (X, p) is said to be *unramified*. Every proper holomorphic map of a pure m-dimensional complex space into a domain $\Omega \subseteq \mathbb{C}^m$ of strict rank m is a semi-Riemann domain ([2], p. 117).

A holomorphic map $\pi : X \to Y$ between complex spaces is called a (finite) *analytic covering* of Y iff π is a finite mapping (i.e., closed with finite fibers) and there exists a thin analytic subset E of Y with the following properties: (i) $E_\pi := \pi^{-1}(E)$ is thin in X; (ii) every point $y \in Y \setminus E$ has an open neighborhood V such that $\pi^{-1}(V)$ is a union of disjoint open subsets U_j for which the restrictions $\pi_{U_j} := \pi : U_j \to V$ are homeomorphisms. An analytic covering $\pi : X \to Y$ is necessarily a proper mapping. If $Y \setminus E$ is connected, then $\#\pi^{-1}(y) = \mathrm{const.} = s,\ \forall y \in Y \setminus E,$ and $\#\pi^{-1}(y) < s,\ \forall y \in E.$ In this case the integer s is called the *sheet number of* π.

Unless otherwise mentioned, let (X, p) be a *semi-Riemann domain* (over Ω) of dimension $m > 0.$ For an open subset $D \subseteq X$ and each $a \in D^0 := D \cap X^0,$ there exists an open neighborhood N with closure in D^0 such that: i) $p^{-1}(a') \cap \overline{N} = \{a\}$; ii) for a sufficiently small ball $U' = \mathbb{B}_{[a']}(\rho)$ in $\mathbb{C}^m,$ $U_a := p^{-1}(U') \cap N = p^{-1}(U') \cap \overline{N}$ is connected and the mapping $p \rfloor U_a : U_a \to U'$ is an analytic covering; iii) every branch $V^k,\ k = 1, \cdots, s_a,$ of U_a contains a; and iv)

$$s_a = \deg(p \rfloor U_a) = \nu_p^{a'}(a) \tag{2.8}$$

([17], Proposition 1.3). For convenience call such U_a a *pseudo-ball* (*of radius* ρ) *at* a. Denote by X^* the largest open subset of X on which p is locally biholomorphic, and set $D^* := D \cap X^*.$ Let $\triangle_p = \triangle_{p_U}$ denote the p_U-pull-back to an open set $U \subseteq X^*$ of the Laplace operator of the Euclidean metric on $\mathbb{C}^m$.

3. Integral Products and Holomorphicity

If V is a complex vector space equipped with a semi-scalar product $\langle\ ,\ \rangle : V \times V \to \mathbb{C},$ denote by $|\langle\phi\rangle| := \sqrt{\langle\phi, \phi\rangle}$ the associated semi-norm of $\phi \in V,$ $\lfloor\mathcal{K}\rfloor$ the subspace generated by a subset $\mathcal{K} \subseteq V,$ and $\lfloor\mathcal{K}\rfloor^\perp$ its orthogonal complement.

Let $\mathcal{A}(D) := C^1(\overline{D}) \cap \mathcal{O}(D^*)$ and $\mathcal{A}^c(D) := C^1(\overline{D}) \cap \overline{\mathcal{O}}(D^*)$. If $D \subseteq X$ is relatively compact and $\phi,\ \psi \in C^1(\overline{D})$, define the *Euler product*

$$\langle \phi, \psi \rangle_D := \frac{1}{2\pi i} \int\limits_D \bar{\partial}\phi \wedge \partial\bar{\psi} \wedge \upsilon_p^{m-1}. \tag{3.1}$$

In terms of the ∂-, respectively, $\bar{\partial}$-, Euler vector field associated to ϕ ([19], (2.6)), the Euler product can be written

$$\langle \phi, \psi \rangle_D = \frac{1}{2m} \int_{D^*} \mathcal{E}_\phi(\bar{\psi})\, \upsilon_p^m.$$

It follows that

$$\langle \phi, \psi \rangle_D = \overline{\langle \psi, \phi \rangle_D}, \tag{3.2}$$

and

$$\langle \phi, \phi \rangle_D = \frac{1}{m} \int_D \sum_{\mu=1}^{m} \|\phi_{\bar{p}_\mu}\|^2\, \upsilon_p^m \geq 0. \tag{3.3}$$

Furthermore, the identity (3.3) and the Cauchy-Riemann equations imply that if $\phi \in C^1(\overline{D})$ with $|\langle \phi \rangle| = 0$, then $\phi \in \mathcal{A}(D)$. Thus the following is proved:

Proposition 3.1. *If $D \subseteq X$ is relatively compact, then the Euler product* (3.1) *is a semi-scalar product on $C^1(\overline{D})$ relative to which $\mathcal{A}(D) = \{\phi \in C^1(\overline{D}) \,|\, |\langle \phi \rangle_D| = 0\}$.*

Of importance to harmonic function theory is the *Dirichlet product*, which, on a semi-Riemann domain, can be defined as follows: if $\eta,\ \phi : D \to \mathbb{C}$ are locally Lipschitz functions ([18], §4), set

$$[\eta, \phi]_D := \int\limits_D d\eta \wedge d^c\bar{\phi} \wedge \upsilon_p^{m-1}, \tag{3.4}$$

provided the integral exists. Alternatively the definition can be written

$$[\phi, \eta]_D = \frac{1}{2}\{\langle \phi, \eta \rangle_D + \langle \bar{\eta}, \bar{\phi} \rangle_D\}. \tag{3.5}$$

Therefore one has

$$[\phi, \eta]_D = \overline{[\eta, \phi]_D} = [\bar{\eta}, \bar{\phi}]_D, \tag{3.6}$$

provided one of the integrals exists. The Dirichlet product is an energy product in that one can define the (*kinetic*) *energy* of a function $\phi \in C^1(\overline{U})$ by setting

$$\mathbb{E}_U(\phi) := [\phi, \phi]_U.$$

Denoting the real and imaginary parts of ϕ by $u = \mathrm{Re}\,(\phi),\ v = \mathrm{Im}\,(\phi),$ one has, by the relations (3.2)-(3.5),

$$\mathbb{E}_U(\phi) = \mathbb{E}_U(u) + \mathbb{E}_U(v) \geq 0. \tag{3.7}$$

If U is a neighborhood of $a,$ denote the semi-scalar product (3.4) (resp. (3.1)) with $D = U_a(r)$ by $[\phi, \eta]_{a,r}$ (resp. $\langle \phi, \eta \rangle_{a,r}$), and similarly for the energy $\mathbb{E}_{a,r}(\phi)$. The identities (3.5) and (3.7) imply the following:

Proposition 3.2. *If $D \subseteq X$ is relatively compact, then the Dirichlet product is a semi-scalar product on $C^1(\overline{D})$ with $\mathcal{A}(D) \subseteq \mathcal{A}^c(D)^\perp$; moreover, if D is connected, then $\{\phi \in C^1(\overline{D}) \,|\, \mathbb{E}_D(\phi) = 0\} = \mathbb{C}$.*

Remark 1. There exists no finite orthonormal subset $\mathcal{F}$ of $\mathcal{A}(D)$ or $\mathcal{A}^c(D)$ relative to either $[\ ,\ ]_D$ or $\langle\ ,\ \rangle_D$ such that $C^1(\overline{D}) = \lfloor \mathcal{F} \rfloor^\perp \oplus (\lfloor \mathcal{F} \rfloor^\perp)^\perp$. The proof goes as follows: Set $\mathfrak{Z}_V := \{\phi \in V \,|\, |\langle \phi \rangle| = 0\}$. Suppose $\mathcal{L}$ is a linear subspace of V spanned by a finite orthonormal subset. Then standard argument and the Schwartz inequality imply that (a) $V = \mathcal{L} \oplus \mathcal{L}^\perp$; (b) if $\phi = g + \eta \in \mathcal{L} \oplus \mathcal{L}^\perp$, then (i) $|\langle \phi \rangle| = |\langle g \rangle| \Leftrightarrow \phi \in (\mathcal{L}^\perp)^\perp$, and (ii) $\mathfrak{Z}_V = \mathcal{L}^\perp \cap (\mathcal{L}^\perp)^\perp$. Hence if $\mathfrak{Z}_V \neq \{0\}$, then $V \neq \mathcal{L}^\perp \oplus (\mathcal{L}^\perp)^\perp$.

Observe that the identity (3.6) yields, $\forall \phi,\ \eta \in C^1(D)$,

$$\begin{aligned} \mathrm{Re}\,[\phi, \bar{\eta}]_U &= \frac{1}{4m} \int_U [\partial_{\nabla\alpha}(u) - \partial_{\nabla\beta}(v)]\, \upsilon_p^m, \\ \mathrm{Im}\,[\phi, \bar{\eta}]_U &= \frac{1}{4m} \int_U [\partial_{\nabla\beta}(u) + \partial_{\nabla\alpha}(v)]\, \upsilon_p^m, \end{aligned} \tag{3.8}$$

for all open sets $U \subseteq D$. Together with the Green's first identity ([19], (5.9)), the Dirichlet product thus provides a natural link, in a sense, between the Cauchy-Riemann and the Laplace operators. Let $\eta = (\eta_1, \cdots, \eta_N) : D \to \mathbb{C}^N$ be a C^1-mapping. Denote by $\lfloor \eta_1, \cdots, \eta_N \rfloor^\perp_{[D],\mathrm{loc}}$ the set of all $\phi \in C^\beta(D)$ such that for some thin analytic subset A of D, $\phi \in C^1(D^* \backslash A)$, and there exists at each $a \in D^* \backslash A$ a pseudo-ball U for which $[\phi, \eta_k]_{a,r} = 0$ for sufficiently small $r > 0$, $\forall k = 1, \cdots, N$. Elements of $\lfloor \bar{\eta}_1, \cdots, \bar{\eta}_N \rfloor^\perp_{[D],\mathrm{loc}}$ are said to be η-*isogenic* (in D). The set $\lfloor \eta_1, \cdots, \eta_N \rfloor^\perp_{\langle D \rangle,\mathrm{loc}}$ is similarly defined. The next Lemma is an immediate consequence of the formulas in (3.8):

Lemma 3.1. *Let $\eta = (\eta_1, \cdots, \eta_N) : D \to \mathbb{C}^N$, $\eta_k = \alpha_k + i\beta_k$, where α_k, β_k are real-valued, be a C^1-mapping. Then a function $\phi \in C^\beta(D)$ is η-isogenic iff the following equations hold*:

$$\partial_{\nabla\alpha_k}(u) = \partial_{\nabla\beta_k}(v),\ \ \partial_{\nabla\beta_k}(u) = -\partial_{\nabla\alpha_k}(v),\ k = 1, \cdots, N, \tag{3.9}$$

locally in $D^*\backslash A,$ *for some thin analytic subset* A *of* $D.$

Remark 2. It follows from the equations in (3.9) that if $\phi \in C^1(D)$ is η-isogenic, then for each $a \in D,$ $[\phi, \eta_k]_{a,r} = 0,$ $\forall k,$ for sufficiently small $r > 0.$ Also, if $\phi = u + iv$ and $\eta = \alpha + i\beta$ (where u, v, α, β are real-valued) are semi-harmonic ([19], §4)) in $D,$ then it is easily shown (using the identity (2.11) of [19]) that ϕ is η-isogenic iff the functions $u\alpha - v\beta$ and $u\beta + v\alpha$ are semi-harmonic.

Given a function $\phi = u + iv \in C^1(D),$ one may also consider the *complex energy*

$$\mathbb{E}^c_U(\phi) := [\phi, \bar{\phi}]_U,$$

for each open subset $U \subseteq D.$ It is easily shown that, if $\psi \in C^1(D),$ then

$$[\phi, \bar{\psi}]_U = \frac{1}{2}\{\mathbb{E}^c_U(\phi + \psi) - \mathbb{E}^c_U(\phi) - \mathbb{E}^c_U(\psi)\}. \tag{3.10}$$

Also, by the relations in (3.8), one has

$$\mathbb{E}^c_U(\phi) = \mathbb{E}_U(u) - \mathbb{E}_U(v) + 2\,i\,[u, v]_U. \tag{3.11}$$

If $a \in U,$ set

$$\mathbb{E}^c_{a,r}(\phi) := \mathbb{E}^c_{U_{[a]}(r)}(\phi), \quad \forall r > 0,$$

(omitting the superscript "c" if ϕ is real-valued). The function ϕ is said to be *self-isogenic at* $a \in D$ iff $\mathbb{E}^c_{a,r}(\phi) = 0$ for sufficiently small $r > 0.$ If $a \in D^*,$ this means that u and v have equal maximal rate of growth at a and the level surfaces $\{u = c_1\}$ and $\{v = c_2\}$ through the point a are orthogonal at a (by the condition (3.9)).

If ϕ and $\eta \in C^1(D)$ are self-isogenic, then by the identity (3.10), ϕ is η-isogenic iff $\phi - \eta$ is self-isogenic. The Cauchy-Riemann equations imply that if $\phi \in \mathcal{O}(D^*) \cup \overline{\mathcal{O}}(D^*),$ then ϕ is self-isogenic in $D.$ The converse statement is valid only for the case $m = 1,$ as is shown by the next example:

Example 3.1. On a semi-Riemann domain (X, p) of dimension $m > 1,$ let $\phi \in \mathcal{O}(D^*)$ such that $\phi_{p_j} \neq 0$ and $\phi_{p_k} = 0$ for some j, k with $j \neq k.$ Let $\psi := \bar{p}_k.$ Then ϕ is ψ-isogenic, hence the function $\Psi := \phi - \psi$ is self-isogenic, but Ψ is neither holomorphic nor anti-holomorphic.

Denote by dD the (maximal) *boundary manifold* of $\mathcal{R}(D)$ in $\mathcal{R}(X),$ the manifold of simple points of $X,$ oriented to the exterior of $\mathcal{R}(D)$ ([18], p. 218), and $d\sigma_{dD}$ the induced (Lebesgue) surface measure on dD ([19], §4). If $\phi \in C^0(D),$ denote by $\mathrm{Zero}(\phi)$ the zero set of $\phi.$ The above example leads to the natural question as to when a function is isogenic to a semi-harmonic function. An answer is given by the following

Proposition 3.3. *Let* (X, p) *be a Riemann domain. Assume that* $\phi = u + iv \in C^\lambda(D)$ *and* $\eta = \alpha + i\beta \in C^{1,1}(D)$*. Then:* (a) *if* η *is semi-harmonic, then* ϕ *is isogenic to* η *iff there exists a thin analytic subset* A *of* D *such that, locally at each* $a \in D^*\backslash A$*, there is a neighborhood basis* $\{U\}$ *consisting of weak Stokes domains* ([19], §4) $U \Subset D^*\backslash A$*, for which the equations*

$$\begin{aligned} \int_{dU} (u\,(\partial_\nu \alpha) - v\,(\partial_\nu \beta))\, d\sigma_{dU} \;&=\; 0, \\ \int_{dU} (u\,(\partial_\nu \beta) + v\,(\partial_\nu \alpha))\, d\sigma_{dU} \;&=\; 0. \end{aligned} \tag{3.12}$$

hold; (b) *if* ϕ *has thin zero set and* η *is isogenic to* ϕ*, then* η *is semi-harmonic in* D *iff* (ϕ, η) *satisfies the equations in* (3.12) *for some neighborhood basis* $\{U\}$ *as above at each point of* $D^*\backslash A$*.*

Proof. The function η being semi-harmonic in D, coincides off a thin analytic subset with a C^∞-function ([19], Theorem 4.2 and Corollary 4.1). Hence by Corollary 4.1, the identities (5.9), (5.7) and Proposition 5.1 of [19], the relation

$$[\phi, \bar{\eta}]_U \;= (-1)^{\frac{m(m-1)}{2}} \frac{1}{2\|\mathbb{S}\|} \int_{dU} \phi\,(\partial_\nu \eta)\, d\sigma_{dU} \tag{3.13}$$

holds for any weak Stokes domain $U \Subset D^*$, whence the assertion (a) follows.

Observe that the semi-harmonicity of α and β in $D^*\backslash A \cup \mathrm{Zero}(\phi)$ (hence also in D) is equivalent to the conditions

$$\begin{aligned} u \triangle_p \alpha - \, v \, \triangle_p \beta = 0, \\ u \triangle_p \beta + \, v \, \triangle_p \alpha = 0, \end{aligned}$$

locally in $D^*\backslash A$. These equations are equivalent, in turn, to the condition

$$\int_U \phi \, dd^c \eta \wedge \upsilon_p^{m-1} = 0. \tag{3.14}$$

This last equation (for any weak Stokes domain $U \Subset D^*$) is, as above, equivalent to the representation (3.13). It follows that η is semi-harmonic in D iff (ϕ, η) satisfies the equations in (3.12). ∎

Examples show that a function isogenic to a holomorphic map needs not be holomorphic. This condition turns out, however, to be sufficient if either the map has maximal *strict rank* ([2], p. 17) or its components are essentially one dimensional:

Theorem 3.1. *Let (X, p) be a normal semi-Riemann domain and $\phi \in C^{\beta}(D)$. Assume $\eta = (\eta_1, \cdots, \eta_N) : D \to \mathbb{C}^N$ such that each $\eta_j \in \mathcal{O}(D)$ (resp. $\eta_j \in \bar{\mathcal{O}}(D)$), and one of the following conditions holds:* (a) *η (resp. $\bar{\eta} = (\bar{\eta}_1, \cdots, \bar{\eta}_N)$) is of strict rank m in D;* (b) *$N = m,\ Z_\eta := \cup_{k=1}^m \mathrm{Zero}((\eta_k)_{p_k})$ (resp. $Z_{\bar{\eta}}$) is thin in $D,$ and with $\eta_j = \alpha^j + i\,\beta^j,\ j = 1, \cdots, m,$*

$$\partial_{\nabla_j \alpha^j}(\eta_k) = \partial_{\nabla_j \beta^j}(\eta_k) = 0, \quad \forall j \neq k, \tag{3.15}$$

holds in $D^* \backslash Z_\eta$ (resp. $D^* \backslash Z_{\bar{\eta}}$). Then $\phi \in \mathcal{O}(D)$ (resp. $\phi \in \bar{\mathcal{O}}(D)$) iff ϕ is η-isogenic.

Proof. Assume that $\phi \in C^{\beta}(D) \cap C^1(D^* \backslash A),$ for some thin analytic subset A of D. Consider at first the case where each $\eta_k \in \mathcal{O}(D)$. Then

$$2 \sum_{j=1}^{m} \partial_j(\eta_k)\, \bar{\partial}_j \phi = \partial_{\nabla \eta_k}(\phi), \quad k = 1, \cdots, N, \tag{3.16}$$

in $D^* \backslash A$ (by the identities (2.7)-(2.8) of [19]). According to Remmert [15], Satz 16, the set $B(m),$ consisting of all simple points $z \in D$ where the Jacobian matrix $J_\eta = \frac{\partial(\eta_1, \cdots, \eta_N)}{\partial(p_1, \cdots, p_m)}$ has rank less than $m,$ has an analytic closure in D. If the restriction $\eta \rfloor D$ has strict rank $m,$ then the set $D^* \backslash \overline{B}(m)$ is dense in D^*. Thus the matrix J_η is row-reducible to an echelon form of rank m almost everywhere in D^*. It follows from this, Lemma 3.1 and the Cauchy-Riemann equations that ϕ is η-isogenic iff ϕ is holomorphic in D^* (off a thin analytic subset of D), and by the Riemann's extension theorem, the latter is equivalent to ϕ being holomorphic in D. The case where the η_j's are anti-holomorphic and $\bar{\eta} \rfloor D$ has strict rank m can be similarly proved.

Assume now that η is holomorphic. Let $a \in D^* \backslash Z_\eta$. Locally the vector fields $\nabla_j \alpha^j,\ \nabla_j \beta^j,\ j = 1, \cdots, m,$ form an orthogonal set in the inner product space $T_z(D^*)$ (for z near a). Hence the vector fields $\nabla_j \alpha^k$ and $\nabla_j \beta^k$ are expressible in $D^* \backslash Z_\eta$ in terms of linear combinations of the $\nabla_\mu \alpha^\mu,\ \nabla_\mu \beta^\mu,\ 1 \leq \mu \leq m$. Under the hypothesis (3.15) one has, $\forall j \neq k,$

$$\nabla_j \alpha^k = c_k \nabla_k \alpha^k + d_k \nabla_k \beta^k,$$

and similarly for $\nabla_j \beta^k$. Thus $\nabla_j \alpha^k$ and $\nabla_j \beta^k$ belong to the span of $\frac{\partial}{\partial \tilde{x}_j}, \frac{\partial}{\partial \tilde{y}_j},$ and that of $\frac{\partial}{\partial \tilde{x}_k}, \frac{\partial}{\partial \tilde{y}_k},$ in $D^* \backslash Z_\eta$; consequently $\nabla_j \alpha^k = \nabla_j \beta^k = 0$ in $D^* \backslash Z_\eta$. Therefore by the equation (3.16),

$$2\,(\eta_k)_{p_k}\, \bar{\partial}_k \phi = \partial_{\nabla \eta_k}(\phi), \quad k = 1, \cdots, m,$$

in $D^*\backslash(Z_\eta \cup A)$. These relations imply, as in the preceding, that ϕ is η-isogenic iff ϕ is holomorphic in D. The remaining case where each η_j is anti-holomorphic can be similarly proved. ∎

Remark 3. The above proof shows that, if η is holomorphic with Z_η thin in D, the condition (3.15) is equivalent to the following: $\frac{\partial \eta_k}{\partial p_j} = 0$ in $D^*\backslash Z_\eta$, $\forall j \neq k$; in particiular, the map $\eta\rfloor D$ is of strict rank m.

Corollary 3.1. *Let* (X, p) *be a normal semi-Riemann domain and* $\eta = (\eta_1, \cdots, \eta_N) : D \to \mathbb{C}^N$ *a holomorphic map satisfying the same conditions as in Theorem* 3.1. *Then:*

(a) $\mathcal{O}(D) = \{\phi \in C^1(D^*\backslash A) \cap C^\beta(D) \,|\, \mathbb{E}^c_{a,r}(\phi + \eta_k) = \mathbb{E}^c_{a,r}(\phi),\ \text{loc. in } D^*\backslash A,\ 1 \leq k \leq N\}$, *for any thin analytic subset* A *of* D.

(b) $\mathcal{O}(D) = \{\phi \in C^1(D^*\backslash A) \cap C^\beta(D) \,|\, \mathbb{E}_{a,r}(\phi + \bar\eta_k) = \mathbb{E}_{a,r}(\phi + i\,\bar\eta_k) = \mathbb{E}_{a,r}(\phi) + \mathbb{E}_{a,r}(\bar\eta_k),\ \text{loc. in } D^*\backslash A,\ 1 \leq k \leq N\}$, *for any thin analytic subset* A *of* D.

(c) $\mathcal{O}(D) = \lfloor \bar\eta_1, \cdots, \bar\eta_N \rfloor^\perp_{\langle D\rangle,\mathrm{loc}}$.

Proof. Since any two holomorphic functions are mutually isogenic, the first assertion is an immediate consequence of the identity (3.10) and Theorem 3.1. To prove the second assertion, observe that the identities (3.10) and (3.7) imply that, for any thin analytic subset A of D, the equations in (3.9) (for the real and imaginary parts of ϕ) are equivalent to the conditions on the pairs (ϕ, η_k) given in (b). Therefore the desired conclusion follows from Lemma 3.1 and Theorem 3.1. The assertion (c) follows from Theorem 3.1 and the identity (for a holomorphic map η) $\lfloor \bar\eta_1, \cdots, \bar\eta_N \rfloor^\perp_{\langle D\rangle,\mathrm{loc}} = \lfloor \bar\eta_1, \cdots, \bar\eta_N \rfloor^\perp_{[D],\mathrm{loc}}$. ∎

Corollary 3.2. *Let* (X, p) *be a normal semi-Riemann domain and* $\eta = (\eta_1, \cdots, \eta_N) : D \to \mathbb{C}^N$ *a holomorphic map satisfying the same conditions as in Theorem* 3.1. *Then*:

(a) $\mathcal{O}(D) = \{\phi \in C^2(D^*\backslash A) \cap C^\beta(D) \,|\, \triangle_p(\eta_k\,\phi) = \eta_k \triangle_p(\phi),\ \text{loc. in } D^*\backslash A,\ 1 \leq k \leq N\}$, *for any thin analytic subset* A *of* D.

(b) $\mathcal{O}(D) = \{\phi \in \mathfrak{H}_w(D^0) \cap C^\beta(D) \,|\, \eta_k\phi \in \mathfrak{H}_w(D^0),\ \ 1 \leq k \leq N\}$.

Proof. By the identity (2.11) of [19], if $\phi \in C^2(D^*\backslash A)$, then

$$\partial_{\nabla \eta_k}(\phi) = \frac{1}{2}\,[\triangle_{p_U}(\eta_k\,\phi) - \eta_k \triangle_{p_U}(\phi)],\ \ 1 \leq k \leq N,$$

in $D^*\backslash A$. Hence the assertion (a) is an immediate consequence of Theorem 3.1. By Theorem 4.2, ibid., every weak solution in D^0 of the semi-Laplace equation

((4.2), ibid.) is semi-harmonic. Therefore the assertion (b) is a consequence of the assertion (a). ■

4. Bochner-martinelli Transforms

If $\overline{D}$ is compact, $a \in D^0$, and $\phi \in C^0(D_{[a]}[r_0])$, define the *solid,* resp. *spherical, mean-value function* of ϕ (with resp. to $p^{[a]}$) by

$$\langle \phi \rfloor D \rangle_{a,r} := \frac{1}{r^{2m}} \int\limits_{D_{[a]}(r)} \phi\, \upsilon_p^m, \quad \forall r \in (0, r_0), \tag{4.1}$$

$$[\phi \rfloor D]_{a,r} := \int\limits_{dD_{[a]}(r)} \phi\, \sigma_a, \quad \forall r \in (0, r_0). \tag{4.2}$$

If $S \subseteq X$, set $\widehat{\partial S} := p^{-1}(p(\partial S))$. Let $\phi \in C^0(\partial D)$. If ∂D is compact, define, $\forall \xi \in X \backslash \widehat{\partial D}$, the *Bochner-Martinelli average* of ϕ on dD (relative to $p^{[\xi]}$) by

$$[\![\phi]\!]_{\partial D}(\xi) := \int\limits_{dD} \phi(\zeta)\, \mathcal{K}_\xi(\zeta).$$

The function

$$\mathfrak{B}^+_{\partial D}(\xi) := [\![\phi]\!]_{\partial D}(\xi), \quad \forall \xi \in D \backslash \widehat{\partial D},$$

respectively,

$$\mathfrak{B}^-_{\partial D}(\xi) := [\![\phi]\!]_{\partial D}(\xi), \quad \forall \xi \in X \backslash (D \cup \widehat{\partial D}),$$

is called the *interior*, respectively, *exterior*, *Bochner-Martinelli transform* of ϕ. By replacing D with $U_{[a]}(r)$, $a \in D$, the functions $[\![\phi \rfloor U]\!]_{a,r}(\xi)$, $\mathfrak{B}^+_{a,r}(\xi)$ and $\mathfrak{B}^-_{a,r}(\xi)$ are similarly defined (for sufficiently small $r > 0$).

Lemma 4.1. *Let $\phi \in C^0(D)$. For each pseudo-ball $U \subseteq D$ at $a \in D^0$ of radius r_a,*

$$[\![\phi \rfloor U]\!]_{a,r}(a) = \int\limits_{dU_{[a]}(r)} \phi\, \overline{\mathcal{K}}_a = [\phi \rfloor U]_{a,r}, \quad \forall r \in (0, r_a). \tag{4.3}$$

Proof. Let $a \in D^0$. Using the notations as in the proof of Lemma 3.1 and Theorem 4.2 of [19], the Stokes' theorem implies that, $\forall r \in (0, r_a)$,

$$\begin{aligned}\int_{\mathbb{S}_{[a']}(r)} \hat{\phi}_{j,\varepsilon}\, d\,\|\mathfrak{z}\|^2 \wedge \upsilon^{m-1} &= \int_{\mathbb{S}_{[a']}(r)} d\,(\hat{\phi}_{j,\varepsilon}\,\|\mathfrak{z}\|^2\,\upsilon^{m-1}) - d\hat{\phi}_{j,\varepsilon} \wedge \|\mathfrak{z}\|^2 \upsilon^{m-1} \\ &= -r^2 \int_{\mathbb{S}_{[a']}(r)} d\hat{\phi}_{j,\varepsilon} \wedge \upsilon^{m-1} = 0,\end{aligned}$$

where $\mathfrak{z} = z' - a'$. Since the functions $\tilde{\phi}_{j,\varepsilon}$ converge to $\tilde{\phi}_j$ uniformly on the closure of $\mathbb{B}_{[a']}(r)$, it follows that

$$\int_{dU_{[a]}(r)} \phi\, d\,\|p^{[a]}\|^2 \wedge \upsilon_p^{m-1} = \sum_{j=1}^{s} \int_{\mathbb{S}_{[a']}(r)} \hat{\phi}_j\, d\,\|\mathfrak{z}\|^2 \wedge \upsilon^{m-1} = 0.$$

Hence

$$\int_{dU_{[a]}(r)} \phi \wedge \partial\,\|p^{[a]}\|^2 \wedge \upsilon_p^{m-1} = \int_{dU_{[a]}(r)} \phi \wedge (-\bar{\partial}\,\|p^{[a]}\|^2) \wedge \upsilon_p^{m-1}.$$

Therefore from this and the identities (2.3)-(2.5) the formula (4.3) follows. ∎

Remark 4. The identities (4.3) and (2.4) imply that (for ϕ, a and U as above)

$$\int_{dU_{[a]}(r)} \phi\, \mathrm{Im}(\mathcal{K}_a) = 0, \quad \forall r \in (0, r_a). \tag{4.4}$$

Hence for every weak Stokes domain $D \subseteq X^0$, $a \in D$, and $w = p(a)$, the Stokes Theorem and Proposition 3.1 of [19] yield a refinement of the assertion (3.1) of Martinelli [12]:

$$\int_{dD} \mathrm{Im}(\mathcal{K}_a) = 0; \quad \int_{dD} \sigma_a = \sum \{\nu_p^w(a_j) \mid a_j \in D\}. \tag{4.5}$$

On account of Theorem 4.1 and Remark 1 to Theorem 4.2 of [19], the formula (4.3) yields another characterization of semi-harmonicity:

Corollary 4.1. *A locally integrable function ϕ in D is semi-harmonic iff $\phi \in C^0(D^*)$ and has the Bochner-Martinelli mean-value property at each $a \in D^*$: there exists a pseudo-ball U at a of radius r_a such that*

$$[\![\phi \rfloor U]\!]_{a,r}(a) = \nu_p(a)\, \phi(a), \quad \forall r \in (0, r_a). \tag{4.6}$$

Let $a \in X$. A solution $\phi \in C^1(D)$ of the equation

$$\bar{\partial}\phi \wedge \mathcal{K}_a = 0$$

can also be regarded as a solution to a $\bar{\partial}$-Euler equation, for, it follows from the identity (2.7) that

$$\bar{\mathcal{E}}_{\mathfrak{g}_a}(\phi)\, dp \wedge d\bar{p} = (-1)^{\frac{m(m+1)}{2}}\, \bar{\partial}\phi \wedge \mathcal{K}_a \tag{4.7}$$

locally in $D^*\backslash p^{-1}(a')$. Alternatively, let $\bar{E}_{p,a}$ denote the $\bar{\partial}$-Euler vector field associated to $\|p^{[a]}\|$ ([19], (5.2)). Then

$$\begin{aligned}\bar{E}_{p,a}(\phi)\, \upsilon^m_{p^{[a]}} &= \frac{-m}{2\pi i} \sum \bar{\mathfrak{z}}_j \,(\frac{\partial \phi}{\partial \bar{\mathfrak{z}}_j})\, d\,\mathfrak{z}_j \wedge d\,\bar{\mathfrak{z}}_j \wedge \upsilon^{m-1}_{p^{[a]}} \\ &= \frac{m}{2\pi i}\, \bar{\partial}\phi \wedge \partial\, \|p^{[a]}\|^2 \wedge \upsilon^{m-1}_{p^{[a]}}\end{aligned}$$

locally in D^*. Hence the identity (2.3) yields the relation

$$\bar{E}_{p,a}(\phi)\, \upsilon^m_{p^{[a]}} \,=\, m\, \|p^{[a]}\|^{2m}\, \bar{\partial}\phi \wedge \mathcal{K}_a \tag{4.8}$$

locally in $D^*\backslash p^{-1}(a')$.

Proposition 4.1. *Let $\phi \in C^1(D)$ and $\xi \in D^*$.* (1) *If $\bar{E}_{p,\xi}(\phi) = 0$ locally in D^*, then $\mathfrak{B}^-_{\partial G}(\xi) = 0$ for any open set $G \Subset D\backslash p^{-1}(\xi')$.* (2) *If at every $z \in D^*\backslash p^{-1}(\xi')$ there is a pseudo-ball $B = B_{[z]}(r_0)$ with $\overline{B} \subseteq D\backslash p^{-1}(\xi')$ such that $\mathfrak{B}^-_{z,r}(\xi) = 0,\ 0 < r < r_0$, then $\bar{E}_{p,\xi}(\phi) = 0$ locally in D^*.*

Proof. The assertion (1) follows from the relations (4.8), (2.3)-(2.4) and the Stokes theorem. To prove the assertion (2), note that, $\forall \xi \in X$, the Bochner-Martinelli form can be written

$$\mathcal{K}_\xi(z) = \text{const.} \sum_{j=1}^{m} (-1)^{j-1} \frac{1}{\|z' - \xi'\|^{2m-1}} \frac{\bar{\mathfrak{z}}_j}{\|z' - \xi'\|}\, d\bar{\mathfrak{z}}_{[j]} \wedge d\mathfrak{z}. \tag{4.9}$$

Hence it follows from [18], Proposition 6.2.8-(1), that, $\forall \psi \in A^{1,\mu\cap\beta}(\overline{D})$, the form $\psi \wedge \mathcal{K}_\xi$ is locally integrable on $\overline{D}$. Let $\xi \in D^*$, $z \in D^*\backslash p^{-1}(\xi')$, and $B = B_{[z]}(r_0)$ be a pseudo-ball at z with closure contained in $D\backslash p^{-1}(\xi')$ such that $\mathfrak{B}^-_{z,r}(\xi) = 0,\ \forall r \in (0, r_0)$. Then it follows as above from (4.8) that

$$\int\limits_{B_{[z]}(r)} \frac{1}{m\, \|p^{[\xi]}\|^{2m}}\, \bar{E}_{p,\xi}(\phi)\, \upsilon^m_{p^{[\xi]}} \,=\, \int\limits_{B_{[z]}(r)} \bar{\partial}\phi \wedge \mathcal{K}_\xi \,=\, 0, \quad \forall r \in (0, r_0).$$

By splitting the function $\bar{E}_{p,\xi}(\phi)$ into real and imaginary parts, this relation implies that $\bar{E}_{p,\xi}(\phi)(z) = 0$. Consequently $\bar{E}_{p,\xi}(\phi) \equiv 0$ (locally) in $D^*\backslash p^{-1}(\xi')$, hence also in D^*. ∎

A function may satisfy a specific $\bar{\partial}$-Euler equation $\bar{E}_{p,a}(\phi) = 0$ without being holomorphic; in fact, it can even be differentialble at a to a high order and real-analytic elsewhere. An example (Rudin [16], p. 63) is given below. It will be shown, however, that a function $\phi \in C^\lambda(D)$ satisfying the equation $\bar{E}_{p,a}(\phi) = 0$ is nearly harmonic at a, if $a \in D^*$ (Proposition 4.2).

Example 4.1. Let $s = \text{const.} \geq 2$. Define $\phi : \mathbb{C}^2 \to \mathbb{C}$ by

$$\phi(z) = \begin{cases} \bar{z}_1 {z_2}^{2+s} \frac{1}{\|z\|^2}, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0. \end{cases}$$

Then: i) $\phi \in C^s(\mathbb{C}^2)$; ii) $\bar{E}_{p,0}(\phi) = 0$; iii) ϕ is real analytic in $\mathbb{C}^2 \backslash \{0\}$; iv) for each fixed $z_2 = c$, the function $\phi(z_1, c)$ is bounded and non-constant, hence not harmonic in z_1, and therefore $\phi \notin \mathcal{O}(\mathbb{C}^2)$.

Observe that if $\phi \in C^\lambda(D)$ and U is a neighborhood of $a \in D^0$ such that $U_{[a]}(r_0) \Subset D$, then the Stokes' theorem ([18], (7.1.3)) and the identity (2.5) imply that

$$[\phi\rfloor U]_{a,r} - [\phi\rfloor U]_{a,s} = \int\limits_{U_{[a]}[s,r)} d\phi \wedge \sigma_a, \tag{4.10}$$

for $0 < s < r < r_0$, where $U_{[a]}[s,r) := U_{[a]}(r) \backslash U_{[a]}(s)$.

The push-forward of a locally Lipschitz function with a vanishing $\bar{\partial}$-Euler derivative admits, as will be shown, a universal Bochner-Martinelli representation. The underlying principle is a fundamental relation which, being likely of some independent interest, is given below in a general form (using the same notations as in the case of a projection mapping p).

Theorem 4.1. *Let X be a complex space of dimension $m > 0$ and $f : X \to \mathbb{C}^m$ a holomorphic map. Let $G \subset X$ be a weak Stokes domain with $dG \neq \emptyset$, and $\phi \in C^\lambda(\overline{G})$.* (1) *Assume that:* (i) $w \in \mathbb{C}^m \backslash f(\text{Spt}_{\partial G}(\phi))$ ([18], §4.2), *and* (ii) $\overline{G} \cap f^{-1}(w)$ *is discrete. Then* $\forall \xi \in G \cap f^{-1}(w)$,

$$\langle\!\langle \phi \rangle\!\rangle_{f,G}(w) = \mathfrak{B}^+_{\partial G}(\xi) - \int\limits_G \bar{\partial}\phi \wedge \mathcal{K}_\xi. \tag{4.11}$$

(2) *For every* $\xi \in X \backslash \widehat{\overline{G}}$,

$$\mathfrak{B}^-_{\partial G}(\xi) = \int\limits_G \bar{\partial}\phi \wedge \mathcal{K}_\xi. \tag{4.12}$$

Proof. Assume that $G \cap f^{-1}(w) = \{a_1, \cdots, a_s\} \neq \emptyset$. Choose at each a_j a pseudo-ball $U_{j,r} := U_{a_j}(r) \Subset G$, $\forall r \in (0, r^*)$, such that the U_j's are pairwise disjoint. Observe that for $\xi = a_1$ and $r \in (0, r^*)$,

$$\int\limits_{d(G \backslash \cup U_{j,r})} \phi\, \mathcal{K}_\xi = \int\limits_{dG} \phi\, \mathcal{K}_\xi - \sum_{j=1}^{s} [\![\phi\rfloor U_{j,r}]\!]_{a_j,r}(\xi).$$

Here the existence of the boundary integrals follows from [18], Lemma 7.1.8. Hence the Stokes' theorem and the equation (2.5) imply that

$$\llbracket\phi\rrbracket_{\partial G}(\xi) \,=\, \sum_{j=1}^{s} \llbracket\phi\rfloor U_{j,r}\rrbracket_{a_j,r}(\xi) \,+ \int\limits_{G\setminus\cup U_{j,r}} \bar\partial\phi\wedge\mathcal{K}_\xi. \tag{4.13}$$

By the identity (2.8) and Proposition 3.1 of [19],

$$\langle\!\langle\phi\rangle\!\rangle_{f,G}(w) \,=\, \lim_{r\to 0}\, \sum_{j=1}^{s} [\phi\rfloor U_{j,r}]_{a_j,r}. \tag{4.14}$$

Consequently, from the relations (4.3), (4.13)-(4.14), and the local integrability of the form $\bar\partial\phi\wedge\mathcal{K}_\xi,$ the push-forward formula (4.11) follows by letting $r\to 0$ in (4.13). The assertion (2) is an immediate consequence of the Stokes' theorem.

Let now $p : X \to \Omega$ be a semi-Riemann domain, and set $\langle\!\langle\phi\rangle\!\rangle_{p,a,r}(w) := \langle\!\langle\phi\rangle\!\rangle_{p,U_{[a]}(r)}(w),$ the local push-forward of $\phi\rfloor U,$ where U is a neighborhood of a and $r>0,$ if defined.

Theorem 4.2. *Let $Q\subseteq X$ be an open set.* (1) *Assume that $\phi\in C^\lambda(\overline{Q})$ and $w\in\Omega_0$. Then the Bochner-Martinelli representation*

$$\langle\!\langle\phi\rangle\!\rangle_{p,G}(w) \,=\, \mathfrak{B}^{+}_{\partial G}(\xi) \tag{4.15}$$

holds for every weak Stokes domain $G\subseteq Q$ with $dG\neq\emptyset$ and $\xi\in G\cap p^{-1}(w)\setminus\widehat{\partial G}$ iff ϕ satisfies the equation $\bar E_{p,\xi}(\phi)=0$ locally in Q^*. (2) *Let $\phi\in C^{1,1}(Q)$ be weakly harmonic in Q and $a\in Q^0$. For any neighborhood D of a and $\forall r_0>0$ such that $D_{[a]}(r_0)\subseteq Q^0$,*

$$p^*\langle\!\langle\phi\rangle\!\rangle_{p,a,r}(a) \,=\, [\phi\rfloor D]_{a,r} \,=\, \langle\phi\rfloor D\rangle_{a,r}, \quad \forall r\in(0,r_0). \tag{4.16}$$

Proof. (1) Let $G\subseteq Q$ be a weak Stokes domain with $dG\neq\emptyset$ and $0\notin p^{[\xi]}(\partial G)$. If $\bar E_{p,\xi}(\phi)=0$ (locally) in G^*, then by the identity (4.8), the form $\mathcal{K}_\xi\wedge\bar\partial\phi=0$ in $G^*\setminus p^{-1}(p(\xi))$. Hence the push-forward formula (4.11) yields the representation (4.15). Conversely, assume the representation (4.15) holds for ϕ. Then by the push-forward formula (4.11) and the identity (4.8), for each $z\in Q*$ and any pseudo-ball $B\Subset Q^*\setminus p^{-1}(p(\xi))$ at z,

$$\int\limits_{B_{[z]}(r)} \frac{1}{m\,\|p^{[\xi]}\|^{2m}}\bar E_{p,\xi}(\phi)\,\upsilon^m_{p^{[\xi]}} \,=\, \int\limits_{B_{[z]}(r)} \bar\partial\phi\wedge\mathcal{K}_{m,\xi} \,=\, 0,$$

for sufficiently small $r > 0$. Hence it follows as in Proposition 4.1 that $\bar{E}_{p,\xi}(\phi) \equiv 0$ in $Q^*\backslash p^{-1}(p(\xi))$, hence also in Q^*.

(2) Assume $\phi \in C^{1,1}(Q)$ is weakly harmonic in Q. Let $a \in Q^0$ and D be a neighborhood of a such that $D_{[a]}(r_0) \Subset Q^0$. By [19], Theorem 4.2, ϕ is semi-harmonic in Q, hence it follows from the formulas (5.9) and (3.7), ibid., and the identity (3.6) that

$$[\phi \rfloor D]_{a,r} = \langle \phi \rfloor D \rangle_{a,r}, \quad \forall r \in (0, r_0). \tag{4.17}$$

Let $D_{[a]}[r_0] \cap p^{-1}(a') = \{c_1, \cdots, c_l\}$, and $D_j \Subset D_{[a]}(r_0)$ a pseudo-ball at c_j such that the D_j's are pairwise disjoint. It can be shown as in Theorem 4.1 that for sufficiently small $r^* > 0$ and $W(r^*) := \cup_{j=1}^{l} D_j(r^*)$,

$$\langle\!\langle \phi \rangle\!\rangle_{p,a,r}(a') = \int\limits_{dD_{[a]}(r)} \phi\, \sigma_a - \int\limits_{D_{[a]}(r)\backslash W(r^*)} d\phi \wedge \sigma_a.$$

Thus $\forall s \in (0, r^*)$,

$$\langle\!\langle \phi \rangle\!\rangle_{p,a,r}(a') = \int\limits_{dD_{[a]}(r)} \phi\, \sigma_a - \int\limits_{D_{[a]}[s,r)} d\phi \wedge \sigma_a + \int\limits_{W(r^*)\backslash D_{[a]}[s]} d\phi \wedge \sigma_a. \tag{4.18}$$

Since

$$\sigma_a = \frac{d^c \|p^{[a]}\|^2}{\|p^{[a]}\|^{2m}} \wedge \upsilon_p^{m-1}, \tag{4.19}$$

and

$$d^c \|p^{[a]}\|^2 \wedge d\phi \wedge \upsilon_p^{m-1} = -\, d \,\|p^{[a]}\|^2 \wedge d^c\phi \wedge \upsilon_p^{m-1},$$

the semi-harmonicity of ϕ and the Stokes theorm imply that the second integral on the right-hand side of the above relation (4.18) vanishes. Also, by [18], Proposition 6.2.8-(1), the last integral in the above relation tends to zero as $r^* \to 0$. Therefore one has

$$\langle\!\langle \phi \rangle\!\rangle_{p,a,r}(a') = [\phi \rfloor D]_{a,r}, \quad \forall r \in (0, r_0). \tag{4.20}$$

Hence the formula (4.16) follows from the relations (4.17) and (4.20). ∎

Remark 5. The proof of the assertion (1) above implies the following: If $\phi \in C^{\lambda}(Q)$ and $\xi \in Q^0$, then ϕ satisfies the equation $\bar{E}_{p,\xi}(\phi) = 0$ locally in Q^* iff for every pseudo-ball $B \Subset Q^*$ (not necessarily centered at ξ) with $0 \notin p^{[\xi]}(\partial B)$, the representation (4.15) holds with $G = B$.

Lemma 4.2. *Let $D \subseteq X$ be a weak Stokes domain with $dD \neq \emptyset$, and $a \in D \cap p^{-1}(w)\backslash\widehat{\partial D} \subseteq X^0$. If $u \in C^{\lambda}(\overline{D})$ is real-valued, then:*

$$\sum \{\nu_p^w(a)\, u(a) \mid a \in D\} = \int\limits_{dD} u\, \sigma_a - \int\limits_{D} \mathrm{Re}\,(\bar{\partial} u \wedge \mathcal{K}_a). \tag{4.21}$$

$$\int_{dD} u\,\mathrm{Im}\,(\mathcal{K}_a) \;=\; \int_{D} \mathrm{Im}\,(\bar\partial u \wedge \mathcal{K}_a), \tag{4.22}$$

Proof. By the push-forward formula (4.11), one has

$$\sum \{\nu_p^w(a)\,u(a) \mid a \in D\} = \int_{dD} u\,\mathcal{K}_a - \int_{D} \bar\partial u \wedge \mathcal{K}_a. \tag{4.23}$$

The formula (4.21), resp. (4.22), follows from this representation by equating the real, resp. imaginary parts, of both sides. ■

Observe that, $\forall \phi \in C^\lambda(\overline{D}),$ the equation

$$\int_{D} \bar\partial\phi \wedge \mathcal{K}_a = \; 0 \tag{4.24}$$

can be written, owing to the identity (4.22), in an equivalent form:

$$\begin{aligned} \int_{D} \mathrm{Re}(\bar\partial u \wedge \mathcal{K}_a) &= \int_{dD} v\,\mathrm{Im}(\mathcal{K}_a), \\ \int_{D} \mathrm{Re}(\bar\partial v \wedge \mathcal{K}_a) &= \; -\int_{dD} u\,\mathrm{Im}(\mathcal{K}_a). \end{aligned} \tag{4.25}$$

Proposition 4.2. *Let $\phi \in C^{1,1}(D)$ and $a \in D^*$. If ϕ satisfies the equation $\bar E_{p,a}(\phi) = 0$ a.e. in a neighborhood of $a,$ then ϕ is nearly harmonic at $a.$*

Proof. Let $U \subseteq D$ be a pseudo-ball at $a \in D^* \cap p^{-1}(w),$ of radius $r_0,$ in which $\bar E_{p,a}(\phi) = 0$ a.e. Then by the relation (4.8),

$$\int_{U_{[a]}[s,r)} \bar\partial\phi \wedge \mathcal{K}_a = 0, \quad 0 < s < r < r_0. \tag{4.26}$$

Thus it follows from the assertion (4.4) that both integrals on the left-hand sides of the relations (4.25) must vanish. Hence by the identity (4.22) and the relations in (4.25), one has

$$\int_{U_{[a]}[s,r)} \partial\phi \wedge \overline{\mathcal{K}_a} = 0. \tag{4.27}$$

Therefore the relations (4.26), (4.27) and (4.10) imply that

$$[\phi\rfloor U]_{a,r} - [\phi\rfloor U]_{a,s} = \int_{U_{[a]}[s,r)} d\phi \wedge \sigma_a = 0, \quad 0 < s < r < r_0,$$

whence

$$[\phi\rfloor U]_{a,r} = \text{const.}, \quad \forall r \in (0, r_0).$$

From this and Lemma 3.3 of [19], the near-harmonicity of ϕ at a follows. ■

The converse of Proposition 4.2 is false, as is shown by the harmonic function $\phi : \mathbb{C}^2 \to \mathbb{C},\ \phi(z) = x_1 y_1 + i\, x_2 y_2,$ where $z = (z_1, z_2),\ z_j = x_j + iy_j,\ x_j, y_j \in \mathbb{R},\ j = 1, 2.$ Clearly one has $\bar{E}_{p,0}(\phi) \not\equiv 0$ in $\mathbb{C}^2$.

5. Applications

For a domain D in $\mathbb{C}^m$ with piece-wise smooth boundary, it is known that the only solutions to the problem of finding harmonic functions ϕ in D subject to the $\bar{\partial}$-Neumann boundary condition $(\bar{\partial}_n \phi)_{\rfloor dD} = 0$ ([10], p. 62, and [19], (5.16)) are the holomorphic functions (see [10], Theorems 14.1, 15.1, and [9], Theorem 1). By modifying the proofs of [10], this assertion can be generalized as follows:

Theorem 5.1. *Let (X, p) be a normal semi-Riemann domain and $G \subset X$ a weak Stokes domain with $dG \neq \emptyset$.* (1) *Assume that $\phi \in C^{1,1}(\overline{G}) \cap (C_0^{\lambda}(G; \mathbb{R}))^{\perp}$* (*with respect to the Dirichlet product*). *If $(\bar{\partial}_n \phi)_{\rfloor dG} = 0$, then $\phi \in \mathcal{O}(G)$.* (2) *Every weak solution $\phi \in C^{1,1}(\overline{G})$ to the homogeneous $\bar{\partial}$-Neumann problem*

$$dd^c(\phi\, \upsilon_p^{m-1}) = 0 \quad \text{in } G^*, \quad (\bar{\partial}_n \phi)_{\rfloor dG} = 0, \tag{5.1}$$

is holomorphic in G. (3) *If $\partial G \subset X^0$, then every $\phi \in \mathcal{O}(G) \cap C^{1,1}(\overline{G})$ is a solution to the above homogeneous $\bar{\partial}$-Neumann problem.*

Proof. (1) If $\phi \in C^{1,1}(\overline{G}) \cap (C_0^{\lambda}(G; \mathbb{R}))^{\perp}$, then ϕ is semi-harmonic in G by [19], Proposition 5.2. Hence by the Stokes' theorem, one has

$$\begin{aligned} \langle \phi, \phi \rangle_G &= \frac{i}{2\pi}\Big(\int_{dG} \bar{\phi}\ \bar{\partial}\phi \wedge \upsilon_p^{m-1} - \int_G \bar{\phi}\, \partial\bar{\partial}\phi \wedge \upsilon_p^{m-1} \Big) \\ &= \text{Const.} \int_{dG} \bar{\phi}\, \mu_\phi, \end{aligned} \tag{5.2}$$

where the $(m-1, m)$-form μ_ϕ is given by

$$\mu_\phi := \sum_{k=1}^{m} (-1)^{m+k-1} \Big(\frac{\partial \phi}{\partial \bar{p}_k}\Big)\, dp_{[k]} \wedge d\bar{p}. \tag{5.3}$$

Thus the identity

$$(5.4)\qquad (\bar\partial_n \phi)\, d\sigma_{U\cap dG} \;=\; 2^{1-m} i^{-m}\, \mathrm{j}^*_{U\cap dG}\, \mu_\phi$$

((5.19), ibid.) and the $\bar\partial$-Neumann condition $(\bar\partial_n \phi)_{\rfloor dG} = 0$ imply that $|\langle\phi\rangle_G| = 0.$ Therefore the desired conclusion follows from Proposition 3.1

(2) If ϕ is a weak solution to the $\bar\partial$-Neumann problem (5.1), then by Theorem 4.2, ibid., ϕ is semi-harmonic in G. Hence it follows as in the preceding that $\phi \in \mathcal{O}(G)$.

(3) Observe that the relation

$$(5.5)\qquad \mu_\phi \;=\; i^m 2^{m-1} * (\bar\partial\phi)$$

holds in G^* for all $\phi \in C^1(G)$. Assume now that $\phi \in \mathcal{O}(G) \cap C^{1,1}(\overline{G})$ and $\xi \in X\backslash\widehat{\partial G}$. There exists a neighborhood W of ∂G such that $\xi \notin p^{-1}(p(W))$. Let $\mathfrak{g}_\xi$, $\xi \notin \widehat{\partial G}$, be the Newtonian functions given by (2.6). By the identities (5.4) and (5.5), for each $\xi \notin \widehat{\partial G}$ the integral

$$\int\limits_{dG} \mathfrak{g}_\xi(z)\, (\bar\partial_n\phi)(z)\, d\sigma_{dG}(z) \;=\; \int\limits_{dG} \mathfrak{g}_\xi(z)\; \mathrm{j}^*_{dD}(*\,\bar\partial\phi)(z)$$

exists. Thus for such ξ the Stokes' theorem and the ∂-closedness of the form $*\,\bar\partial\phi$ yield

$$\int\limits_{dG} \mathfrak{g}_\xi(z)\, (\bar\partial_n\phi)(z)\, d\sigma_{dG}(z) \;=\; \int\limits_{G} \partial\, \mathfrak{g}_\xi \wedge *\,\bar\partial\phi.$$

Therefore if $\phi \in \mathcal{A}(G)$, then

$$(5.6)\qquad \int_{dG} \mathfrak{g}_\xi(z)\, (\bar\partial_n\phi)(z)\, d\sigma_{dG}(z) = 0, \quad \forall \xi \notin \widehat{\partial G}.$$

To show that the function $(\bar\partial_n\phi)_{\rfloor dG}$ annihilates (as a distribution) all functions $f \in C_0^\infty(dG) \cap G^*$, by using a C^∞-partition of unity one needs only consider the case where the f is supported in a branch U^μ of a pseudo-ball at a point of $(dG) \cap G^*$. Note that the set U^μ being biholomorphic under p to an open ball in $\mathbb{C}^m$, the Keldysh-Lavrentiev Lemma ([1], 1.5, and [11], p. 347, if m = 1) asserts that the function f can be uniformly approximated on dG by elements of the form $\mathfrak{g}_{\xi_k}$, $\xi_k \in U^\mu\backslash\widehat{\partial D}$. Consequently in view of the equation (5.6), one concludes that the $\bar\partial$-Neumann condition $(\bar\partial_n\phi)_{\rfloor dG} = 0$ is satisfied. ∎

Remark 6. Let (X, p) and G be as in the assertion (2) above. Assume that $\rho \in A^{2m,0}(G\backslash S)$, where S is thin analytic in G, and $\eta \in C^\lambda(\partial G)$. If $\psi = \psi_0 \in C^{1,1}(\overline{G})$ is a weak solution to the $\bar\partial$-Neumann problem

$$dd^c(\psi\, \upsilon_p^{m-1}) = \rho \quad \text{in } G^*, \qquad (\bar\partial_n\psi)_{\rfloor dG} = \eta,$$

then the set of all weak solutions in $C^{1,1}(\overline{G})$ to the $\bar{\partial}$-Neumann problem is $\{\psi_0 + \phi \mid \phi \in \mathcal{O}(G) \cap C^{1,1}(\overline{G})\}$.

Corollary 5.1. *Let (X, p) be a normal semi-Riemann domain and T a thin analytic subset of an open set $D \subseteq X$ with $G := D \backslash T \subseteq X^*$. Assume that $\phi \in C^{\lambda}(G) \cap C^{\beta}(D)$. If $\bar{E}_{p,\xi}(\phi) = 0$ in G, $\forall \xi \in G$, then ϕ is holomorphic in D.*

Proof. For each $a \in G$ choose a pseudo-ball B_a with $\overline{B}_a \subseteq G$. It follows from the identities (5.2) and (5.17) of [19] that

$$(\bar{\partial}_n \phi)\rfloor_{\partial B_a} = \frac{1}{\rho_a} \bar{E}_{p,a}(\phi).$$

Since the function ϕ satisfies in B_a the equation $\bar{E}_{p,\xi}(\phi) = 0$, $\forall \xi \in B_a$, the push-forwatd formula (4.11) and the identity (4.8) imply that

$$\phi(\xi) = \int_{dB_a} \phi(\zeta)\, \mathcal{K}_\xi(\zeta), \quad \forall \xi \in B_a \backslash \widehat{\partial B_a}.$$

This relation implies that the function ϕ is semi-harmonic in B_a. Thus ϕ is a solution to the homogeneous $\bar{\partial}$-Neumann problem (5.1). Therefore by Theorem 5.1-(2), ϕ is holomorphic in B_a. It follows then from the Riemann's extension theorem that ϕ is holomorphic in D.

Theorem 5.2. (A jump formula) *Let (X, p) be a Riemann domain and $G \subset X$ a weak Stokes domain with $dG \neq \emptyset$. If $F \in C^{1,1}(\overline{G})$, then the Bochner-Martinelli transforms of $\phi := F\rfloor \partial G$, $\mathfrak{B}^+_{\partial G}(\xi)$ and $\mathfrak{B}^-_{\partial G}(\xi)$, have continuous extensions (denoted by the same) to $\overline{G} \backslash (\widehat{\partial G} \cap G)$, resp. X. Moreovr, the function $p^* \langle\!\langle \phi \rangle\!\rangle_{p,G}$ has a continuous extension (denoted by the same) to $\overline{G} \backslash (\widehat{\partial G} \cap G)$ with*

$$p^* \langle\!\langle F \rangle\!\rangle_{p,G}(\xi) = \mathfrak{B}^+_{\partial G}(\xi) - \mathfrak{B}^-_{\partial G}(\xi), \quad \forall \xi \in \overline{G} \backslash (\widehat{\partial G} \cap G). \tag{5.7}$$

Proof. Choose an open covering $\{U_\mu\}_{\mu=1}^{l}$, consisting of pseudo-balls U_μ, of $\overline{G}$, and a subordinated C^∞-partition of unity $\{\alpha_\mu\}_{\mu=1}^{l}$ on $\overline{G}$. Set $F_\mu := \alpha_\mu F \in C_0^{1,1}(U_\mu)$. It is shown in Proposition 4.1 that the integral

$$T_G(\xi) := \int_G \bar{\partial} F \wedge \mathcal{K}_\xi = \sum_{\mu,j} \int_{G \cap V^{\mu,j}} \bar{\partial} F_\mu \wedge \mathcal{K}_\xi, \quad \forall \xi \in X,$$

exists, where $V^{\mu,j}$, $1 \leq j \leq s_a$, are the branches of U_μ. Moreover, the push-forward formula (4.11) applied to the function F yields

$$\langle\!\langle F\rangle\!\rangle_{p,G}(p(\xi)) \,=\, \mathfrak{B}^+_{\partial G}(\xi) \,-\, T_G(\xi), \quad \forall \xi \in G\backslash\widehat{\partial G}.$$

Also, if $\xi \in X\backslash(G \cup \widehat{\partial G})$, one has, by the formula (4.12),

$$\mathfrak{B}^-_{\partial G}(\xi) \,=\, T_G(\xi).$$

On account of the expression (4.9), the function $T_G(\xi)$, hence also $\mathfrak{B}^-_{\partial G}(\xi)$, extends continuously to X according to [18], Proposition 6.2.8-(2). Therefore, it follows from the continuity of the fiber sum $p^*\langle\!\langle F\rangle\!\rangle_{p,G}$ on $X\backslash\widehat{\partial G}$ (Theorem 5.1.2, ibid.) that the function $\mathfrak{B}^+_{\partial G}(\xi)$ is continuous in $G\backslash\widehat{\partial G}$.

Let $\{\lambda_\mu\}^l_{\mu=1}$ be a C^∞-partition of unity on ∂G subordinated to an open covering $\{U_\mu\}^l_{\mu=1}$ by open sets of the same type as above. Set $\tilde F_\mu := \lambda_\mu F \in C^{1,1}_0(U_\mu)$. Let $V^{\mu,j}$, $1 \le j \le s_a$, be the branches of U_μ. By [1], Proposition 0.10, the function

$$M_{\mu,j}(\xi) \,:=\, \int\limits_{dG\cap V^{\mu,j}} \tilde F_\mu(\zeta)\, \mathfrak{g}_\xi(\zeta)\, d\bar p_{[k]} \wedge dp, \quad \xi \in G\backslash\widehat{\partial G},$$

has a C^1-extension to U_μ, if $m > 1$; the case $m = 1$ is similar (cf. [6], pp. 169-174; a different proof will be given elsewhere). The Bochner-Martinelli transform $\mathfrak{B}^+_{\partial G}(\xi)$ is a linear combination of derivatives of functions of the form $M_{\mu,j}(\xi)$, and consequently extends continuously to a neighborhood of ∂G. It follows that the function $p^*\langle\!\langle F\rangle\!\rangle_{p,G}$ has a continuous extension to $\overline{G}\backslash(\widehat{\partial G} \cap G)$, in terms of which the jump relation (5.7) holds. ■

Lemma 5.1. (Cf. Aronov-Kytmanov Theorem [10], pp. 159-160) *Let (X,p) be a Riemann domain and $G \subset X$ a weak Stokes domain with $dG \neq \emptyset$. If $\phi \in \mathfrak{H}_w(G) \cap C^{1,1}(\overline{G})$ and the push-forward of ϕ admits the Bochner-Martinelli representation*

(5.8) $$p^*\langle\!\langle \phi\rangle\!\rangle_{p,G}(\xi) \,=\, \mathfrak{B}^+_{\partial G}(\xi), \quad \forall \xi \in G\backslash\widehat{\partial G},$$

then $(\bar\partial_n \phi)_{\rfloor dG} = 0.$

Proof. As in Theorem 5.2, if $\phi \in C^1(\overline{G})$ the integral

$$T_G(\xi) := \int\limits_G \bar\partial\phi \wedge \mathcal{K}_\xi, \quad \xi \in X,$$

exists. Also, it follows from the identity (4.7) and the definition (5.3) that

(5.9) $$\bar\partial\phi \wedge \mathcal{K}_\xi \,=\, \partial\, \hat{\mathfrak{g}}_\xi \wedge \mu_\phi,$$

locally in $G^*\backslash p^{-1}(\xi')$, where $\hat{\mathfrak{g}}_\xi := (-1)^{\frac{m(m-1)}{2}}\,\mathfrak{g}_\xi$. The Stokes' theorem together with the identities (5.4) and (5.9) imply that

$$\int_{dG} (\bar{\partial}_n\phi)(z)\,\hat{\mathfrak{g}}_\xi(z)\,d\sigma_{dG}(z) = (-1)^m\,2^{1-m}i^m\,T_G(\xi) \tag{5.10}$$

for every $\xi \notin \widehat{\partial G}$. The push-forward formula (4.11) and the representation (5.8) imply that $T_G(\xi) = 0$ on $G\backslash\widehat{\partial G}$. Let $\{\xi_n\}$ be a sequence in $G\backslash\widehat{\partial G}$ converging to a point $\xi \in \partial G$. Then the Bochner-Martinelli transform $\mathfrak{B}^+_{\partial G}(\xi_n) = p^*\langle\!\langle\phi\rangle\!\rangle_{p,G}(\xi_n)$. Hence by the jump formula (5.7), the continuous extension of the exterior transform

$$\mathfrak{B}^-_{\partial G}(\xi) = [\![\phi]\!]_{\partial G}(\xi), \quad \xi \in Y := X\backslash(G\cup\widehat{\partial G}),$$

vanishes on ∂G. Also, $\lim_{\|\xi'\|\to\infty}\mathfrak{B}^-_{\partial G}(\xi) = 0$. Let Y_0 be the union of all component(s) Y^j of Y with $Y^j\cap\partial G \neq \emptyset$. Since the function $\mathfrak{B}^-_{\partial G}$ is semi-harmonic in Y, Corollary 4.2 of [19] implies that $\mathfrak{B}^-_{\partial G} \equiv 0$ in Y_0. For each $\xi \in Y$, the form $\phi\,\mathcal{K}_{m,\xi}$ being smooth in $\overline{G}$, the Stokes' theorem shows that $T_G(\xi) = 0$, $\forall\xi \in Y_0$. Thus the relation (5.10) implies that, for every ξ in a neighborhood of ∂G, ϕ satisfies the equation (5.6). Consequently it follows as in Theorem 5.1 that $(\bar{\partial}_n\phi)_{\rfloor dG} = 0$. ∎

Remark 7. Let (X,p) be a Riemann domain and $G \subset X$ a weak Stokes domain with $dG \neq \emptyset$. Assume that $\phi \in C^{1,1}(\overline{G})$. Then by the identity (5.9),

$$\bar{\partial}\phi\wedge\mathcal{K}_\xi = d\,(\hat{\mathfrak{g}}_\xi\,\mu_\phi) - \frac{1}{4}\,\hat{\mathfrak{g}}_\xi\,(\triangle_p\phi)\,d\bar{p}\wedge dp$$

locally in $G^*\backslash p^{-1}(\xi')$. Therefore for each $w \in \mathbb{C}^m\backslash p\,(\mathrm{Spt}_{\partial G}(\phi))$, the push-forward formula (4.11) admits an alternative form: for all $\xi \in G\cap p^{-1}(w)$,

$$\langle\!\langle\phi\rangle\!\rangle_{p,G}(w) = \int_{dG}(\phi\,\mathcal{K}_\xi - \hat{\mathfrak{g}}_\xi\,\mu_\phi) - \frac{1}{4}\int_G \hat{\mathfrak{g}}_\xi\,(\triangle_p\phi)\,d\bar{p}\wedge dp. \tag{5.11}$$

Theorem 5.3. (Cf. Aronov-Kytmanov [3], Theorem 1). *Let (X,p) be a normal Riemann domain and $G \subset X$ a weak Stokes domain with $dG \neq \emptyset$. Assume that $\phi \in \mathfrak{H}_w(G)\cap C^{1,1}(\overline{G})$. Then the push-forward of ϕ admits the Bochner-Martinelli representation* (5.8) *iff $\phi \in \mathcal{O}(G)$.*

Proof. If $\phi \in \mathcal{O}(G)$, then the push-forward of ϕ admits the representation (5.8) by the formula (4.11). Conversely, by Lemma 5.1, the Bochner-Martinelli representation (5.8) implies that the $\bar{\partial}$-Neumann derivative $(\bar{\partial}_n\phi)_{\rfloor dG} = 0$. Hence from part (2) of Theorem 5.1 the holomorphicity of ϕ in G follows. ∎

In the following let $\pi : X \to \mathbb{C}^m$ be a (finite) analytic covering map. For a continuous function $\phi : X \to \mathbb{R}$, set $M(\phi, r) := \sup_{X[r]} \phi$. A well-known theorem of Bôcher ([4], p. 50) characterizes real-valued harmonic functions which are positive near an isolated singularity. It is of some interest to see to what extent this theorem, in its generalized form given by Axler, Bourdon and Ramey ([4], 9.11), carries over to a semi-harmonic function. A step in helping towards this goal might be the following (cf. [4], 9.10):

Theorem 5.4. (Generalized Liouville property I). *Assume that:* (i) $\phi : X \to \mathbb{R}$ *is semi-harmonic;* (ii) *there exists* $a_0 \in X$ *such that*

$$\liminf_{\|\pi^{[a_0]}(\xi)\| \to \infty} \frac{\phi(\xi)}{\|\pi^{[a_0]}(\xi)\|} \geq 0.$$

Then the push-forward $\langle\!\langle \phi \rangle\!\rangle_\pi$ *is constant on* $\mathbb{C}^m$.

Proof. Let $a \in X$ and $a' = \pi(a)$. For a given $\varepsilon > 0$, choose $R > \|a'\|$ such that $\phi(\xi) \geq -\varepsilon \|\pi^{[a_0]}(\xi)\|$, $\forall \xi \in X$ with $\|\pi^{[a_0]}(\xi)\| > R - \|a'\|$. Let $S_{[a,a_0]}(R)$ be the symmetric difference of $B_{[a]}(R)$ and $B_{[a_0]}(R)$. The push-forward formula (4.16) (with $U = X$) implies that

$$|\langle\!\langle \phi \rangle\!\rangle_{\pi,a,R}(a') - \langle\!\langle \phi \rangle\!\rangle_{\pi,a_0,R}(a_0')| \leq \frac{1}{R^{2m}} \int\limits_{S_{[a,a_0]}(R)} |\phi| \, \upsilon_\pi^m.$$

The above last integral can be estimated by following an idea of [4], p. 198. Note that for each y in the annulus $A(a_0) := B_{[a_0]}(R + \|a'\|) \backslash B_{[a_0]}(R - \|a'\|)$, one has $R - \|a'\| < \|\pi^{[a_0]}(y)\| < R + \|a'\|$; hence $\phi(y) \geq -\varepsilon \|\pi^{[a_0]}(y)\|$. Therefore, for such y,

$$|\phi(y)| \leq (\phi(y) + \varepsilon \|\pi^{[a_0]}(y)\|) + \varepsilon \|\pi^{[a_0]}(y)\| \leq \phi(y) + 4\varepsilon R.$$

Hence

$$|\langle\!\langle \phi \rangle\!\rangle_{\pi,a,R}(a') - \langle\!\langle \phi \rangle\!\rangle_{\pi,a_0,R}(a_0')| \leq \frac{1}{R^{2m}} \int\limits_{A(a_0)} (\phi + 4\varepsilon R) \upsilon_\pi^m.$$

Consequently it follows from Theorem 5.2.2 of [18] and the sheet number formula ([19], (2.4)) that

$$\begin{aligned} &|\langle\!\langle \phi \rangle\!\rangle_{\pi,a,R}(a') - \langle\!\langle \phi \rangle\!\rangle_{\pi,a_0,R}(a_0')| \\ &\qquad \leq \frac{(R + \|a'\|)^{2m} - (R - \|a'\|)^{2m}}{R^{2m}} \deg(\pi) \, (M + 4\varepsilon R), \end{aligned}$$

where $M := \sup\{|\phi(y)| \mid \|\pi^{[a_0]}(y)\| \leq R + \|a'\|\}$. Thus letting $R \to \infty$ yields

$$|\langle\!\langle \phi \rangle\!\rangle_\pi(a') - \langle\!\langle \phi \rangle\!\rangle_\pi(a'_0)| \leq 8\, m\, \varepsilon \deg(\pi)\, \|a'\|.$$

Now taking the limit as $\varepsilon \to 0$, one has $|\langle\!\langle \phi \rangle\!\rangle_\pi(a') - \langle\!\langle \phi \rangle\!\rangle_\pi(a'_0)| \leq 0$. Thus $\langle\!\langle \phi \rangle\!\rangle_\pi(a') = \langle\!\langle \phi \rangle\!\rangle_\pi(a'_0) =$ constant. ∎

Theorem 5.5. (Generalized Liouville property II). *Assume that:* (i) $\phi = (\phi_1, \cdots, \phi_N) : X \to \mathbb{C}^N$ *is semi-harmonic* (*i.e., each* ϕ_j *is semi-harmonic*)*;* (ii) *there exists* $\alpha \in (0,1)$ *such that either* $\limsup_{r\to\infty} M(\|\phi\|, r)/r^\alpha < \infty$ *or each* ϕ_j *is real-valued wih* $\limsup_{r\to\infty} M(\phi_j, r)/r^\alpha < \infty$. *Then the push-forward* $\langle\!\langle \phi \rangle\!\rangle_\pi$ *is constant on* $\mathbb{C}^m$.

Proof. Let a_0 and a be distinct points of X and set $d = \|a' - a'_0\|$. By the assumption (ii), there exists a constant K such that, for sufficiently large $r > d$,

$$\|\phi_j(\xi)\| \leq K\, r^\alpha \quad (\text{resp.,}\ \ \phi_j(\xi) \leq K\, r^\alpha), \quad 1 \leq j \leq N, \tag{5.12}$$

on $X(r)$. Since $X_{[a]}(r) \subseteq X(r + \|a'\|)$, the inequality $\|\phi_j(\xi)\| \leq K(r + \|a'\|)^\alpha$ (resp., $\phi_j(\xi) \leq K(r + \|a'\|)^\alpha$) holds for every $\xi \in X_{[a]}(r)$. Hence the inequality (5.12) remains valid on $X_{[a]}(r)$ (with a suitable constant K).

Let $\tilde{\pi} = \pi - a'_0$, $S = \{\xi \in X \mid \nu_{\tilde{\pi}}(\xi) > 1\}$, and $E = \tilde{\pi}(S)$. Then the mapping $\tilde{\pi} : \hat{X} = X \backslash E_\pi \to \mathbb{C}^m \backslash E$ is proper, holomorphic, and locally topological. Setting $u_j := -\mathrm{Re}\,(\phi_j)$ and noting that $X_{[a_0]}(r-d) \subseteq X_{[a]}(r)$, one has

$$\int\limits_{\hat{X}_{[a]}(r)} (u_j + K\, r^\alpha)\, \upsilon_\pi^m \ \geq \int\limits_{\hat{X}_{[a_0]}(r-d)} (u_j + K\, r^\alpha)\, \upsilon_\pi^m.$$

Thus by the definition (4.1) (with $U = X$),

$$r^{2m}\langle u_j \rangle_{a,r} - (r-d)^{2m}\langle u_j \rangle_{a_0, r-d} + K\, r^\alpha \Big(\int\limits_{\hat{X}_{[a]}(r)} \upsilon_\pi^m - \int\limits_{\hat{X}_{[a_0]}(r-d)} \upsilon_\pi^m \Big) \geq 0.$$

The above volume difference can be estimated by using an idea of [6], p. 94: setting $\tilde{\xi} = \tilde{\pi}(\xi)$, $\xi \in \hat{X}$, one has

$$\begin{aligned}
\int\limits_{\hat{X}_{[a]}(r)} \upsilon_\pi^m - \int\limits_{\hat{X}_{[a_0]}(r-d)} \upsilon_\pi^m &= \int\limits_{\{\|\tilde{\xi}\| > r-d,\ \|\tilde{\xi} - \tilde{a}\| < r\}} \upsilon_{\tilde{\pi}}^m \\
&\leq \int\limits_{\{r-d < \|\tilde{\xi}\| < r + \|\tilde{a}\|\}} \upsilon_{\tilde{\pi}}^m \\
&= 2m \deg(\pi) \int_{r-d}^{r+\|\tilde{a}\|} \rho^{2m-1}\, d\rho.
\end{aligned}$$

Therefore by the push-forward formula (4.16),

$$\begin{aligned}\langle\!\langle u_j \rangle\!\rangle_{\pi,a,r}(a') - (\frac{r-d}{r})^{2m} \langle\!\langle u_j \rangle\!\rangle_{\pi,a_0,r-d}(a'_0) \\ \geq - \frac{K \deg(\pi)}{r^{2m-\alpha}} ((r + \|\tilde{a}\|)^{2m} - (r-d)^{2m}).\end{aligned}$$

Letting $r \to \infty$, the above inequality implies that $\langle\!\langle u_j \rangle\!\rangle_\pi (a') - \langle\!\langle u_j \rangle\!\rangle_\pi (a'_0) \geq 0.$ Since $X_{[a]}(r-d) \subseteq X_{[a_0]}(r)$, the preceding argument shows that $\langle\!\langle u_j \rangle\!\rangle_\pi (a'_0) - \langle\!\langle u_j \rangle\!\rangle_\pi (a') \geq 0$. Therefore $\langle\!\langle u_j \rangle\!\rangle_\pi (a') =$ constant on $\mathbb{C}^m$. Clearly the same is true for the imaginary part of ϕ_j. ∎

References

1. L. A. Aizenberg and A. P. Yuzhakov, Integral representations and residues in multidimensional complex analysis, Translations of Mathematical Monographs, *Amer. Math. Soc.*, **58** (1983).

2. A. Andreotti and W. Stoll, *Analytic and algebraic dependence of meromorphic functions*, Lecture Notes in Math. **234**, Springer, Berlin-Heidelberg-New York, 1971.

3. A. M. Aronov and A. M. Kytmanov, The holomorphy of functions that are representable by the Martinelli-Bochner Integral, *Funkcional Anal. i Priložen*, **9(3)** (1975), 83-84; *English Transl. in Functional Ana. Appl.*, **9** (1975), 254-255.

4. S. P. Bourdon Axler and W. Ramey, *Harmonic Function Theory*, Graduate Texts in Math., **137**, Second edition, Springer, Berlin-Heidelberg-New York, 2001.

5. H. Behnke and H. Grauert, *Analysis in non-compact complex spaces*, Analytic functions (H. Behnke and H. Grauert, ed.), Princeton Univ. Press, Princeton, N.J., 1960, pp. 11-44.

6. G. B. Folland, *Introduction to Partial Differential Equations*, Math. Notes **17**, Princeton Univ. Press, Princeton, N.J, 1976.

7. J. Fornæss and B. Stensønes, *Lectures on counterexamples in several complex variables*, Math. Notes **33** Princeton Univ. Press, Princeton, N.J., 1987.

8. H. Grauert and R. Remmert, *Theory of Stein Spaces*, Grundl. Math. Wiss. **236**, Springer, Berlin-Heidelberg-New York, 1979.

9. A. M. Kytmanov, On the $\bar{\partial}$-Neumann problem for smooth functions and distributions, *Mat. Sb.* **181(5)** (1990), 656-668; *Engl. translation, Math. USSR Sb.* **70(1)** (1991), 79-92.

10. A. M. Kytmanov, *The Bochner-Martinelli Integrals and its Applications*, Birkhauser-Verlag, Basel, 1995.

11. N. S. Landkof, *Foundations of modern potential theory*, Nauka, Moskow 1966; English translation, Springer-Verlag, 1972.

12. E. Martinelli, Onalche riflessione sulla rapresentazione integrale di massima dimensione per le funzioni di più variabili complesse, *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur.*, **76(4)** (1984), 235-242, (1985).

13. O. Martio, S. Rickman and J. Väisälä, Definitions for quasiregular mappings, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **448** (1969), 1-40.

14. T. Radò and P. Reichelderfer, *Continuous transformations in analysis*, Springer-Verlag, Berlin-Heidelberg, 1955.

15. R. Remmert, Holomorphe und meromorphe Abbildungen komplexer Räume, *Math. Ann.*, **133** (1957), 328-370.

16. W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Grundl. Math. Wiss., **241**, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

17. W. Stoll, The multiplicity of a holomorphic map, *Invent. Math.*, **2** (1966), 15-58.

18. C. Tung, *The first main theorem of value distribution on complex spaces*, Memoire dell'Accademia Nationale dei Lincei, Serie VIII, Vol. **XV** (1979), Sez. 1, 91-263.

19. C. Tung, *Semi-harmonicity, integral means and Euler type vector fields*, Advances in Applied Clifford Algebras, **17** (2007), 555-573.

Chia-Chi Tung
Department of Mathematics and Statistics,
Minnesota State University,
Mankato, MN 56001,
U.S.A.
E-mail: chia.tung@mnsu.edu