TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 5, pp. 1537-1582, October 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

SOME NEW ITERATIVE ALGORITHMS FOR GENERALIZED MIXED EQUILIBRIUM PROBLEMS WITH STRICT PSEUDO-CONTRACTIONS AND MONOTONE MAPPINGS

Jian-Wen Peng[1] and Jen-Chih Yao[2]*

Dedicated to Professor Wataru Takahashi on the occasion of his 65^{th} birthday.

Abstract. In this paper, we propose some parallel and cyclic algorithms based on the extragradient method (nonextragradient method) for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a finite family of strict pseudo-contractions and the set of the variational inequality for a monotone, Lipschitz continuous mapping (an inverse strongly monotone mapping). We obtain some weak and strong convergence theorems for the sequences generated by these processes in Hilbert spaces. The results in this paper generalize, improve and unify some well-known results in the literature.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and induced norm $\|.\|$ and let C be a nonempty closed convex subset of H. let $B : C \to H$ be a nonlinear mapping and let $\varphi : C \to R \cup \{+\infty\}$ be a function and F be a bifunction from $C \times C$ to R, where R is the set of real numbers. Peng and Yao [1] considered the following generalized mixed equilibrium problem:

$$(1.1) \quad \text{Finding } x \in C \text{ such that } F(x, y) + \varphi(y) + \langle Bx, y - x\rangle \geq \varphi(x), \forall y \in C.$$

Received May 3, 2008, accepted July 15, 2008.
Communicated by Mau-Hsiang Shih.
2000 *Mathematics Subject Classification*: 47H05, 47H10, 47H17.
Key words and phrases: Generalized mixed equilibrium problem, Extragradient method, Parallel algorithms, Cyclic algorithms, Strict pseudo-contraction, Monotone mapping.
[1]The first author was supported by the National Natural Science Foundation of China (Grant No. 10471159), the Science and Technology Research Project of Chinese Ministry of Education (Grant No. 206123), the Education Committee project Research Foundation of Chongqing (Grant No. KJ070816).
[2]The second author was partially supported by the grant NSC96-2628-E-110-014-MY3.
*Corresponding author.

The set of solutions of (1.1) is denoted by $GMEP(F, \varphi, B)$. It is easy to see that $x \in \mathrm{GMEP}(F, \varphi, B)$ implies that $x \in \mathrm{dom}\varphi = \{x \in C | \varphi(x) < +\infty\}$.

If $B = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following mixed equilibrium problem:

$$\text{(1.2)} \qquad \text{Find } \ x \in C \text{ such that } F(x, y) + \varphi(y) \geq \varphi(x), \ \forall y \in C.$$

Problem (1.2) was studied by Ceng and Yao [2] and Bigi, Castellani and Kassay [3]. The set of solutions of (1.2) is denoted by $MEP(F, \varphi)$.

If $\varphi = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following generalized equilibrium problem:

$$\text{(1.3)} \qquad \text{Finding } \ x \in C \text{ such that } F(x, y) + \langle Bx, y - x\rangle \geq 0, \forall y \in C.$$

Problem (1.2) was studied by Takahashi and Takahashi [4]. The set of solutions of (1.3) is denoted by $GEP(F, B)$.

If $\varphi = 0$ and $B = 0$, then the generalized mixed equilibrium problem (1.1) becomes the following equilibrium problem:

$$\text{(1.4)} \qquad \text{Finding } \ x \in C \text{ such that } F(x, y) \geq 0, \forall y \in C.$$

The set of solutions of (1.4) is denoted by $EP(F)$.

If $F(x, y) = 0$ for all $x, y \in C$, the generalized mixed equilibrium problem (1.1) becomes the following generalized variational inequality problem:

$$\text{(1.5)} \qquad \text{Finding } \ x \in C \text{ such that } \varphi(y) + \langle Bx, y - x\rangle \geq \varphi(x), \ \forall y \in C.$$

The set of solutions of (1.5) is denoted by $GVI(C, B, \varphi)$.

If $\varphi = 0$ and $F(x, y) = 0$ for all $x, y \in C$, the generalized mixed equilibrium problem (1.1) becomes the following variational inequality problem:

$$\text{(1.6)} \qquad \text{Finding } \ x \in C \text{ such that } \langle Bx, y - x\rangle \geq 0, \forall y \in C.$$

The set of solutions of (1.6) is denoted by $VI(C, B)$.

If $B = 0$ and $F(x, y) = 0$ for all $x, y \in C$, the generalized mixed equilibrium problem (1.1) becomes the following minimize problem:

$$\text{(1.7)} \qquad \text{Finding } \ x \in C \text{ such that } \varphi(y) \geq \varphi(x), \forall y \in C.$$

The set of solutions of (1.7) is denoted by $Argmin(\varphi)$.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others; see for instance, [1-6].

Recall that a mapping $T : C \to C$ is said to be a κ-strict pseudo-contraction [7] if there exists $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \forall x, y \in C,$$

where I denotes the identity operator on C. When $\kappa = 0$, T is said to be nonexpansive [8], and it is said to be a pseudo-contraction if $\kappa = 1$. Clearly, the class of κ-strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contractions. It is easy to see that T is a pseudo-contraction if and only if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2, \forall x, y \in C.$$

We denote the set of fixed points of T by $Fix(T)$.

Peng and Yao [1] introduced an iterative scheme for finding a common element of the set of solution of problem (1.1), the set of fixed points of a nonexpansive mapping and the set of the variational inequality for a monotone, Lipschitz continuous mapping and obtain a strong convergence theorem. Ceng and Yao [2] introduced an iterative scheme for finding a common element of the set of solution of problem (1.2) and the set of common fixed points of a family of finitely nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem. Takahashi and Takahashi [4] introduced an iterative scheme for finding a common element of the set of solution of problem (1.3) and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a strong convergence theorem. Some methods have been proposed to solve the problem (1.4); see, for instance, [5, 6, 9-16, 27-29] and the references therein. Recently, Combettes and Hirstoaga [9] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Takahashi and Takahashi [10] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution of problem (1.4) and the set of fixed points of a nonexpansive mapping and proved a strong convergence theorem in a Hilbert space. Peng and Yao [11] introduced a hybrid iterative scheme for finding the common element of the set of fixed points of a family of infinitely nonexpansive mappings, the set of an equilibrium problem and the set of solutions of variational inequality problem for an α-inverse strongly monotone mapping. Tada and Takahashi [12] introduced some iterative schemes for finding a common element of the set of solution of problem (1.4) and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem. Ceng, AI-Homidan, Ansari and Yao [14] introduced an iterative algorithm for finding a common element of the set of solution of problem (1.4) and the set of fixed points of a strict pseudo-contraction mapping. Plubtieng and Punpaeng [15] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings, the set of an equilibrium problem and the set of solutions of variational inequality problem for α-inverse strongly monotone mappings. Chang, Joseph Lee and Chan [16] introduced some iterative processes based on the extragradient method for finding the common element of the set of fixed points of a family of infinitely nonexpansive mappings, the set of an equilibrium problem and the set of solutions of variational inequality problem for an α-inverse strongly monotone mapping. Yao, Liou and Yao [27] and Ceng and Yao [28] introduced some iterative viscosity approximation schemes for finding a common element of the set of an equilibrium problem and the set of fixed points of infinitely nonexpansive mappings in a Hilbert space.

On the other hand, Marino and Xu [17] and Zhou [18] introduced and researched an iterative scheme for finding a fixed point of a strict pseudo-contraction mapping. Acedoa and Xu [19] introduced the following parallel algorithm for finding a common fixed point of a family of finite strict pseudo-contraction mappings $\{T_j\}_{j=1}^{N}$:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{j=1}^{N} \zeta_j^{(n)} T_j x_n. \tag{1.8}$$

Acedoa and Xu [19] also introduced the following cyclic algorithm for finding a common fixed point of a family of finite strict pseudo-contraction mappings $\{T_j\}_{j=0}^{N-1}$:

$$\begin{cases} x_0 &= x \in C, \\ x_1 &= \lambda_0 x_0 + (1-\lambda_0)T_0 x_0, \\ x_2 &= \lambda_1 x_1 + (1-\lambda_1)T_1 x_1, \\ &\vdots, \\ x_N &= \lambda_{N-1} x_{N-1} + (1-\lambda_{N-1})T_{N-1} x_{N-1}, \\ x_{N+1} &= \lambda_N x_N + (1-\lambda_N)T_0 x_N, \end{cases}$$

In a more compact form, x_{n+1} can be written as

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)T_{[n]} x_n, \tag{1.9}$$

where $T_{[n]} = T_i$, with $i = n(\mathrm{mod} N)$, $0 \le i \le N-1$.

Acedoa and Xu obtained weak convergence theorems for the sequences generated by the algorithms (1.8) and (1.9). Furthermore, Acedoa and Xu [19] proposed the modifications for the algorithms (1.8) and (1.9), respectively, as follows:

$$\begin{cases} x_1 &= x \in C, \\ z_n &= \alpha_n x_n + (1-\alpha_n)W_n x_n, \\ C_n &= \{z \in C : \|z_n - z\|^2 \\ &\quad \le \|x_n - z\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|x_n - W_n x_n\|^2\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n} x \end{cases} \tag{1.10}$$

for every $n = 1, 2, \dots$, where $W_n = \sum_{j=1}^{N} \zeta_j^{(n)} T_j$.

And

$$(1.11)\qquad \begin{cases} x_0 &= x \in C, \\ y_n &= \lambda_n x_n + (1-\lambda_n) T_{[n]} x_n, \\ C_n &= \{z \in C : \|y_n - z\|^2 \\ &\quad \le \|x_n - z\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|x_n - T_{[n]} x_n\|^2\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x). \end{cases}$$

They proved the strong convergence theorems of the algorithms (1.10) and (1.11). Ceng, Petrusel and Yao [27] introduced some parallel algorithms and cyclic algorithms based on extragradient method for finding a common fixed point of a family of finite strict pseudo-contraction mappings and a monotone and lipschitz continuous mapping and obtained some weak convergence theorems and strong convergence theorems.

In the present paper, inspired and motivated by the above ideas, we introduce some parallel and cyclic algorithms based on the extragradient method (nonextragradient method) for finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a finite family of strict pseudo-contractions and the set of the variational inequality for a monotone, Lipschitz continuous mapping (an inverse strongly monotone mapping). We obtain some weak convergence theorems and strong convergence theorems for the sequences generated by these processes. The results in this paper generalize, improve and unify some well-known results in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \le \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is known that $P_C(x) \in C$ and

$$(2.1)\qquad \langle x - P_C(x), P_C(x) - y\rangle \ge 0$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.1) is equivalent to

$$\|x-y\|^2 \geq \|x-P_C(x)\|^2 + \|y-P_C(x)\|^2 \tag{2.2}$$

for all $x \in H$ and $y \in C$. It is also known that

$$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2 \tag{2.3}$$

A mapping A of C into H is called monotone if

$$\langle Ax-Ay, x-y\rangle \geq 0$$

for all $x, y \in C$. A mapping A of C into H is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2$$

for all $x, y \in C$. A mapping $A : C \to H$ is called k-Lipschitz continuous if there exists a positive real number k such that

$$\|Ax-Ay\| \leq k\|x-y\|$$

for all $x, y \in C$. If A is α-inverse-strongly monotone of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} &\|(I-\lambda A)x-(I-\lambda A)y\|^2 \\ &= \|x-y-\lambda(Ax-Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda\langle x-y, Ax-Ay\rangle + \lambda^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \lambda(\lambda-2\alpha)\|Ax-Ay\|^2. \end{aligned} \tag{2.4}$$

So, if $\lambda \leq 2\alpha$, then $I-\lambda A$ is a nonexpansive mapping of C into H. It is easy to see that if A is an α-inverse strongly monotone mapping, then A is monotone and Lipschitz continuous. The converse is not true in general. The class of α-inverse strongly monotone mappings does not contain some important classes of mappings even in a finite-dimensional case. For example, if the matrix in the corresponding linear complementarity problem is positively semidefinite, but not positively definite, then the mapping A will be monotone and Lipschitz continuous, but not α-inverse strongly monotone.

Let A be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (2.1) implies the following:

$$u \in VI(C,A) \;\Rightarrow\; u = P_C(u-\lambda Au),\; \lambda > 0,$$

and

$$u = P_C(u - \lambda Au) \text{ for some } \lambda > 0 \Rightarrow \ u \in VI(C, A).$$

It is also known that H satisfies the Opial's condition [21], i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, k-Lipschitz continuous mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, i.e, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$(see [21]).

We shall use the following results in the sequel.

Lemma 2.1. [22] *Let H be a real Hilbert space, let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|,$$

for every $n = 0, 1, 2, ...$ Then, the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

Lemma 2.2. [17, 19] *Assume C is a closed convex subset of a Hilbert space H.*

- (*i*) *If $T : C \to C$ is a κ-strict pseudo-contraction, then T satisfies the Lipschitz condition*
$$\|Tx - Ty\| \leq \frac{1+\kappa}{1-\kappa}\|x - y\|, \forall x, y \in C.$$
- (*ii*) *If $T : C \to C$ is a κ-strict pseudo-contraction, then the mapping $I - T$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \bar{x}$ and $(I - T)x_n \to 0$, then $(I - T)\bar{x} = 0$.*
- (*iii*) *If $T : C \to C$ is a κ-strict pseudo-contraction, then the fixed point set $Fix(T)$ of T is closed and convex so that the projection $P_{Fix(T)}$ is well defined.*

(*iv*) *Given an integer* $N \geq 1$, *assume, for each* $1 \leq i \leq N$, $T_i : C \to C$ *is a* κ_i *-strict pseudo-contraction for some* $0 < \kappa_i < 1$. *Assume* $\{\zeta_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \zeta_i = 1$. *Then* $\sum_{i=1}^N \zeta_i T_i$ *is a* κ*-strict pseudo-contraction, with* $\kappa = max\{\kappa_i : 1 \leq i \leq N\}$.

(*v*) *Let* $\{T_i\}_{i=1}^N$ *and* $\{\zeta_i\}_{i=1}^N$ *be given as in (iv) above. Suppose that* $\{T_i\}_{i=1}^N$ *has a common fixed point. Then* $Fix(\sum_{i=1}^N \zeta_i T_i) = \bigcap_{i=1}^N Fix(T_i)$.

For solving the generalized mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ *and the set* C:

$(A1)$ $F(x, x) = 0$ *for all* $x \in C$;

$(A2)$ F *is monotone, i.e.* $F(x, y) + F(y, x) \leq 0$ *for any* $x, y \in C$;

$(A3)$ *for each* $y \in C$, $x \mapsto F(x, y)$ *is weakly upper semicontinuous;*

$(A4)$ *for each* $x \in C, y \mapsto F(x, y)$ *is convex;*

$(A5)$ *for each* $x \in C, y \mapsto F(x, y)$ *is lower semicontinuous;*

$(B1)$ *For each* $x \in H$ *and* $r > 0$, *there exist a bounded subset* $D_x \subseteq C$ *and* $y_x \in C$ *such that for any* $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

$(B2)$ C *is a bounded set;*

$(B3)$ *For each* $x \in H$ *and* $r > 0$, *there exist a bounded subset* $D_x \subseteq C$ *and* $y_x \in C$ *such that for any* $z \in C \setminus D_x$,

$$F(z, y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

$(B4)$ *For each* $x \in H$ *and* $r > 0$, *there exists a bounded subset* $D_x \subseteq C$ *and* $y_x \in C$ *such that for any* $z \in C \setminus D_x$,

$$\varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z).$$

3. Strong Convergence Theorems

We first derive two strong convergence theorems of some parallel and cyclic algorithms based on both hybrid method and extragradient method which solves the problem of finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a finite family of strict pseudo-contractions and the set of the variational inequality for a monotone, Lipschitz continuous mapping in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A5) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H and B be a β-inverse strongly monotone mapping of C into H. Let $N \geq 1$ be an integer. For each $1 \leq j \leq N$, let $T_j : C \to C$ be an ε_j-strict pseudo-contraction for some $0 \leq \varepsilon_j < 1$ such that $\Gamma_1 = \cap_{j=1}^{N} Fix(T_j) \cap VI(C, A) \cap GMEP(F, \varphi, B) \neq \emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{j=1}^{N} \zeta_j^{(n)} = 1$ for all n and $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ for all $0 \leq j \leq N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. Assume also that either (B1) or (B2) holds. Let the mapping W_n be defined by*

$$W_n x = \sum_{j=1}^{N} \zeta_j^{(n)} T_j x, \forall x \in C.$$

Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1 &= x \in C, F(u_n, y) + \varphi(y) - \varphi(u_n) \\ & \quad +\langle Bx_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n &= P_C(u_n - \lambda_n A u_n), \\ t_n &= P_C(u_n - \lambda_n A y_n), \\ z_n &= \alpha_n t_n + (1 - \alpha_n) W_n t_n, \\ C_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \varepsilon)\|t_n - W_n t_n\|^2\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some $\gamma, \tau \in (0, 2\alpha)$. Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ and $\{z_n\}$ converge strongly to $w = P_{\Gamma_1}(x)$.*

Proof. From observe that C_n is closed and convex by Lemma 1.3 in [23] and Q_n is closed and convex for every $n = 1, 2, \ldots$. It is easy to see that $\langle x_n - z, x - x_n\rangle \geq 0$ for all $z \in Q_n$ and by (2.1), $x_n = P_{Q_n}(x)$. Let $u \in \Gamma_1$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1 in [31]. Then $u = P_C(u - \lambda_n A u) = T_{r_n}(u - r_n B u)$. From $u_n = T_{r_n}(x_n - r_n B x_n) \in C$ and the β-inverse-strongly monotonicity of B and (2.4), we have

$$\begin{aligned}
\|u_n - u\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(u - r_nBu)\|^2 \\
&\le \|x_n - r_nBx_n - (u - r_nBu)\|^2 \\
&\le \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2 \\
&\le \|x_n - u\|^2.
\end{aligned} \tag{3.1}$$

From (2.2), the monotonicity of A, and $u \in VI(C, A)$, we have

$$\begin{aligned}
&\|t_n - u\|^2 \le \|u_n - \lambda_nAy_n - u\|^2 - \|u_n - \lambda_nAy_n - t_n\|^2 \\
&= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - t_n\rangle \\
&= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n(\langle Ay_n - Au, u - y_n\rangle \\
&\quad +\langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\
&\le \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle \\
&\quad -\|y_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \lambda_nAy_n - y_n, t_n - y_n\rangle.
\end{aligned}$$

Further, Since $y_n = P_C(u_n - \lambda_nAu_n)$ and A is k-Lipschitz continuous, we have

$$\begin{aligned}
&\langle u_n - \lambda_nAy_n - y_n, t_n - y_n\rangle \\
&= \langle u_n - \lambda_nAu_n - y_n, t_n - y_n\rangle + \langle \lambda_nAu_n - \lambda_nAy_n, t_n - y_n\rangle \\
&\le \langle \lambda_nAu_n - \lambda_nAy_n, t_n - y_n\rangle \\
&\le \lambda_nk\|u_n - y_n\|\|t_n - y_n\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\|t_n - u\|^2 \le \|u_n - u\|^2 - \|u_n - y_n\|^2 \\
&\quad -\|y_n - t_n\|^2 + 2\lambda_nk\|u_n - y_n\|\|t_n - y_n\| \\
&\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad +\lambda_n{}^2k^2\|u_n - y_n\|^2 + \|t_n - y_n\|^2 \\
&= \|u_n - u\|^2 + (\lambda_n{}^2k^2 - 1)\|u_n - y_n\|^2. \\
&\le \|u_n - u\|^2.
\end{aligned} \tag{3.2}$$

By Lemma 2.2, we know that W_n is an ε-strict pseudo-contraction and $F(W_n) = \cap_{j=1}^N Fix(T_j)$. It follows from (3.1), (3.2), $z_n = \alpha_nt_n + (1-\alpha_n)W_nt_n$ and $u = W_nu$

that

$$
\begin{aligned}
\|z_n - u\|^2 &= \alpha_n\|t_n - u\|^2 + (1-\alpha_n)\|W_n t_n - u\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|t_n - W_n t_n\|^2 \\
&\le \alpha_n\|t_n - u\|^2 + (1-\alpha_n)[\|t_n - u\|^2 + \varepsilon\|t_n - W_n t_n\|^2] \\
&\quad -\alpha_n(1-\alpha_n)\|t_n - W_n t_n\|^2 \\
&= \|t_n - u\|^2 + (1-\alpha_n)(\varepsilon - \alpha_n)\|t_n - W_n t_n\|^2 \qquad (3.3)\\
&\le \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 + (1-\alpha_n)(\varepsilon - \alpha_n)\|t_n - W_n t_n\|^2 \\
&\le \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\le \|x_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\le \|x_n - u\|^2,
\end{aligned}
$$

for every $n = 1, 2, ...$

From (3.3) and (3.1), we know that

$$\|z_n - u\|^2 \le \|x_n - u\|^2 + (1-\alpha_n)(\varepsilon - \alpha_n)\|t_n - W_n t_n\|^2, \tag{3.4}$$

for every $n = 1, 2, ...$, and hence $u \in C_n$. So, $\Gamma_1 \subset C_n$ for every $n = 1, 2, ...$ Next, let us show by mathematical induction that $\{x_n\}$ is well defined and $\Gamma_1 \subset C_n \cap Q_n$ for every $n = 1, 2, ...$ For $n = 1$ we have $x_1 = x \in H$ and $Q_1 = H$. Hence we obtain $\Gamma_1 \subset C_1 \cap Q_1$. Suppose that x_k is given and $\Gamma_1 \subset C_k \cap Q_k$ for some positive integer k. Since Γ_1 is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of H. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x)$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in C_k \cap Q_k$. Since $\Gamma_1 \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in \Gamma_1$ and hence $\Gamma_1 \subset Q_{k+1}$. Therefore, we obtain $\Gamma_1 \subset C_{k+1} \cap Q_{k+1}$.

Let $l_0 = P_{\Gamma_1}x$. From $x_{n+1} = P_{C_n \cap Q_n}x$ and $l_0 \in \Gamma_1 \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \le \|l_0 - x\| \tag{3.5}$$

for every $n = 1, 2, ...$ Therefore, $\{x_n\}$ is bounded. From (3.1)-(3.3), we also obtain that $\{t_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}(x)$, we have

$$\|x_n - x\| \le \|x_{n+1} - x\|$$

for every $n = 1, 2, ...$ Therefore, $\lim_{n\to\infty} \|x_n - x\|$ exists.

Since $x_n = P_{Q_n}(x)$ and $x_{n+1} \in Q_n$, using (2.2), we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 1, 2, \ldots$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - W_n t_n\|^2 \le \|x_n - x_{n+1}\|^2$$

and hence

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \le 2\|x_n - x_{n+1}\|$$

for every $n = 1, 2, \ldots$. From $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have $\|x_n - z_n\| \to 0$.

For $u \in \Gamma_1$, from (3.3) we obtain

$$\|z_n - u\|^2 \le \|x_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2.$$

Thus, we have

$$\|u_n - y_n\|^2 \le \frac{1}{(1 - {\lambda_n}^2 k^2)}\Big(\|x_n - u\|^2 - \|z_n - u\|^2\Big)$$
$$\le \frac{1}{(1 - b^2 k^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.$$

It follows from $\|x_n - z_n\| \to 0$, $\{x_n\}$ and $\{z_n\}$ are bounded that $\|u_n - y_n\| \to 0$. From the the definition of t_n and y_n, we have

$$\|t_n - y_n\| = \|P_C(u_n - \lambda_n A y_n) - P_C(u_n - \lambda_n A u_n)\|$$
$$\le \|(u_n - \lambda_n A y_n) - (u_n - \lambda_n A u_n)\| \le \lambda_n k \|y_n - u_n\|,$$

which implies that $\lim_{n\to\infty} \|t_n - y_n\| = 0$. From $\|u_n - t_n\| \le \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is k-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$.

From $\varepsilon < c \le \alpha_n \le d < 1$ and (3.4), we have

$$(1-d)(c-\varepsilon)\|t_n - W_n t_n\|^2 \le (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - W_n t_n\|^2$$
$$\le \|x_n - u\|^2 - \|z_n - u\|^2 \le (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.$$

This implies that

$$\lim_{n\to\infty} \|t_n - W_n t_n\| = 0. \tag{3.6}$$

Also by (3.3) and (3.1), we have

$$\|z_n - u\|^2 \le \|u_n - u\|^2 \le \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2.$$

Thus, we have

$$\gamma(2\beta-\tau)\|Bx_n - Bu\|^2 \leq r_n(2\beta - r_n)\|Bx_n - Bu\|^2$$

$$\leq \|x_n - u\|^2 - \|z_n - u\|^2 \leq (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.$$

It follows from $\|x_n - z_n\| \to 0$, $\{x_n\}$ and $\{z_n\}$ are bounded that $\|Bx_n - Bu\| \to 0$.

For $u \in \Gamma_1$, we have, from Lemma 2.1 in [31],

$$\begin{aligned}
&\|u_n - u\|^2 = \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(u - r_nBu)\|^2 \\
&\leq \langle T_{r_n}(x_n - r_nBx_n) - T_{r_n}(u - r_nBu), x_n - r_nBx_n - (u - r_nBu)\rangle \\
&= \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - r_nBx_n - (u - r_nBu)\|^2 \\
&\quad -\|x_n - r_nBx_n - (u - r_nBu) - (u_n - u)\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - r_nBx_n - (u - r_nBu) - (u_n - u)\|^2\} \\
&= \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - u_n\|^2 \\
&\quad +2r_n\langle Bx_n - Bu, x_n - u_n\rangle - r_n^2\|Bx_n - Bu\|^2.
\end{aligned}$$

Hence,

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle. \tag{3.7}$$

It follows from (3.3) and (3.7) that

$$\|z_n - u\|^2 \leq \|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle.$$

Hence,

$$\|x_n - u_n\|^2 \leq \|x_n - u\|^2 - \|z_n - u\|^2 + 2r_n\|Bx_n - Bu\|\|x_n - u_n\|.$$

$$\leq (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\| + 2r_n\|Bx_n - Bu\|\|x_n - u_n\|.$$

Since $\|Bx_n - Bu\| \to 0$, $\|x_n - z_n\| \to 0$, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0$. From $\|t_n - x_n\| \leq \|t_n - u_n\| + \|x_n - u_n\|$ we also have $\|t_n - x_n\| \to 0$.

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$ and $\|t_n - x_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$ and $t_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

In order to show that $w \in \Gamma_1$, we first show that $w \in \cap_{k=1}^N Fix(T_k)$. To see this, we observe that we may assume (by passing to a further subsequence if necessary)

$\zeta_k^{(n_i)} \to \zeta_k$ (as $i \to \infty$) for $k = 1, 2, ..., N$. It is easy to see that $\zeta_k > 0$ and $\sum_{k=1}^{N} \zeta_k = 1$. We also have

$$W_{n_i}x \to Wx \text{ (as } i \to \infty) \text{ for all } x \in C,$$

where $W = \sum_{k=1}^{N} \zeta_k T_k$. Note that by Lemma 2.2, W is an ε-strict pseudo-contraction and $Fix(W) = \cap_{i=1}^{N} Fix(T_i)$. Since

$$\|t_{n_i} - Wt_{n_i}\| \le \|t_{n_i} - W_{n_i}t_{n_i}\| + \|W_{n_i}t_{n_i} - Wt_{n_i}\|$$
$$\le \|t_{n_i} - W_{n_i}t_{n_i}\| + \sum_{k=1}^{N} |\zeta_k^{(n_i)} - \zeta_k| \|T_k t_{n_i}\|.$$

It follows from (3.6) and $\zeta_k^{(n_i)} \to \zeta_k$ that

$$\|t_{n_i} - Wt_{n_i}\| \to 0.$$

So by the demiclosedness principle (Lemma 2.2(ii)), it follows that $w \in Fix(W) = \cap_{i=1}^{N} Fix(T_i)$.

By the similar argument as in the proof of Theorem 3.1 in [1], we can show $w \in GMEP(F, \varphi, B)$ and $w \in VI(C, A)$, which implies $w \in \Gamma_1$.

From $l_0 = P_{\Gamma_1}(x)$, $w \in \Gamma_1$ and (3.5), we have

$$\|l_0 - x\| \le \|w - x\| \le \liminf_{i\to\infty} \|x_{n_i} - x\| \le \limsup_{i\to\infty} \|x_{n_i} - x\| \le \|l_0 - x\|.$$

So, we obtain

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|w - x\|.$$

From $x_{n_i} - x \rightharpoonup w - x$ we have $x_{n_i} - x \to w - x$ and hence $x_{n_i} \to w$. Since $x_n = P_{Q_n}(x)$ and $l_0 \in \Gamma_1 \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|l_0 - x_{n_i}\|^2 = \langle l_0 - x_{n_i}, x_{n_i} - x\rangle + \langle l_0 - x_{n_i}, x - l_0\rangle \ge \langle l_0 - x_{n_i}, x - l_0\rangle.$$

As $i \to \infty$, we obtain $-\|l_0 - w\|^2 \ge \langle l_0 - w, x - l_0\rangle \ge 0$ by $l_0 = P_{\Gamma_1}(x)$ and $w \in \Gamma_1$. Hence we have $w = l_0$. This implies that $x_n \to l_0$. It is easy to see $u_n \to l_0$, $y_n \to l_0$, $t_n \to l_0$, and $z_n \to l_0$. The proof is now complete.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A5) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H and B be a β-inverse strongly monotone mapping of C into H. Let $N \ge 1$ be an integer. For each $0 \le j \le N-1$, let $T_j : C \to C$ be an ε_j-strict pseudo-contraction for some*

$0 \leq \varepsilon_j < 1$ *such that* $\Gamma_2 = \cap_{j=0}^{N-1} Fix(T_j) \cap VI(C,A) \cap GMEP(F,\varphi,B) \neq \emptyset$. *Let* $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N-1\}$. *Assume that either (B1) or (B2) holds. Let* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle \\ \quad\quad + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ z_n = \alpha_n t_n + (1-\alpha_n) T_{[n]} t_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - T_{[n]}t_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n}(x) \end{cases}$$

for every $n = 0, 1, 2, ...$ *If* $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c,d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_{\Gamma_2}(x)$.

Proof. From the proof of Theorem 3.1, we know that both C_n and Q_n are closed and convex for every $n = 0, 1, 2, ...$, $x_n = P_{Q_n}(x)$ and for $u \in \Gamma_2$, the following formula hold

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2 \leq \|x_n - u\|^2. \tag{3.1}$$

$$\|t_n - u\|^2 \leq \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \leq \|u_n - u\|^2. \tag{3.2}$$

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle. \tag{3.7}$$

And

$$\|t_n - y_n\| \leq \lambda_n k \|y_n - u_n\|. \tag{3.8}$$

Since for each $j = 0, 1, ..., N-1$, T_j is ε_j-strictly pseudocontractive and $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N-1\} \in [0,1)$, we have

$$\|T_{[n]}x - T_{[n]}y\|^2 \leq \|x - y\|^2 + \varepsilon\|x - T_{[n]}x - (y - T_{[n]}y)\|^2, \forall x, y \in C. \tag{3.9}$$

It follows from (3.1), (3.2) and (3.9), $z_n = \alpha_n t_n + (1-\alpha_n)T_{[n]}t_n$ and $u = T_{[n]}u$ that

$$
\begin{aligned}
&\|z_n - u\|^2 \\
&= \alpha_n\|t_n - u\|^2 + (1-\alpha_n)\|T_{[n]}t_n - u\|^2 - \alpha_n(1-\alpha_n)\|t_n - T_{[n]}t_n\|^2 \\
&\le \alpha_n\|t_n - u\|^2 + (1-\alpha_n)[\|t_n - u\|^2 \\
&\quad +\varepsilon\|t_n - T_{[n]}t_n\|^2] - \alpha_n(1-\alpha_n)\|t_n - T_{[n]}t_n\|^2 \\
&= \|t_n - u\|^2 + (1-\alpha_n)(\varepsilon - \alpha_n)\|t_n - T_{[n]}t_n\|^2 \\
&\le \|u_n - u\|^2 + ({\lambda_n}^2k^2 - 1)\|u_n - y_n\|^2 + (1-\alpha_n)(\varepsilon - \alpha_n)\|t_n - T_{[n]}t_n\|^2 \\
&\le \|x_n - u\|^2 + ({\lambda_n}^2k^2 - 1)\|u_n - y_n\|^2 \\
&\le \|x_n - u\|^2,
\end{aligned}
\tag{3.10}
$$

for every $n = 0, 1, 2, \ldots$.

From (3.10) and (3.1), we can obtain that

$$\|z_n - u\|^2 \le \|x_n - u\|^2 + (1-\alpha_n)(\varepsilon - \alpha_n)\|t_n - T_{[n]}t_n\|^2, \tag{3.11}$$

for every $n = 0, 1, 2, \ldots$, and hence $u \in C_n$. So, $\Gamma_2 \subset C_n$ for every $n = 0, 1, 2, \ldots$. Next, let us show by mathematical induction that $\{x_n\}$ is well defined and $\Gamma_2 \subset C_n \cap Q_n$ for every $n = 0, 1, 2, \ldots$. For $n = 0$ we have $x_0 = x \in H$ and $Q_0 = H$. Hence we obtain $\Gamma_2 \subset C_0 \cap Q_0$. Suppose that x_k is given and $\Gamma_2 \subset C_k \cap Q_k$ for some integer $k \ge 0$. Since Γ_2 is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of H. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x)$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in C_k \cap Q_k$. Since $\Gamma_2 \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in \Gamma_2$ and hence $\Gamma_2 \subset Q_{k+1}$. Therefore, we obtain $\Gamma_2 \subset C_{k+1} \cap Q_{k+1}$.

Let $l_0 = P_{\Gamma_2}(x)$. From $x_{n+1} = P_{C_n \cap Q_n}x$ and $l_0 \in \Gamma_2 \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \le \|l_0 - x\| \tag{3.12}$$

for every $n = 0, 1, 2, \ldots$. Therefore, $\{x_n\}$ is bounded. From (3.1), (3.2) and (3.10), we also obtain that $\{t_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$ and $x_n = P_{Q_n}(x)$, we have

$$\|x_n - x\| \le \|x_{n+1} - x\|$$

for every $n = 0, 1, 2, \ldots$. Therefore, $\lim_{n\to\infty}\|x_n - x\|$ exists.

Since $x_n = P_{Q_n}(x)$ and $x_{n+1} \in Q_n$, using (2.2), we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x\|^2 - \|x_n - x\|^2$$

for every $n = 0, 1, 2, \ldots$. This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

It follows that for all $j = 0, 1, ..., N-1$,

$$\lim_{n\to\infty} \|x_{n+j} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - T_{[n]}t_n\|^2 \le \|x_n - x_{n+1}\|^2$$

and hence

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \le 2\|x_n - x_{n+1}\|$$

for every $n = 0, 1, 2, ...$ From $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have $\|x_n - z_n\| \to 0$.

For $u \in \Gamma_2$, from (3.10) we obtain

$$\|z_n - u\|^2 \le \|x_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2.$$

Thus, we have

$$\begin{aligned} \|u_n - y_n\|^2 &\le \frac{1}{(1 - {\lambda_n}^2 k^2)}\Big(\|x_n - u\|^2 - \|z_n - u\|^2\Big) \\ &\le \frac{1}{(1 - b^2 k^2)}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|. \end{aligned}$$

It follows from $\|x_n - z_n\| \to 0$, $\{x_n\}$ and $\{z_n\}$ are bounded that $\|u_n - y_n\| \to 0$.

It follows from (3.8) that $\lim_{n\to\infty} \|t_n - y_n\| = 0$. From $\|u_n - t_n\| \le \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is k-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$.

From $\varepsilon < c \le \alpha_n \le d < 1$ and (3.11), we have

$$(1-d)(c-\varepsilon)\|t_n - T_{[n]}t_n\|^2 \le (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - T_{[n]}t_n\|^2$$

$$\le \|x_n - u\|^2 - \|z_n - u\|^2 \le (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.$$

This implies that

$$\lim_{n\to\infty} \|t_n - T_{[n]}t_n\| = 0. \tag{3.13}$$

Let $L_j = \frac{1-\varepsilon_j}{1+\varepsilon_j}$, by Lemma 2.2, we have $\|T_j x - T_j y\| \le L_j\|x - y\|, \ \forall j = 0, 1, ..., N-1$.

If we choose $L = max_{0\le j\le N-1}\{L_j\}$, then

$$\|T_j x - T_j y\| \le L\|x - y\|, \forall j = 0, 1, ..., N-1. \tag{3.14}$$

Also by (3.10) and (3.1), we have

$$\|z_n - u\|^2 \le \|u_n - u\|^2 \le \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2.$$

Thus, we have

$$\gamma(2\beta - \tau)\|Bx_n - Bu\|^2 \le r_n(2\beta - r_n)\|Bx_n - Bu\|^2$$
$$\le \|x_n - u\|^2 - \|z_n - u\|^2 \le (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.$$

It follows from $\|x_n - z_n\| \to 0$, $\{x_n\}$ and $\{z_n\}$ are bounded that $\|Bx_n - Bu\| \to 0$.

It follows from (3.10) and (3.7) that

$$\|z_n - u\|^2 \le \|u_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle.$$

Hence,

$$\|x_n - u_n\|^2 \le (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\| + 2r_n\|Bx_n - Bu\|\|x_n - u_n\|.$$

Since $\|Bx_n - Bu\| \to 0$, $\|x_n - z_n\| \to 0$, $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0$. From $\|t_n - x_n\| \le \|t_n - u_n\| + \|x_n - u_n\|$ we also have $\|t_n - x_n\| \to 0$.

By (3.14), we have

$$\begin{aligned} \|x_n - T_{[n]}x_n\| &\le \|x_n - t_n\| + \|t_n - T_{[n]}t_n\| + \|T_{[n]}t_n - T_{[n]}x_n\| \\ &\le (1 + L)\|x_n - t_n\| + \|t_n - T_{[n]}t_n\|. \end{aligned} \tag{3.15}$$

It follows from (3.13), (3.15) and $\|t_n - x_n\| \to 0$ that $\|x_n - T_{[n]}x_n\| \to 0$.

We observe that for each $j = 0, 1, ..., N-1$,

$$\begin{aligned} \|x_n - T_{[n+j]}x_n\| &\le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{[n+j]}x_{n+j}\| \\ &\quad + \|T_{[n+j]}x_{n+j} - T_{[n+j]}x_n\| \\ &\le (1 + L)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{[n+j]}x_{n+j}\|. \end{aligned} \tag{3.16}$$

Thus, we get for each $j = 0, 1, ..., N-1$,

$$\lim_{n\to\infty} \|x_n - T_{[n+j]}x_n\| = 0. \tag{3.17}$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$ and $\|t_n - x_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$ and $t_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

In order to show that $w \in \Gamma_2$, we first show that $w \in \cap_{j=0}^{N-1} Fix(T_j)$. In fact, it follows from (3.17) that for each $l = 0, 1, ..., N-1$

$$\|x_{n_i} - T_l x_{n_i}\| \to 0.$$

So by the demiclosedness principle, it follows that $w \in Fix(T_l)$. Since l is an arbitrary element in the finite set $\{0, 1, ..., N-1\}$, we get $w \in \cap_{j=0}^{N-1} Fix(T_j)$. The rest of the proof is similar with that of Theorem 3.1. The proof is now complete.

Now we derive a strong convergence theorem of a cyclic algorithm based on hybrid method but not extragradient method which solves the problem of finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a finite family of strict pseudo-contractions and the set of the variational inequality for an inverse strongly monotone mapping in a Hilbert space.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A5) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $A : C \to H$ and $B : C \to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $N \geq 1$ be an integer. For each $0 \leq j \leq N-1$, let $T_j : C \to C$ be an ε_j-strict pseudo-contraction for some $0 \leq \varepsilon_j < 1$ such that $\Gamma_2 = \cap_{j=0}^{N-1} Fix(T_j) \cap VI(C, A) \cap GMEP(F, \varphi, B) \neq \emptyset$. Let $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N-1\}$. Assume also that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle \\ \quad + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ z_n = \alpha_n y_n + (1 - \alpha_n) T_{[n]} y_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \varepsilon) \|y_n - T_{[n]} y_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, ...$ If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ for some $\gamma, \tau \in (0, 2\beta)$. Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $w = P_{\Gamma_2}(x)$.

Proof. From the proof of Theorem 3.1 and 3.2, we know that both C_n and Q_n are closed and convex for every $n = 0, 1, 2, ...$, $x_n = P_{Q_n}(x)$ and for $u \in \Gamma_2$, the following formula hold

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2 \leq \|x_n - u\|^2. \tag{3.1}$$

$$\|u_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle. \tag{3.7}$$

$$\|T_{[n]}x - T_{[n]}y\|^2 \le \|x - y\|^2 + \varepsilon\|x - T_{[n]}x - (y - T_{[n]}y)\|^2, \forall x, y \in C. \tag{3.9}$$

$$\|T_j x - T_j y\| \le L\|x - y\|, \forall j = 0, 1, ..., N-1, \tag{3.14}$$

where $L = max_{0\le j\le N-1}\{\frac{1-\varepsilon_j}{1+\varepsilon_j}\}$. And

$$\|x_n - T_{[n+j]}x_n\| \le (1 + L)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{[n+j]}x_{n+j}\|. \tag{3.16}$$

Since A is an α-inverse strongly monotone mapping, from (2.4), we have

$$\begin{aligned} \|y_n - u\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(u - \lambda_n Au)\|^2 \\ &\le \|u_n - \lambda_n Au_n - (u - \lambda_n Au)\|^2 \\ &\le \|u_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2 \\ &\le \|u_n - u\|. \end{aligned} \tag{3.18}$$

It follows from (3.1) and (3.18), $z_n = \alpha_n y_n + (1 - \alpha_n)T_{[n]}y_n$ and $u = T_{[n]}u$ that

$$\begin{aligned} \|z_n - u\|^2 &= \alpha_n\|y_n - u\|^2 + (1 - \alpha_n)\|T_{[n]}y_n - u\|^2 \\ &\quad -\alpha_n(1 - \alpha_n)\|y_n - T_{[n]}y_n\|^2 \\ &\le \alpha_n\|y_n - u\|^2 + (1 - \alpha_n)[\|y_n - u\|^2 + \varepsilon\|y_n - T_{[n]}y_n\|^2] \\ &\quad -\alpha_n(1 - \alpha_n)\|y_n - T_{[n]}y_n\|^2 \\ &= \|y_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|y_n - T_{[n]}y_n\|^2 \\ &\le \|u_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2 \\ &\quad +(1 - \alpha_n)(\varepsilon - \alpha_n)\|y_n - T_{[n]}y_n\|^2 \\ &\le \|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2 \\ &\le \|x_n - u\|^2, \end{aligned} \tag{3.19}$$

for every $n = 0, 1, 2, ...$

From (3.19) and (3.1), we know that

$$\|z_n - u\|^2 \le \|x_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|y_n - T_{[n]}y_n\|^2, \tag{3.20}$$

for every $n = 0, 1, 2, ...$, and hence $u \in C_n$. So, $\Gamma_2 \subset C_n$ for every $n = 0, 1, 2, ...$ By the similar argument in the proof of Theorem 3.2, we can show by mathematical induction that $\{x_n\}$ is well defined and $\Gamma_2 \subset C_n \cap Q_n$ for every $n = 0, 1, 2, ..., \{x_n\}$,

$\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded, $\lim_{n\to\infty}\|x_n-x\|$ exists, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, $\|x_n - z_n\| \to 0$ and for each $j = 0, 1, ..., N-1$, $\lim_{n\to\infty}\|x_{n+j} - x_n\| = 0$.

From $\varepsilon < c \le \alpha_n \le d < 1$ and (3.20), we have

$$\begin{aligned}(1-d)(c-\varepsilon)\|y_n - T_{[n]}y_n\|^2 &\le (1-\alpha_n)(\alpha_n-\varepsilon)\|y_n - T_{[n]}y_n\|^2 \\ \le \|x_n-u\|^2 - \|z_n-u\|^2 &\le (\|x_n-u\| + \|z_n-u\|)\|x_n-z_n\|.\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|y_n - T_{[n]}y_n\| = 0. \tag{3.21}$$

By similar argument with that in the proof of Theorem 3.2, we know that $\|Bx_n - Bu\| \to 0$ and $\|x_n - u_n\| \to 0$. From (2.3) and (2.4), we have

$$\begin{aligned}\|y_n-u\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(u-\lambda_n Au)\|^2 \\ &\le \langle (u_n-\lambda_n Au_n) - (u-\lambda_n Au), y_n - u\rangle \\ &= \frac{1}{2}\{\|(u_n-\lambda_n Au_n) - (u-\lambda_n Au)\|^2 + \|y_n-u\|^2 \\ &\quad -\|[(u_n-\lambda_n Au_n) - (u-\lambda_n Au)] - (y_n-u)\|^2\} \\ &\le \frac{1}{2}\{\|u_n-u\|^2 + \|y_n-u\|^2 - \|(u_n-y_n) - \lambda_n(Au_n - Au)\|^2\} \\ &= \frac{1}{2}\{\|u_n-u\|^2 + \|y_n-u\|^2 - \|u_n-y_n\|^2 \\ &\quad +2\lambda_n\langle u_n-y_n, Au_n-Au\rangle - \lambda_n^2\|Au_n-Au\|^2\}.\end{aligned}$$

Hence,

$$\begin{aligned}\|y_n-u\|^2 &\le \|u_n-u\|^2 - \|u_n-y_n\|^2 \\ &\quad +2\lambda_n\langle u_n-y_n, Au_n-Au\rangle - \lambda_n^2\|Au_n-Au\|^2.\end{aligned} \tag{3.22}$$

From (3.19), (3.1) and (3.22), we have

$$\begin{aligned}\|z_n-u\|^2 \le \|y_n-u\|^2 &\le \|x_n-u\|^2 - \|u_n-y_n\|^2 \\ +2\lambda_n\langle u_n-y_n, Au_n-Au\rangle &- \lambda_n^2\|Au_n-Au\|^2.\end{aligned}$$

And hence

$$\begin{aligned}\|u_n-y_n\|^2 &\le \|x_n-u\|^2 - \|z_n-u\|^2 + 2\lambda_n\langle u_n-y_n, Au_n-Au\rangle - \lambda_n^2\|Au_n-Au\|^2 \\ &\le (\|x_n-u\| + \|z_n-u\|)\|x_n-z_n\| + 2\lambda_n\langle u_n-y_n, Au_n-Au\rangle - \lambda_n^2\|Au_n-Au\|^2.\end{aligned}$$

Since $\|x_n - z_n\| \to 0$ and $\|Au_n - Au\| \to 0$, we obtain $\|u_n - y_n\| \to 0$. It follows from the Lipschitz-continuity of A that $\|Au_n - Ay_n\| \to 0$. From $\|y_n - x_n\| \le \|y_n - u_n\| + \|x_n - u_n\|$ we also have $\|y_n - x_n\| \to 0$.

By (3.14), we have

$$\begin{aligned}\|x_n - T_{[n]}x_n\| &\le \|x_n - y_n\| + \|y_n - T_{[n]}y_n\| + \|T_{[n]}y_n - T_{[n]}x_n\| \\ &\le (1+L)\|x_n - y_n\| + \|y_n - T_{[n]}y_n\|.\end{aligned}$$

It follows from (3.21) and $\|y_n - x_n\| \to 0$ that $\|x_n - T_{[n]}x_n\| \to 0$.
It follows from (3.16) that for each $j = 0, 1, ..., N-1$,

$$\lim_{n\to\infty} \|x_n - T_{[n+j]}x_n\| = 0. \tag{3.23}$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$ and $\|y_n - x_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$ and $y_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$. The rest of the proof is similar with that of Theorem 3.2. The proof is now complete.

Let $A = 0$, by Theorem 3.1 and 3.2, respectively, we obtain the following results:

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A5) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B be a β-inverse strongly monotone mapping of C into H. Let $N \ge 1$ be an integer. For each $1 \le j \le N$, let $T_j : C \to C$ be an ε_j-strict pseudo-contraction for some $0 \le \varepsilon_j < 1$ such that $\Delta_1 = \cap_{j=1}^N Fix(T_j) \cap GMEP(F, \varphi, B) \ne \emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^N$ is a finite sequence of positive numbers such that $\sum_{j=1}^N \zeta_j^{(n)} = 1$ for all n and $\inf_{n\ge1} \zeta_j^{(n)} > 0$ for all $0 \le j \le N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \le j \le N\}$. Assume also that either (B1) or (B2) holds. Let the mapping W_n be defined by*

$$W_n x = \sum_{j=1}^N \zeta_j^{(n)} T_j x, \forall x \in C.$$

Let $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle \\ \quad +\dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \qquad \forall y \in C, \\ z_n = \alpha_n u_n + (1-\alpha_n)W_n u_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|u_n - W_n u_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n=1,2,\ldots$. *If* $\{\alpha_n\}\subset[c,d]$ for some $c,d\in(\varepsilon,1)$ and $\{r_n\}\subset[\gamma,\tau]$ *for some* $\gamma,\tau\in(0,2\beta)$. *Then,* $\{x_n\}$, $\{u_n\}$ *and* $\{z_n\}$ *converge strongly to* $w=P_{\Delta_1}(x)$.

Theorem 3.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C\times C$ *to* R *satisfying (A1)-(A5) and* $\varphi: C\to R\cup\{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* B *be a* β*-inverse strongly monotone mapping of* C *into* H. *Let* $N\geq 1$ *be an integer. For each* $0\leq j\leq N-1$, *let* $T_j: C\to C$ *be an* ε_j*-strict pseudo-contraction for some* $0\leq\varepsilon_j<1$ *such that* $\Delta_2=\cap_{j=0}^{N-1}Fix(T_j)\cap VI(C,A)\cap GMEP(F,\varphi,B)\neq\emptyset$. *Let* $\varepsilon=\max\{\varepsilon_j: 0\leq j\leq N-1\}$. *Assume that either (B1) or (B2) holds. Let* $\{x_n\}$, $\{u_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_0 = x\in C,\\ \quad F(u_n,y)+\varphi(y)-\varphi(u_n)+\langle Bx_n, y-u_n\rangle\\ \quad +\dfrac{1}{r_n}\langle y-u_n, u_n-x_n\rangle\geq 0, \qquad \forall y\in C,\\ z_n = \alpha_n u_n+(1-\alpha_n)T_{[n]}u_n,\\ C_n=\{z\in C:\|z_n-z\|^2\leq\|x_n-z\|^2-(1-\alpha_n)(\alpha_n-\varepsilon)\|t_n-T_{[n]}t_n\|^2\},\\ Q_n=\{z\in H:\langle x_n-z, x-x_n\rangle\geq 0\},\\ x_{n+1}=P_{C_n\bigcap Q_n}x \end{cases}$$

for every $n=0,1,2,\ldots$. *If* $\{\alpha_n\}\subset[c,d]$ for some $c,d\in(\varepsilon,1)$ and $\{r_n\}\subset[\gamma,\tau]$ *for some* $\gamma,\tau\in(0,2\beta)$. *Then,* $\{x_n\}$, $\{u_n\}$ *and* $\{z_n\}$ *converge strongly to* $w=P_{\Delta_2}(x)$.

Theorem 3.6. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C\times C$ *to* R *satisfying (A1)-(A5) and* $\varphi: C\to R\cup\{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* S *be a pseudo-contraction and* m*-Lipschitz-continuous mapping of* C *into itself and* B *be a* β*-inverse strongly monotone mapping of* C *into* H. *Let* $N\geq 1$ *be an integer. For each* $1\leq j\leq N$, *let* $T_j: C\to C$ *be an* ε_j*-strict pseudo-contraction for some* $0\leq\varepsilon_j<1$ *such that* $\Omega_1=\cap_{j=1}^{N}Fix(T_j)\cap Fix(S)\cap GMEP(F,\varphi,B)\neq\emptyset$. *Assume for each* n, $\{\zeta_j^{(n)}\}_{j=1}^N$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^N\zeta_j^{(n)}=1$ *for all* n *and* $\inf_{n\geq 1}\zeta_j^{(n)}>0$ *for all* $0\leq j\leq N$. *Let* $\varepsilon=\max\{\varepsilon_j: 1\leq j\leq N\}$. *Assume also that either (B1) or (B2) holds. Let the mapping* W_n *be defined by*

$$W_n x=\sum_{j=1}^{N}\zeta_j^{(n)}T_j x, \forall x\in C.$$

Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_1 &= x \in C, \\ & F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle \\ & + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \qquad \forall y \in C, \\ y_n &= u_n - \lambda_n (u_n - Su_n), \\ t_n &= P_C(u_n - \lambda_n (y_n - Sy_n)), \\ z_n &= \alpha_n t_n + (1 - \alpha_n) W_n t_n, \\ C_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \varepsilon) \|t_n - W_n t_n\|^2\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{m+1})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\alpha)$*. Then,* $\{x_n\}$*,* $\{u_n\}$*,* $\{y_n\}$ *,* $\{t_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_{\Omega_1}(x)$*.*

Proof. Let $A = I - S$. From the proof of Theorem 4.5 in [26], we know that the mapping A is monotone and $(m+1)$-Lipschitz-continuous and $Fix(S) = VI(C, A)$. By Theorem 3.1 we obtain the desired result.

Theorem 3.7. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C$ *to* R *satisfying (A1)-(A5) and* $\varphi : C \to R \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* S *be a pseudo-contraction and* m*-Lipschitz-continuous mapping of* C *into itself and* B *be a* β*-inverse strongly monotone mapping of* C *into* H*. Let* $N \geq 1$ *be an integer. For each* $0 \leq j \leq N - 1$*, let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Omega_2 = \cap_{j=0}^{N-1} Fix(T_j) \cap Fix(S) \cap GMEP(F, \varphi, B) \neq \emptyset$*. Let* $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N - 1\}$*. Assume that either (B1) or (B2) holds. Let* $\{x_n\}$*,* $\{u_n\}$*,* $\{y_n\}$*,* $\{t_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_0 &= x \in C, \\ & F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle \\ & + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \qquad \forall y \in C, \\ y_n &= u_n - \lambda_n (u_n - Su_n), \\ t_n &= P_C(u_n - \lambda_n (y_n - Sy_n)), \\ z_n &= \alpha_n t_n + (1 - \alpha_n) T_{[n]} t_n, \\ C_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \varepsilon) \|t_n - T_{[n]} t_n\|^2\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n}(x) \end{cases}$$

for every $n = 0, 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{m+1}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_{\Omega_2}(x)$.

Proof. By Theorem 3.2 and the proof of Theorem 3.6, we know that the conclusion holds.

Remark 3.1.

(i) Let $A = 0$ in Theorem 3.3, we can also recover Theorem 3.5.

(ii) In Theorems 3.1 - 3.7, if we let part of the mappings F, B, φ be zero mappings, we can obtain many new and interesting strong convergence theorems for some algorithms for the special case of problem (1.1) (i.e., problems (1.2)-(1.7)). Now we only give five examples as follows:

Corollary 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *to* R *satisfying (A1)-(A5) and* $\varphi : C \to R \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H. *Let* $N \geq 1$ *be an integer. For each* $1 \leq j \leq N$, *let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Lambda = \cap_{j=1}^N Fix(T_j) \cap VI(C, A) \cap MEP(F, \varphi) \neq \emptyset$. *Assume for each* n, $\{\zeta_j^{(n)}\}_{j=1}^N$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^N \zeta_j^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ *for all* $0 \leq j \leq N$. *Let* $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. *Assume also that either (B1) or (B2) holds. Let the mapping* W_n *be defined by*

$$W_n x = \sum_{j=1}^N \zeta_j^{(n)} T_j x, \forall x \in C.$$

Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ z_n = \alpha_n t_n + (1 - \alpha_n) W_n t_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \varepsilon)\|t_n - W_n t_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ *for some* $\gamma > 0$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_\Lambda(x)$.

Proof. Putting $B = 0$, by Theorem 3.1 we obtain the desired result.

Corollary 3.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *to* R *satisfying (A1)-(A5). Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H. *Let* $N \geq 1$ *be an integer. For each* $0 \leq j \leq N-1$, *let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Sigma = \cap_{j=0}^{N-1} Fix(T_j) \cap VI(C, A) \cap EP(F) \neq \emptyset$. *Let* $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N-1\}$. *Assume that either (B3) or (B2) holds. Let* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ \quad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ z_n = \alpha_n t_n + (1-\alpha_n) T_{[n]} t_n, \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - T_{[n]}t_n\|^2\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n}(x) \end{cases}$$

for every $n = 0, 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ *for some* $\gamma > 0$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_\Sigma(x)$.

Proof. Putting $F = 0$ and $\varphi = 0$. By Theorem 3.2 we obtain the desired result.

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H *and* B *be a* β*-inverse strongly monotone mapping of* C *into* H. *Let* $N \geq 1$ *be an integer. For each* $1 \leq j \leq N$, *let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Theta_1 = \cap_{j=1}^{N} Fix(T_j) \cap VI(C, A) \cap \cap VI(C, B) \neq \emptyset$. *Assume for each* n, $\{\zeta_j^{(n)}\}_{j=1}^N$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^N \zeta_j^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ *for all* $0 \leq j \leq N$. *Let* $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. *Let the mapping* W_n *be defined by*

$$W_n x = \sum_{j=1}^{N} \zeta_j^{(n)} T_j x, \forall x \in C.$$

Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_1 &= x \in C, \\ \quad u_n &= P_C(x_n - r_n B x_n), \\ y_n &= P_C(u_n - \lambda_n A u_n), \\ t_n &= P_C(u_n - \lambda_n A y_n), \\ z_n &= \alpha_n t_n + (1-\alpha_n) W_n t_n, \\ C_n &= \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - W_n t_n\|^2\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$*. Then,* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_{\Theta_1}(x)$*.*

Proof. In Theorem 3.1, put $F = 0$ and $\varphi = 0$. Then, we obtain that

$$\langle Bx_n, y - u_n \rangle + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \qquad \forall y \in C, \forall n \ge 1.$$

This implies that

$$\langle y - u_n, u_n - (x_n - r_n B x_n) \rangle \ge 0, \qquad \forall y \in C, \forall n \ge 1.$$

So, we get that $u_n = P_C(x_n - r_n B x_n)$ for all $n \ge 1$. Then we obtain the desired result from Theorem 3.1.

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H *and* B *be a* β*-inverse strongly monotone mapping of* C *into* H*. Let* $N \ge 1$ *be an integer. For each* $0 \le j \le N-1$*, let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \le \varepsilon_j < 1$ *such that* $\Theta_2 = \cap_{j=0}^{N-1} Fix(T_j) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$*. Let* $\varepsilon = \max\{\varepsilon_j : 0 \le j \le N-1\}$*. Let* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_0 &= x \in C, \\ u_n &= P_C(x_n - r_n B x_n), \\ y_n &= P_C(u_n - \lambda_n A u_n), \\ t_n &= P_C(u_n - \lambda_n A y_n), \\ z_n &= \alpha_n t_n + (1-\alpha_n) T_{[n]} t_n, \\ C_n &= \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - T_{[n]} t_n\|^2\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 0, 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_{\Theta_2}(x)$.

Proof. Putting $F = 0$ and $\varphi = 0$, by Theorem 3.2 and the proof of Corollary 3.4, we obtain the desired result.

Corollary 3.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\varphi : C \to R \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H. *Let* $N \geq 1$ *be an integer. For each* $1 \leq j \leq N$, *let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Xi = \cap_{j=1}^N Fix(T_j) \cap VI(C, A) \cap Argmin(\varphi) \neq \emptyset$. *Assume for each* n, $\{\zeta_j^{(n)}\}_{j=1}^N$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^N \zeta_j^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ *for all* $0 \leq j \leq N$. *Let* $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. *Assume also that either (B4) or (B2) holds. Let the mapping* W_n *be defined by*

$$W_n x = \sum_{j=1}^N \zeta_j^{(n)} T_j x, \forall x \in C.$$

Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_1 &= x \in C, \\ & \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n &= P_C(u_n - \lambda_n A u_n), \\ t_n &= P_C(u_n - \lambda_n A y_n), \\ z_n &= \alpha_n t_n + (1 - \alpha_n) W_n t_n, \\ C_n &= \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \varepsilon)\|t_n - W_n t_n\|^2\}, \\ Q_n &= \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \bigcap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ *for some* $\gamma > 0$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$ *and* $\{z_n\}$ *converge strongly to* $w = P_{\Xi}(x)$.

Proof. Putting $F = 0$ and $B = 0$, by Theorem 3.1 we obtain the desired result.

Remark 3.2.

(i) Since the nonexpansive mappings has been replaced by the strict pseudo-contraction mappings, Theorems 3.1-3.7 improve Theorem 3.1 in [1] and Theorem 3.1 in [4]. Theorems 3.1 -3.7 extend and improve Theorem 4.1 in [2], Theorem 4.1 in [4], Theorem 3.2 in [10], Theorem 3.1 in [11], Theorem 3.1 in [12]. Corollary 3.3 and 3.4 generalize and improve Theorem 3.1 in [24] and Theorem 3.1 in [25].

(ii) Since the inverse strongly monotonicity of the mapping A has been weakened by the monotonicity of A, Theorem 3.1, 3.2, Corollary 3.1 and 3.2 also extend and improve Theorem 3.1 in [15] and Theorem 3.1 in [16].

4. Weak Convergence Theorems

we first show weak convergence theorems of the parallel and cyclic algorithms based on extragradient method which solves the problem of finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a finite family of strict pseudo-contractions and the set of the variational inequality for a monotone, Lipschitz continuous mapping in a Hilbert space.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A5) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H and B be a β-inverse strongly monotone mapping of C into H. Let $N \geq 1$ be an integer. For each $1 \leq j \leq N$, let $T_j : C \to C$ be an ε_j-strict pseudo-contraction for some $0 \leq \varepsilon_j < 1$ such that $\Gamma_1 = \cap_{j=1}^{N} Fix(T_j) \cap VI(C, A) \cap GMEP(F, \varphi, B) \neq \emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{j=1}^{N} \zeta_j^{(n)} = 1$ for all n and $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ for all $0 \leq j \leq N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. Assume also that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle \\ \quad + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n)\displaystyle\sum_{j=1}^{N} \zeta_j^{(n)} T_j t_n, \end{cases}$$

for every $n = 1, 2, \ldots$. *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \Gamma_1$, *where* $w = \lim_{n\to\infty} P_{\Gamma_1} x_n$.

Proof. Let $u \in \Gamma_1$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1 in [31]. Then $u = P_C(u - \lambda_n Au) = T_{r_n}(u - r_n Bu)$. From the proof of Theorem 3.1, we have

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2 \leq \|x_n - u\|^2. \tag{4.1}$$

$$\|t_n - u\|^2 \leq \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \leq \|u_n - u\|^2. \tag{4.2}$$

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle. \tag{4.3}$$

And

$$\|t_n - y_n\| \leq \lambda_n k\|y_n - u_n\|. \tag{4.4}$$

For each $n \geq 1$, let $W_n = \sum_{j=1}^{N} \zeta_j^{(n)} T_j$. By Lemma 2.2, we know that W_n is ε-strict pseudo-contraction and $F(W_n) = \cap_{j=1}^{N} Fix(T_j)$. It follows from (4.1), (4.2), $x_{n+1} = \alpha_n t_n + (1 - \alpha_n)W_n t_n$ and $u = W_n u$ that

$$\begin{aligned}
&\|x_{n+1} - u\|^2 \\
&= \alpha_n\|t_n - u\|^2 + (1 - \alpha_n)\|W_n t_n - u\|^2 - \alpha_n(1 - \alpha_n)\|t_n - W_n t_n\|^2 \\
&\leq \alpha_n\|t_n - u\|^2 + (1 - \alpha_n)[\|t_n - u\|^2 \\
&\quad + \varepsilon\|t_n - W_n t_n\|^2] - \alpha_n(1 - \alpha_n)\|t_n - W_n t_n\|^2 \\
&= \|t_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - W_n t_n\|^2 \qquad (4.5)\\
&\leq \|u_n - u\|^2 \\
&\quad + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - W_n t_n\|^2 \\
&\leq \|x_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\leq \|x_n - u\|^2,
\end{aligned}$$

for every $n = 1, 2, \ldots$. Therefore, there exists $\theta = \lim_{n\to\infty} \|x_n - u\|$ and $\{x_n\}$ is bounded. From (4.1) and (4.2), we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded.

By (4.5), we have

$$\|u_n - y_n\|^2 \leq \frac{1}{(1 - \alpha_n)(1 - {\lambda_n}^2 k^2)}\Big(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\Big).$$

Hence, $\|u_n - y_n\| \to 0$. It follows from (4.4) that $\lim_{n\to\infty} \|t_n - y_n\| = 0$. From $\|u_n - t_n\| \leq \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is k-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$.

From (4.5) and (4.1), we also have

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - W_n t_n\|^2, \tag{4.6}$$

for every $n = 1, 2, ...$

From $\varepsilon < c \leq \alpha_n \leq d < 1$ and (4.6), we have

$$(1-d)(c-\varepsilon)\|t_n - W_n t_n\|^2 \leq (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - W_n t_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

This implies that

$$\lim_{n\to\infty} \|t_n - W_n t_n\| = 0. \tag{4.7}$$

Also by (4.5) and (4.1), we have

$$\|x_{n+1} - u\|^2 \leq \|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2.$$

Thus, we have

$$\gamma(2\beta - \tau)\|Bx_n - Bu\|^2 \leq r_n(2\beta - r_n)\|Bx_n - Bu\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

It follows that $\|Bx_n - Bu\| \to 0$.

By (4.5) and (4.3),

$$\|x_{n+1} - u\|^2 \leq \|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle.$$

Hence,

$$\|x_n - u_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + 2r_n\|Bx_n - Bu\|\|x_n - u_n\|.$$

Since $\|Bx_n - Bu\| \to 0$, $\{x_n\}$ and $\{u_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0$. From $\|t_n - x_n\| \leq \|t_n - u_n\| + \|x_n - u_n\|$ we also have $\|t_n - x_n\| \to 0$.

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$ and $\|t_n - x_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$ and $t_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$. By the similar argument with that in the proof of Theorem 3.1, we can obtain that $w \in \cap_{k=1}^N Fix(T_k)$. And by the similar argument as in the proof of Theorem 3.1 in [1], we can show $w \in GMEP(F, \varphi, B)$ and $w \in VI(C, A)$, which implies $w \in \Gamma_1$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$. Then $z \in \Gamma_1$. Let us show $w = z$. Assume that $w \neq z$. From the Opial condition, we have

$$\lim_{n\to\infty} \|x_n - w\| = \liminf_{i\to\infty} \|x_{n_i} - w\| < \liminf_{i\to\infty} \|x_{n_i} - z\|$$

$$= \lim_{n\to\infty} \|x_n - z\| = \liminf_{j\to\infty} \|x_{n_j} - z\|$$
$$< \liminf_{j\to\infty} \|x_{n_j} - w\| = \lim_{n\to\infty} \|x_n - w\|.$$

This is a contradiction. Thus, we have $w = z$. This implies that $x_n \rightharpoonup w \in \Gamma_1$. Since $\|x_n - u_n\| \to 0$, we have $u_n \rightharpoonup w \in \Gamma_1$. Since $\|y_n - u_n\| \to 0$, we have also $y_n \rightharpoonup w \in \Gamma_1$.

Now put $w_n = P_{\Gamma_1}(x_n)$. We show that $w = \lim_{n\to\infty} w_n$.

From $w_n = P_{\Gamma_1}(x_n)$ and $w \in \Gamma_1$, we have

$$\langle w - w_n, w_n - x_n\rangle \geq 0.$$

From (4.5) and Lemma 2.1, we know that $\{w_n\}$ converges strongly to some $w_0 \in \Gamma_1$. Then, we have

$$\langle w - w_0, w_0 - w\rangle \geq 0$$

and hence $w = w_0$. The proof is now complete.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A5) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H and B be a β-inverse strongly monotone mapping of C into H. Let $N \geq 1$ be an integer. For each $0 \leq j \leq N-1$, let $T_j : C \to C$ be an ε_j-strict pseudo-contraction for some $0 \leq \varepsilon_j < 1$ such that $\Gamma_2 = \cap_{j=0}^{N-1} Fix(T_j) \cap VI(C, A) \cap GMEP(F, \varphi, B) \neq \emptyset$. Let $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N-1\}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle \\ \quad + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n t_n + (1-\alpha_n) T_{[n]} t_n, \end{cases}$$

for every $n = 0, 1, 2, \ldots$. If $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some $\gamma, \tau \in (0, 2\beta)$. Then, $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ and $\{y_n\}$ converge weakly to $w \in \Gamma_2$, where $w = \lim_{n\to\infty} P_{\Gamma_2}(x_n)$.*

Proof. Let $u \in \Gamma_2$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1 in [31]. Then $u = P_C(u - \lambda_n Au) = T_{r_n}(u - r_n Bu)$. From the proof of Theorem 3.2, we have:

$$\|u_n - u\|^2 \le \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2 \le \|x_n - u\|^2. \tag{4.1}$$

$$\|t_n - u\|^2 \le \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \le \|u_n - u\|^2. \tag{4.2}$$

$$\|u_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle. \tag{4.3}$$

$$\|t_n - y_n\| \le \lambda_n k\|y_n - u_n\|. \tag{4.4}$$

$$\|T_{[n]}x - T_{[n]}y\|^2 \le \|x - y\|^2 + \varepsilon\|x - T_{[n]}x - (y - T_{[n]}y)\|^2, \forall x, y \in C. \tag{4.8}$$

$$\|x_n - T_{[n]}x_n\| \le (1 + L)\|x_n - t_n\| + \|t_n - T_{[n]}t_n\|, \tag{4.9}$$

where $L = max_{0\le j\le N-1}\{\frac{1-\varepsilon_j}{1+\varepsilon_j}\}$. And

$$\|x_n - T_{[n+j]}x_n\| \le (1 + L)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{[n+j]}x_{n+j}\|. \tag{4.10}$$

It follows from (4.1), (4.2) and (4.8), $x_{n+1} = \alpha_n t_n + (1 - \alpha_n)T_{[n]}t_n$ and $u = T_{[n]}u$ that

$$\begin{aligned}
&\|x_{n+1} - u\|^2 \\
&= \alpha_n\|t_n - u\|^2 + (1 - \alpha_n)\|T_{[n]}t_n - u\|^2 - \alpha_n(1 - \alpha_n)\|t_n - T_{[n]}t_n\|^2 \\
&\le \alpha_n\|t_n - u\|^2 + (1 - \alpha_n)[\|t_n - u\|^2 \\
&\quad + \varepsilon\|t_n - T_{[n]}t_n\|^2] - \alpha_n(1 - \alpha_n)\|t_n - T_{[n]}t_n\|^2 \\
&= \|t_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - T_{[n]}t_n\|^2 \\
&\le \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\quad + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - T_{[n]}t_n\|^2 \\
&\le \|x_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\le \|x_n - u\|^2,
\end{aligned} \tag{4.11}$$

for every $n = 0, 1, 2, ...$ Therefore, there exists $\theta = \lim_{n\to\infty} \|x_n - u\|$ and $\{x_n\}$ is bounded. From (4.1) and (4.2), we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded.

It follows from (4.11) that

$$\|u_n - y_n\|^2 \le \frac{1}{(1 - \alpha_n)(1 - {\lambda_n}^2 k^2)}\Big(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\Big).$$

Hence, $\|u_n - y_n\| \to 0$. It follows from (4.4) that $\lim_{n\to\infty} \|t_n - y_n\| = 0$. From $\|u_n - t_n\| \le \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is k-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$.

From (4.10) and (4.1), we also obtain

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 + (1-\alpha_n)(\varepsilon - \alpha_n)\|t_n - T_{[n]}t_n\|^2, \tag{4.12}$$

for every $n = 0, 1, 2, ...$

From $\varepsilon < c \leq \alpha_n \leq d < 1$ and (4.12), we have

$$(1-d)(c-\varepsilon)\|t_n - T_{[n]}t_n\|^2 \leq (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - T_{[n]}t_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

This implies that

$$\lim_{n\to\infty} \|t_n - T_{[n]}t_n\| = 0. \tag{4.13}$$

By (4.11) and (4.1), we have

$$\|x_{n+1} - u\|^2 \leq \|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2.$$

Thus, we have

$$\gamma(2\beta - \tau)\|Bx_n - Bu\|^2 \leq r_n(2\beta - r_n)\|Bx_n - Bu\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

Thus, we have $\|Bx_n - Bu\| \to 0.$

It follows from (4.11) and (4.3) that

$$\|x_{n+1} - u\|^2 \leq \|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle.$$

Hence,

$$\|x_n - u_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + 2r_n\|Bx_n - Bu\|\|x_n - u_n\|.$$

Since $\|Bx_n - Bu\| \to 0$, $\{x_n\}$ and $\{u_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0.$ From $\|t_n - x_n\| \leq \|t_n - u_n\| + \|x_n - u_n\|$ we also have $\|t_n - x_n\| \to 0.$ It follows from (4.9) and (4.13) that

$$\lim_{n\to\infty} \|x_n - T_{[n]}x_n\| = 0. \tag{4.14}$$

Since $\|T_{[n]}t_n - x_n\| \leq \|T_{[n]}t_n - t_n\| + \|t_n - x_n\|$, it follows from (4.13) that

$$\lim_{n\to\infty} \|T_{[n]}t_n - x_n\| = 0. \tag{4.15}$$

We observe that

$$\|x_{n+1} - x_n\|^2 \leq \alpha_n\|t_n - x_n\|^2 + (1-\alpha_n)\|T_{[n]}t_n - x_n\|^2.$$

It follows from (4.15) that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. It is easy to see that

$$\lim_{n\to\infty}\|x_{n+j}-x_n\|=0, \forall j=0,1,...,N-1. \tag{4.16}$$

By (4.10), (4.14) and (4.16), we get

$$\lim_{n\to\infty}\|x_n-T_{[n+j]}x_n\|=0.$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni}\rightharpoonup w$. From $\|x_n-u_n\|\to 0$ and $\|t_n-x_n\|\to 0$, we obtain that $u_{ni}\rightharpoonup w$ and $t_{ni}\rightharpoonup w$. Since $\{u_{ni}\}\subset C$ and C is closed and convex, we obtain $w\in C$. By similar argument with that in the proof of Theorem 3.2, we know that $w\in\cap_{j=0}^{N-1}Fix(T_j)$. The rest of the proof is similar with that in the proof of Theorem 4.1. The proof is now complete.

we now derive a weak convergence theorem of the cyclic algorithm based on nonextragradient method which solves the problem of finding a common element of the set of solutions of a generalized mixed equilibrium problem, the set of fixed points of a finite family of strict pseudo-contractions and the set of the variational inequality for an inverse strongly monotone mapping in a Hilbert space.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying (A1)-(A5) and $\varphi: C\to R\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let $A: C\to H$ and $B: C\to H$ be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $N\geq 1$ be an integer. For each $0\leq j\leq N-1$, let $T_j: C\to C$ be an ε_j-strict pseudo-contraction for some $0\leq\varepsilon_j<1$ such that $\Gamma_2=\cap_{j=0}^{N-1}Fix(T_j)\cap VI(C,A)\cap GMEP(F,\varphi,B)\neq\emptyset$. Let $\varepsilon=\max\{\varepsilon_j: 0\leq j\leq N-1\}$. Assume also that either (B1) or (B2) holds. Let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0=x\in C, \\ \quad F(u_n,y)+\varphi(y)-\varphi(u_n)+\langle Bx_n,y-u_n\rangle \\ \quad +\dfrac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, \qquad \forall y\in C, \\ y_n=P_C(u_n-\lambda_n Au_n), \\ x_{n+1}=\alpha_n y_n+(1-\alpha_n)T_{[n]}y_n, \end{cases}$$

for every $n=0,1,2,....$ If $\{\lambda_n\}\subset[a,b]$ for some $a,b\in(0,2\alpha),\{\alpha_n\}\subset[c,d]$ for some $c,d\in(\varepsilon,1)$ and $\{r_n\}\subset[\gamma,\tau]$ for some $\gamma,\tau\in(0,2\beta)$. Then, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge weakly to $w\in\Gamma_2$, where $w=P_{\Gamma_2}(x_n)$.

Proof. Let $u \in \Gamma_2$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1 in [31]. Then $u = P_C(u - \lambda_n Au) = T_{r_n}(u - r_n Bu)$. From the proof of Theorem 3.3, we have:

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2 \leq \|x_n - u\|^2. \tag{4.1}$$

$$\|T_{[n]}x - T_{[n]}y\|^2 \leq \|x - y\|^2 + \varepsilon\|x - T_{[n]}x - (y - T_{[n]}y)\|^2, \forall x, y \in C. \tag{4.8}$$

$$\|x_n - T_{[n+j]}x_n\| \leq (1 + L)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{[n+j]}x_{n+j}\|, \tag{4.10}$$

where $L = max_{0\leq j\leq N-1}\{\frac{1-\varepsilon_j}{1+\varepsilon_j}\}$.

$$\|x_n - T_{[n]}x_n\| \leq (1 + L)\|x_n - y_n\| + \|y_n - T_{[n]}y_n\|. \tag{4.17}$$

$$\|y_n - u\|^2 \leq \|u_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2 \leq \|u_n - u\|. \tag{4.18}$$

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle. \tag{4.19}$$

And

$$\begin{aligned} \|y_n - u\|^2 \leq& \|u_n - u\|^2 - \|u_n - y_n\|^2 \\ &+2\lambda_n\langle u_n - y_n, Au_n - Au\rangle - \lambda_n^2\|Au_n - Au\|^2. \end{aligned} \tag{4.20}$$

It follows from (4.1), (4.8), (4.18), $x_{n+1} = \alpha_n y_n + (1-\alpha_n)T_{[n]}y_n$ and $u = T_{[n]}u$ that

$$\begin{aligned} &\|x_{n+1} - u\|^2 \\ &= \alpha_n\|y_n - u\|^2 + (1-\alpha_n)\|T_{[n]}y_n - u\|^2 - \alpha_n(1-\alpha_n)\|y_n - T_{[n]}y_n\|^2 \\ &\leq \alpha_n\|y_n - u\|^2 + (1 - \alpha_n)[\|y_n - u\|^2 \\ &\quad +\varepsilon\|y_n - T_{[n]}y_n\|^2] - \alpha_n(1 - \alpha_n)\|y_n - T_{[n]}y_n\|^2 \\ &= \|y_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|y_n - T_{[n]}y_n\|^2 \\ &\leq \|u_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2 \\ &\quad +(1 - \alpha_n)(\varepsilon - \alpha_n)\|y_n - T_{[n]}y_n\|^2 \\ &\leq \|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2 \\ &\leq \|x_n - u\|^2, \end{aligned} \tag{4.21}$$

for every $n = 0, 1, 2, \ldots$. Therefore, there exists $\theta = \lim_{n\to\infty}\|x_n - u\|$ and $\{x_n\}$ is bounded. From (4.1) and (4.18), we also obtain that $\{y_n\}$ and $\{u_n\}$ are bounded.

It follows from (4.21) that

$$\|Au_n - Au\|^2 \leq \frac{1}{\lambda_n(2\alpha - \lambda_n)}\Big(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\Big)$$
$$\leq \frac{1}{a(2\alpha - b)}\Big(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\Big).$$

Hence, $\|Au_n - Au\| \to 0$.

From (4.21) and (4.1), we also have

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|y_n - T_{[n]}y_n\|^2, \tag{4.22}$$

for every $n = 0, 1, 2, ...$ From $\varepsilon < c \leq \alpha_n \leq d < 1$ and (4.22), we have

$$(1-d)(c-\varepsilon)\|y_n - T_{[n]}y_n\|^2 \leq (1-\alpha_n)(\alpha_n - \varepsilon)\|y_n - T_{[n]}y_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

This implies that

$$\lim_{n\to\infty} \|y_n - T_{[n]}y_n\| = 0. \tag{4.23}$$

Also by (4.21) and (4.1), we have

$$\|x_{n+1} - u\|^2 \leq \|u_n - u\|^2 \leq \|x_n - u\|^2 + r_n(r_n - 2\beta)\|Bx_n - Bu\|^2.$$

Thus, we have

$$\gamma(2\beta - \tau)\|Bx_n - Bu\|^2 \leq r_n(2\beta - r_n)\|Bx_n - Bu\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

It follows that $\|Bx_n - Bu\| \to 0$.

By (4.21) and (4.19), we have

$$\|x_{n+1} - u\|^2 \leq \|u_n - u\|^2 \leq \|x_n - u\|^2 - \|x_n - u_n\|^2 + 2r_n\langle Bx_n - Bu, x_n - u_n\rangle.$$

Hence,

$$\|x_n - u_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + 2r_n\|Bx_n - Bu\|\|x_n - u_n\|.$$

Since $\|Bx_n - Bu\| \to 0$, $\{x_n\}$ and $\{u_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0$.

From (4.21), (4.1) and (4.20), we have

$$\|x_{n+1} - u\|^2 \leq \|y_n - u\|^2 \leq \|x_n - u\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Au\|.$$

Thus, we have

$$\|u_n - y_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2 + 2\lambda_n\|u_n - y_n\|\|Au_n - Au\|.$$

It follows from $\|Au_n - Au\| \to 0$ that $\|u_n - y_n\| \to 0$. From $\|y_n - x_n\| \le \|y_n - u_n\| + \|x_n - u_n\|$ we also have $\|y_n - x_n\| \to 0$. It follows from (4.17) and (4.23) that

$$\lim_{n\to\infty} \|x_n - T_{[n]}x_n\| = 0. \tag{4.24}$$

Since $\|T_{[n]}y_n - x_n\| \le \|T_{[n]}y_n - y_n\| + \|y_n - x_n\|$, it follows from (4.23) that

$$\lim_{n\to\infty} \|T_{[n]}y_n - x_n\| = 0. \tag{4.25}$$

We observe that

$$\|x_{n+1} - x_n\|^2 \le \alpha_n \|y_n - x_n\|^2 + (1-\alpha_n)\|T_{[n]}y_n - x_n\|^2.$$

It follows from (4.25) that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. It is easy to see that

$$\lim_{n\to\infty} \|x_{n+j} - x_n\| = 0, \forall j = 0, 1, ..., N-1. \tag{4.26}$$

It follows from (4.10) and (4.26) that for each $j = 0, 1, ..., N-1$

$$\lim_{n\to\infty} \|x_n - T_{[n+j]}x_n\| = 0. \tag{4.27}$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$ and $\|y_n - x_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$ and $y_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$. The rest of the proof is similar with that in the proof of Theorem 4.2. The proof is now complete.

Let $A=0$, by Theorem 4.1 and 4.2, respectively, we obtain the following results:

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying (A1)-(A5) and $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B be a β-inverse strongly monotone mapping of C into H. Let $N \ge 1$ be an integer. For each $1 \le j \le N$, let $T_j : C \to C$ be an ε_j-strict pseudo-contraction for some $0 \le \varepsilon_j < 1$ such that $\Delta_1 = \cap_{j=1}^N Fix(T_j) \cap GMEP(F, \varphi, B) \ne \emptyset$. Assume for each n, $\{\zeta_j^{(n)}\}_{j=1}^N$ is a finite sequence of positive numbers such that $\sum_{j=1}^N \zeta_j^{(n)} = 1$ for all n and $\inf_{n\ge1} \zeta_j^{(n)} > 0$ for all $0 \le j \le N$. Let $\varepsilon = \max\{\varepsilon_j : 1 \le j \le N\}$. Assume also that either (B1) or (B2) holds. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle \\ \quad + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \qquad \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)\displaystyle\sum_{j=1}^N \zeta_j^{(n)} T_j u_n, \end{cases}$$

for every $n = 1, 2, \ldots$. *If* $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$. *Then,* $\{x_n\}$ *and* $\{u_n\}$ *converge weakly to* $w \in \Delta_1$*, where* $w = \lim_{n\to\infty} P_{\Delta_1} x_n$.

Theorem 4.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *to* R *satisfying (A1)-(A5) and* $\varphi : C \to R \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* B *be a* β*-inverse strongly monotone mapping of* C *into* H. *Let* $N \geq 1$ *be an integer. For each* $0 \leq j \leq N-1$*, let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Delta_2 = \cap_{j=0}^{N-1} Fix(T_j) \cap GMEP(F, \varphi, B) \neq \emptyset$. *Let* $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N-1\}$. *Assume that either (B1) or (B2) holds. Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) T_{[n]} u_n, \end{cases}$$

for every $n = 0, 1, 2, \ldots$. *If* $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \Delta_2$*, where* $w = \lim_{n\to\infty} P_{\Delta_2}(x_n)$.

Theorem 4.6. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *to* R *satisfying (A1)-(A5) and* $\varphi : C \to R \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* S *be a pseudo-contraction and* m*-Lipschitz-continuous mapping of* C *into itself and* B *be a* β*-inverse strongly monotone mapping of* C *into* H. *Let* $N \geq 1$ *be an integer. For each* $1 \leq j \leq N$*, let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Omega_1 = \cap_{j=1}^{N} Fix(T_j) \cap Fix(S) \cap GMEP(F, \varphi, B) \neq \emptyset$. *Assume for each* n, $\{\zeta_j^{(n)}\}_{j=1}^N$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^N \zeta_j^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ *for all* $0 \leq j \leq N$. *Let* $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. *Assume also that either (B1) or (B2) holds. Let* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \ \forall y \in C, \\ y_n = u_n - \lambda_n(u_n - Su_n), \\ t_n = P_C(u_n - \lambda_n(y_n - Sy_n)), \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n) \displaystyle\sum_{j=1}^N \zeta_j^{(n)} T_j t_n, \end{cases}$$

for every $n = 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{m+1}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$*. Then,* $\{x_n\}$*,* $\{u_n\}$ *,* $\{t_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \Omega_1$*, where* $w = \lim_{n\to\infty} P_{\Omega_1} x_n$*.*

Proof. By Theorem 4.1 and the proof of Theorem 3.6, we know the conclusion holds.

Theorem 4.7. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C$ *to* R *satisfying (A1)-(A5) and* $\varphi : C \to R \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let Let* S *be a pseudo-contraction and* m*-Lipschitz-continuous mapping of* C *into itself and* B *be a* β*-inverse strongly monotone mapping of* C *into* H*. Let* $N \geq 1$ *be an integer. For each* $0 \leq j \leq N - 1$*, let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Omega_2 = \cap_{j=0}^{N-1} Fix(T_j) \cap Fix(S) \cap GMEP(F, \varphi, B) \neq \emptyset$*. Let* $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N - 1\}$*. Assume that either (B1) or (B2) holds. Let* $\{x_n\}$*,* $\{u_n\}$*,* $\{t_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ \qquad F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle \\ \qquad + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = u_n - \lambda_n(u_n - Su_n), \\ t_n = P_C(u_n - \lambda_n(y_n - Sy_n)), \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n)T_{[n]}t_n, \end{cases}$$

for every $n = 0, 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{m+1}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$*. Then,* $\{x_n\}$*,* $\{u_n\}$ *,* $\{t_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \Omega_2$*, where* $w = \lim_{n\to\infty} P_{\Omega_2}(x_n)$*.*

Proof. From Theorem 4.2 and the proof of Theorem 4.6, we know that the conclusion holds.

Remark 4.1. In Theorems 4.1-4.7, if we assume some of the mappings F, B, φ equal to zero mappings, we can obtain many new and interesting weak convergence theorems for some algorithms for the special case of problem (1.1) (i.e., Problems (1.2)-(1.7)). Now we only give five examples as follows:

Corollary 4.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C$ *to* R *satisfying (A1)-(A5) and* φ :

$C \to R \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H*. Let* $N \geq 1$ *be an integer. For each* $1 \leq j \leq N$*, let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Lambda = \cap_{j=1}^{N} Fix(T_j) \cap VI(C, A) \cap MEP(F, \varphi) \neq \emptyset$*. Assume for each* n*,* $\{\zeta_j^{(n)}\}_{j=1}^{N}$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^{N} \zeta_j^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ *for all* $0 \leq j \leq N$*. Let* $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$*. Assume also that either (B1) or (B2) holds. Let* $\{x_n\}$*,* $\{u_n\}$*,* $\{t_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C, \\ \quad F(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n) \displaystyle\sum_{j=1}^{N} \zeta_j^{(n)} T_j t_n, \end{cases}$$

for every $n = 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some$c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ *for some* $\gamma > 0$*. Then,* $\{x_n\}$*,* $\{u_n\}$ *,* $\{t_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \Lambda$*, where* $w = \lim_{n\to\infty} P_\Lambda x_n$*.*

Proof. Putting $B = 0$, by Theorem 4.1 we obtain the desired result.

Corollary 4.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H*. Let* F *be a bifunction from* $C \times C$ *to* R *satisfying (A1)-(A5). Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H*. Let* $N \geq 1$ *be an integer. For each* $0 \leq j \leq N-1$*, let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Sigma = \cap_{j=0}^{N-1} Fix(T_j) \cap VI(C, A) \cap EP(F) \neq \emptyset$*. Let* $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N - 1\}$*. Assume that either (B3) or (B2) holds. Let* $\{x_n\}$*,* $\{u_n\}$*,* $\{t_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ \quad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ t_n = P_C(u_n - \lambda_n A y_n), \\ x_{n+1} = \alpha_n t_n + (1 - \alpha_n) T_{[n]} t_n, \end{cases}$$

for every $n = 0, 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ *for some* $\gamma > 0$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \Sigma$, *where* $w = \lim_{n\to\infty} P_\Sigma(x_n)$.

Proof. Putting $B = 0$ and $\varphi = 0$. By Theorem 4.2 we obtain the desired result.

Corollary 4.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H *and* B *be a* β*-inverse strongly monotone mapping of* C *into* H. *Let* $N \geq 1$ *be an integer. For each* $1 \leq j \leq N$, *let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Theta_1 = \cap_{j=1}^{N} Fix(T_j) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. *Assume for each* n, $\{\zeta_j^{(n)}\}_{j=1}^{N}$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^{N} \zeta_j^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ *for all* $0 \leq j \leq N$. *Let* $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. *Let* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_1 &= x \in C, \\ u_n &= P_C(x_n - r_n Bx_n), \\ y_n &= P_C(u_n - \lambda_n Au_n), \\ t_n &= P_C(u_n - \lambda_n Ay_n), \\ x_{n+1} &= \alpha_n t_n + (1 - \alpha_n) \displaystyle\sum_{j=1}^{N} \zeta_j^{(n)} T_j t_n, \end{cases}$$

for every $n = 1, 2, ...$ *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k}), \{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \Theta_1$, *where* $w = \lim_{n\to\infty} P_{\Theta_1} x_n$.

Proof. Putting $F = 0$ and $\varphi = 0$, by Theorem 4.1 and the proof of Corollary 3.3, we obtain the desired result.

Corollary 4.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H *and* B *be a* β*-inverse strongly monotone mapping of* C *into* H. *Let* $N \geq 1$ *be an integer. For each* $0 \leq j \leq N-1$, *let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Theta_2 = \cap_{j=0}^{N-1} Fix(T_j) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$. *Let* $\varepsilon = \max\{\varepsilon_j : 0 \leq j \leq N - 1\}$. *Let* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_0 &= x \in C, \\ u_n &= P_C(x_n - r_n B x_n), \\ y_n &= P_C(u_n - \lambda_n A u_n), \\ t_n &= P_C(u_n - \lambda_n A y_n), \\ x_{n+1} &= \alpha_n t_n + (1-\alpha_n) T_{[n]} t_n, \end{cases}$$

for every $n = 0, 1, 2, \ldots$. *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, \tau]$ *for some* $\gamma, \tau \in (0, 2\beta)$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \Theta_2$, *where* $w = \lim_{n\to\infty} P_{\Theta_2}(x_n)$.

Proof. Putting $F = 0$ and $\varphi = 0$, by Theorem 4.2 and the proof of Corollary 3.3, we obtain the desired result.

Corollary 4.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\varphi : C \to R \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H. *Let* $N \geq 1$ *be an integer. For each* $1 \leq j \leq N$, *let* $T_j : C \to C$ *be an* ε_j*-strict pseudo-contraction for some* $0 \leq \varepsilon_j < 1$ *such that* $\Xi = \cap_{j=1}^N Fix(T_j) \cap VI(C, A) \cap Argmin(\varphi) \neq \emptyset$. *Assume for each* n, $\{\zeta_j^{(n)}\}_{j=1}^N$ *is a finite sequence of positive numbers such that* $\sum_{j=1}^N \zeta_j^{(n)} = 1$ *for all* n *and* $\inf_{n\geq 1} \zeta_j^{(n)} > 0$ *for all* $0 \leq j \leq N$. *Let* $\varepsilon = \max\{\varepsilon_j : 1 \leq j \leq N\}$. *Assume also that either (B4) or (B2) holds. Let* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *be sequences generated by*

$$\begin{cases} x_1 &= x \in C, \\ & \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \qquad \forall y \in C, \\ y_n &= P_C(u_n - \lambda_n A u_n), \\ t_n &= P_C(u_n - \lambda_n A y_n), \\ x_{n+1} &= \alpha_n t_n + (1-\alpha_n) \displaystyle\sum_{j=1}^N \zeta_j^{(n)} T_j t_n, \end{cases}$$

for every $n = 1, 2, \ldots$. *If* $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset [\gamma, +\infty)$ *for some* $\gamma > 0$. *Then,* $\{x_n\}$, $\{u_n\}$, $\{t_n\}$ *and* $\{y_n\}$ *converge weakly to* $w \in \Xi$, *where* $w = \lim_{n\to\infty} P_\Xi x_n$.

Proof. Putting $F = 0$ and $B = 0$, by Theorem 4.1 we obtain the desired result.

Remark 4.2.

(i) Theorems 4.1-4.7 generalize, extend and improve Theorem 4.1 in [12] and Theorem 3.1 in [14]. Corollary 4.3 and 4.4 generalize and improve Theorem 3.1 in [26].

(ii) Let $A = B = 0$, by Corollary 3.3, 3.4, 4.3 and 4.4, respectively, we can recover Theorem 5.1, 5.2, 3.3 and 4.1 in [19] with modified condition $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$.

(iii) Let $A = 0$, by Corollary 3.3, 3.4, 4.3 and 4.4, respectively, we can recover Theorem 5.1, 5.2, 3.1 and 4.1 in [30].

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Jian-Wen Peng
College of Mathematics and Computer Science,
Chongqing Normal University,
Chongqing 400047,
P. R. China

Jen-Chih Yao
Department of Applied Mathematics,
National Sun Yat-sen University,
Kaohsiung 804, Taiwan.
E-mail: yaojc@math.nsysu.edu.tw