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SYSTEMS OF GENERALIZED VECTOR QUASI-VARIATIONAL INCLUSION PROBLEMS AND APPLICATION TO MATHEMATICAL PROGRAMS

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Abstract. In this paper, we introduce and study some new systems of generalized vector quasi-variational inclusion problems involving condensing mappings in locally FC-uniform spaces. These systems contain many known systems of generalized vector quasi-variational inclusion problems, systems of generalized vector quasi-equilibrium problems and systems of vector quasi-optimization problems as special cases. By applying an existence theorem of maximal elements of a family of set-valued mappings involving condensing mapping due to author, we prove some new existence theorems of solutions for the systems of generalized quasi-variational inclusion problems. As applications, some existence results of solutions of the mathematical programs with systems of generalized vector quasi-variational inclusion constraints are established in noncompact locally FC-uniform spaces.

1. Introduction

Recently Lin [22] and Lin and Tu [23] studied some systems of generalized quasi-variational inclusion problems with applications in locally convex topological vector spaces. Hai and Khanh [17] studied some systems of set-valued quasivariational inclusion problems in topological vector spaces with applications.

Inspired by this line of research works, we introduce and study some new systems of generalized quasi-variational inclusion problems involving condensing set-valued mappings in locally FC-uniform spaces without convexity structure.

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For a nonempty set X, we denote by 2^X the family of all subsets of X. Let I be any index set. For each $i \in I$, let X_i, Y_i and Z_i be topological spaces. Let $X = \prod_{i \in I} X_i$, $Y = \prod_{i \in I} Y_i$ and for $x \in X$, $x_i = \pi_i(x)$ be the projection of x onto X_i. For each $i \in I$, let $A_i : X \times Y \to 2^{X_i}$, $T_i, S_i : X \times Y \to 2^{Y_i}$ and $\Phi_i, \Psi_i : Y_i \times X_i \times X \to 2^{Z_i}$ be set-valued mappings. We consider the following systems of generalized vector quasi-variational inclusions problems (SGVQVIP):

(I) Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x}, \hat{y})$, $\hat{y}_i \in S_i(\hat{x}, \hat{y})$ and

$$\text{SGVQVIP(I)} \;\; \Phi_i(v_i, u_i, \hat{x}) \subseteq \Psi_i(v_i, \hat{x}_i, \hat{x}), \; \forall \, v_i \in T_i(\hat{x}, \hat{y}), \; u_i \in A_i(\hat{x}, \hat{y}).$$

(II) Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x}, \hat{y})$, $\hat{y}_i \in S_i(\hat{x}, \hat{y})$ and

$$\text{SGVQVIP(II)} \;\; \Phi_i(v_i, u_i, \hat{x}) \nsubseteq \Psi_i(v_i, \hat{x}_i, \hat{x}), \; \forall \, v_i \in T_i(\hat{x}, \hat{y}), \; u_i \in A_i(\hat{x}, \hat{y}).$$

(III) Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x}, \hat{y})$, $\hat{y}_i \in S_i(\hat{x}, \hat{y})$ and
SGVQVIP(III)

$$\Phi_i(v_i, u_i, \hat{x}) \bigcap \Psi_i(v_i, \hat{x}_i, \hat{x}) = \emptyset, \; \forall \, v_i \in T_i(\hat{x}, \hat{y}), \; u_i \in A_i(\hat{x}, \hat{y}).$$

(IV) Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x}, \hat{y})$, $\hat{y}_i \in S_i(\hat{x}, \hat{y})$ and
SGVQVIP(IV)

$$\Phi_i(v_i, u_i, \hat{x}) \bigcap \Psi_i(v_i, \hat{x}_i, \hat{x}) \neq \emptyset, \; \forall \, v_i \in T_i(\hat{x}, \hat{y}), \; u_i \in A_i(\hat{x}, \hat{y}).$$

If for each $i \in I$ and for all $(x, y) \in X \times Y$, $A_i(x, y) = A_i(x)$, $S_i(x, y) = S_i(x)$ and $T_i(x, y) = T_i(x)$, then the SGQVIP (I)- SGQVIP (VI) reduce to the following problems:

$(\text{I})'$ Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$, $\hat{y}_i \in S_i(\hat{x})$ and

$$\text{SGVQVIP(I)}' \quad \Phi_i(v_i, u_i, \hat{x}) \subseteq \Psi_i(v_i, \hat{x}_i, \hat{x}), \; \forall \, v_i \in T_i(\hat{x}), \; u_i \in A_i(\hat{x}).$$

$(\text{II})'$ Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$, $\hat{y}_i \in S_i(\hat{x})$ and

$$\text{SGVQVIP(II)}' \quad \Phi_i(v_i, u_i, \hat{x}) \nsubseteq \Psi_i(v_i, \hat{x}_i, \hat{x}), \; \forall \, v_i \in T_i(\hat{x}), \; u_i \in A_i(\hat{x}).$$

$(\text{III})')$ Find $(\hat{x}, \hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$, $\hat{y}_i \in S_i(\hat{x})$ and

$$\text{SGVQVIP(III)}' \;\; \Phi_i(v_i, u_i, \hat{x}) \bigcap \Psi_i(v_i, \hat{x}_i, \hat{x}) = \emptyset, \; \forall \, v_i \in T_i(\hat{x}), \; u_i \in A_i(\hat{x}).$$

(IV)′ Find $(\hat{x},\hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$, $\hat{y}_i \in S_i(\hat{x})$ and

$$\mathrm{SGVQVIP(IV)'}\ \ \Phi_i(v_i,u_i,\hat{x})\bigcap\Psi_i(v_i,\hat{x}_i,\hat{x})\neq\emptyset,\ \forall\ v_i\in T_i(\hat{x}),\ u_i\in A_i(\hat{x}).$$

It is easy to see that the SGVQVIP (I)’ and SGVQVIP (IV)’ contain respectively the (SQVIP 2) and (SQVIP 4) introduced and studied by Hai and Khanh [17] in topological vector spaces as special cases. The SGVQVIP (II)’ and SGQVIP (III)’ are new.

If for each $i \in I$, Z_i is a topological vector space and let $\Psi_i(y_i,x_i,x) = \{0\}$ for all $(y_i,x_i,x) \in Y_i \times X_i \times X$, then the SGVQVIP (III) and SGVVQVIP (IV) reduce to the following problems respectively:

(a) Find $(\hat{x},\hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x},\hat{y})$, $\hat{y}_i \in S_i(\hat{x},\hat{y})$ and

$$\mathrm{(SVDP(a))}\qquad 0 \notin \Phi_i(v_i,u_i,\hat{x}),\ \forall\ v_i \in T_i(\hat{x},\hat{y}),\ u_i \in A_i(\hat{x},\hat{y}).$$

(b) Find $(\hat{x},\hat{y}) \in X \times Y$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x},\hat{y})$, $\hat{y}_i \in S_i(\hat{x},\hat{y})$ and

$$\mathrm{(SVIP(b))}\qquad 0 \in \Phi_i(v_i,u_i,\hat{x}),\ \forall\ v_i \in T_i(\hat{x},\hat{y}),\ u_i \in A_i(\hat{x},\hat{y}).$$

It is easy to see that the SVIP (b) and SVDP (a) contain respectively the (SVIP1) and (SVIP2), introduced and studied by Lin and Tu [23] in locally convex topological vector spaces as special cases.

Hence the SGVQVIP (I)-SGVQVIP(VI) include many known systems of generalized vector quasi-variational inclusion problems, systems of quasi-variational disclusion problems and generalized vector quasi-equilibrium problems with wide applications as very special cases, for example, see [2-4, 8-10, 12, 15, 19-21] and the references therein.

In this paper, we introduce the new notions of Ψ_i-FC-quasiconvexity for set-valued mappings $\Phi_i, \Psi_i : Y_i \times X_i \times X \to 2^{Z_i}$ in FC-space. by using these notions and an existence theorem of maximal elements for a family of set-valued mappings involving condensing mappings due to author [11], some new existence theorems of solutions for the SGVQVIP (I)-SGVQVIP (VI) are proved in noncompact locally FC-uniform spaces. These results improve, unify and generalize many known results in recent literature. As applications, some existence results of solutions of the mathematical programs with systems of generalized vector quasi-variational inclusion constraints are established in noncompact locally FC-uniform spaces.

2. Preliminaries

For a nonempty set X, we denote by $< X >$ the family of all nonempty finite subsets of X. Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \cdots, e_n$. If J is a nonempty subset of $\{0, 1, \cdots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j : j \in J\}$.

The following notion was introduced by Ben-El-Mechaiekh et al. [5].

Definition 2.1. (X, Γ) is called a L-convex space if X is a topological space and $\Gamma :< X >\to 2^X$ is a mapping such that for each $N \in< X >$ with $|N| = n+1$, there exists a continuous mapping $\varphi_N : \Delta_n \to \Gamma(N)$ satisfying $A \in< N >$ with $|A| = J + 1$ implies $\varphi_N(\Delta_J) \subseteq \Gamma(A)$, where Δ_J is the face of Δ_N corresponding to A.

The following notion of a finitely continuous topological space (in short, FC-space) was introduced by Ding [7].

Definition 2.2. (X, φ_N) is said to be a FC-space if X is a topological space and for each $N = \{x_0, \cdots, x_n\} \in< X >$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to X$. A subset D of (X, φ_N) is said to be a FC-subspace of X if for each $N = \{x_0, \cdots, x_n\} \in< X >$ and for each $(x_{i_0}, \cdots, x_{i_k}\} \subseteq N \bigcap D$, $\varphi_N(\Delta_k) \subseteq D$ where $\Delta_k = \text{co}(\{e_{i_j} : j = 0, \cdots, k\})$.

Comparing the definitions of L-convex spaces and FC-spaces, it is clear that each L-convex space must be a FC-space. The following examples show that there exists a FC-space which is not a L-convex space.

Example 2.1. Let $(X, \|\cdot\|)$ be a strictly convex and reflexive Banach space and X_1 is a nonempty closed convex subset of X and X_2 be a nonempty convex subset of X with $X_1 \bigcap X_2 = \emptyset$. Then $E = X_1 \bigcup X_2$ is not convex. For each $N = \{x_0, \cdots, x_n\} \in< E >$, define a mapping $\varphi_N : \Delta_n \to 2^X$ by

$$\varphi_N(\alpha) = \begin{cases} \displaystyle\sum_{i=0}^{n} \alpha_i x_i, & \text{if } N \subset X_1 \text{ or } N \subset X_2, \\ \displaystyle\sum_{i=0}^{j} \alpha_i x_i + \sum_{i=j+1}^{n} \alpha_i P_{X_1}(x_i), & \text{if } N = N_1 \bigcup N_2, \end{cases}$$

for all $\alpha = (\alpha_0, \cdots, \alpha_n) \in \Delta_n$ where $N_1 = \{x_0, \cdots, x_j\} \subseteq X_1$, $N_2 = \{x_{j+1}, \cdots, x_n\} \subseteq X_2$ and $P_{X_1}(x_i)$ is the metric projection of x_i onto X_1. It is easy to see that φ_N is continuous and hence (E, φ_N) is a FC-space. For any convex subset A of X_1 with $A \neq X_1$ and any subset B of X_2, it is easy to check that the sets A, X_1 and $X_1 \bigcup B$ are all FC-subspaces of E. But the sets X_2, B and $A \bigcup B$ are not

FC-subspaces of E. If we define a set-valued mapping $\Gamma :< E >\to 2^E$ by

$$\Gamma(N) = \varphi_N(\Delta_n),\ \forall\ N = \{x_0, \cdots, x_n\} \in< E >,$$

then we have that for each $N = \{x_0, \cdots, x_n\} \in< E >$, $\varphi_N(\Delta_n) \subseteq \Gamma(N)$. But if $N = N_1 \bigcup N_2$ where $N_1 = \{x_0, \cdots, x_j\} \subseteq X_1$ and $N_2 = \{x_{j+1}, \cdots, x_n\} \subseteq X_2$, then we have $\Gamma(N_2) = \varphi_{N_2}(\Delta_J) \subseteq X_2$ and $\varphi_N(\Delta_J) \subseteq X_1$ where $\Delta_J = \text{co}\{e_k : k = j+1, \cdots, n\}$. Hence we have $\varphi_N(\Delta_J) \not\subseteq \Gamma(N_2)$ and so (E, Γ) is not a L-convex space.

It is clear that any convex subset of a topological vector space, any H-space introduced by Horvath [18], any G-convex space introduced by Park and Kim [25] and any L-convex spaces introduced by Ben-El-Mechaiekh et al. [5] are all FC-space. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces.

By the definition of FC-subspaces of a FC-space, it is easy to see that if $\{B_i\}_{i\in I}$ is a family of FC-subspaces of a FC-space (Y, φ_N) and $\bigcap_{i\in I} B_i \neq \emptyset$, then $\bigcap_{i\in I} B_i$ is also a FC-subspace of (Y, φ_N) where I is any index set. For a subset A of (Y, φ_N), we can define the FC-hull of A as follows:

$$FC(A) = \bigcap\{B \subset Y : A \subseteq B \text{ and } B \text{ is } FC - \text{subspace of } Y\}.$$

Clearly, $FC(A)$ is the smallest FC-subspace of Y containing A and each FC-subspace of a FC-space is also a FC-space

Lemma 2.1. [13]. *Let (Y, φ_N) be a FC-space and A be a nonempty subset of Y. Then*

$$FC(A) = \bigcup\{FC(N) : N \in< A >\}.$$

Lemma 2.2. [13]. *Let X be a topological space, (Y, φ_N) be a FC-space and $G : X \to 2^Y$ be such that $G^{-1}(y) = \{x \in X : y \in G(x)\}$ is compactly open in X for each $y \in Y$. Then the mapping $FC(G) : X \to 2^Y$ defined by $FC(G)(x) = FC(G(x))$ for each $x \in X$ satisfies that $(FC(G))^{-1}(y)$ is also compactly open in X for each $y \in Y$.*

The following notion was introduced by Ding [15].

Definition 2.3. $(X, \mathcal{U}, \varphi_N)$ is said to be a locally FC-uniform space if $(X, \mathcal{U})$ is a uniform space, and (X, φ_N) is an FC-space such that $\mathcal{U}$ has a basis $\mathcal{B}$ consisting of entourages satisfying that for each $V \in \mathcal{B}$, the set $V[M] = \{x \in X : M \bigcap V[x] \neq \emptyset\}$ is an FC-subspace of X whenever $M \subseteq X$ is an FC-subspace of X.

Example 2.2. Let (E, φ_N) is the FC-space given in Example 2.1. Note that E is a metric space and each metric space is a Hausdorff uniform space with uniform structure $\mathcal{U} = \{V(\epsilon) : \epsilon > 0\}$ where $V(\epsilon) = \{(x, y) \in E \times E : d(x, y) < \epsilon\}$, see [16, p. 201]. For each $V(\epsilon) \in \mathcal{U}$, we have $V[x] = \{y \in X : |x - y| < \epsilon\}$ and so $V[M] = \{x \in X : M \bigcap V[x] \neq \emptyset\} = \bigcup_{x \in M} V[x]$ for each FC-subspace M of E. It is easy to check that $(E, \mathcal{U}, \varphi_N)$ is locally FC-uniform space.

Definition 2.4. [11] Let C be a lattice with a least element, denoted by 0, $(X, \mathcal{U}, \varphi_N)$ be a locally FC-uniform space and $\Phi : 2^X \to C$ be a mapping. Then Φ is said to be a measure of noncompactness on X if the following conditions are satisfied:

(i) for any $A \subset X$, $\Phi(A) = 0$ if and only if A is relatively compact,

(ii) $\Phi(\overline{FC}(A)) = \Phi(A)$, where $\overline{FC}(A)$ denotes the closure of FC-hull of A,

(iii) $\Phi(A \bigcup B) = \max\{\Phi(A), \Phi(B)\}$.

It follows from (iii) that if $A \subseteq B$, then $\Phi(A) \leq \Phi(B)$.

Definition 2.5. [11] Let $\Phi : 2^X \to C$ be a measure of noncompactness on $(X, \mathcal{U}, \varphi_N)$. A mapping $G : X \to 2^X$ is said to be Φ-condensing if, whenever $A \subseteq X$ with $\Phi(G(A)) \geq \Phi(A)$ then A is relatively compact.

Remark 2.1. It is clear that if $G : X \to 2^X$ is Φ-condensing and $G^* : X \to 2^X$ satisfies $G^*(x) \subseteq G(x),\ \forall\ x \in X$, then G^* is also Φ-condensing.

The following result is Theorem 2.2 of Ding [15].

Lemma 2.3. *Let I be any index set. For each $i \in I$, let $(X_i, \mathcal{U}_i, \varphi_{N_i})$ be a locally FC-uniform spaces with each $(X_i, \mathcal{U}_i)$ having a basis $\mathcal{B}_i$ consisting of symmetric entourages, and $X = \prod_{i \in I} X_i$, $\mathcal{U} = \prod_{i \in I} \mathcal{U}_i$ and $\varphi_N = \prod_{i \in I} \varphi_{N_i}$ for any $N \in < X >$ where N_i is the projection of N onto X_i. Then $(X, \mathcal{U}, \varphi_N)$ is also a locally FC-uniform space.*

The following result is a special case of Corollary 3.3 of Ding [13].

Lemma 2.4. *Let I be any index set. For each $i \in I$, let (X_i, φ_{N_i}) be a compact FC-space and $X = \prod_{i \in I} X_i$. For each $i \in I$, let $G_i : X \to 2^{X_i}$ be such that*

(i) *for each $x \in X$, $G_i(x)$ is a FC-subspace of X_i,*

(ii) *for each $x \in X$, $\pi_i(x) \notin G_i(x)$,*

(iii) *for each $y_i \in X_i$, $G_i^{-1}(y_i)$ is compactly open in X*

Then there exists $\hat{x} \in X$ *such that* $G_i(\hat{x}) = \emptyset$ *for each* $i \in I$.

The following result is Lemma 2.2 of Ding [11].

Lemma 2.5. *Let* I *be any index set. For each* $i \in I$*, let* $(X_i, \mathcal{U}_i, \varphi_{N_i})$ *be a locally* FC*-uniform space and* $G_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ *be a set-valued mapping such that* $G : X \to 2^X$ *defined by* $G(x) = \prod_{i \in I} G_i(x)$ *is* Φ*-condensing. Then there exists a nonempty compact* FC*-subspace* $K = \prod_{i \in I} K_i$ *of* X *where each* K_i *is a compact* FC*-subspace of* X_i *such that* $G(K) \subseteq K$.

By Lemma 2.4 and Lemma 2.5, we have the following result.

Theorem 2.1. *Let* I *be any index set. For each* $i \in I$*, let* $(X_i, \mathcal{U}_i, \varphi_{N_i})$ *be a locally* FC*-uniform space. Let* $X = \prod_{i \in I} X_i$ *be the locally* FC*-uniform space defined as in Lemma 2.3 and* Φ *is a measure of noncompactness on* X*. For each* $i \in I$*, let* $G_i : X \to 2^{X_i}$ *be such that*

(i) *for each* $x \in X$*,* $G_i(x)$ *is a* FC*-subspace of* X_i*,*

(ii) *for each* $x \in X$*,* $\pi_i(x) \notin G_i(x)$*,*

(iii) *for each* $y_i \in X_i$*,* $G_i^{-1}(y_i)$ *is compactly open in* X*,*

(iv) *the mapping* $G : X \to 2^X$ *defined by* $G(x) = \prod_{i \in I} G_i(x)$ *for each* $x \in X$ *is* Φ*-condensing.*

Then there exist a compact subset $K = \prod_{i \in I} K_i$ *of* X *and a point* $\hat{x} \in K$ *such that* $G_i(\hat{x}) = \emptyset$*, for each* $i \in I$.

Proof. By (iv) and Lemma 2.5, there exists a compact FC-subspace $K = \prod_{i \in I} K_i$ of X where each K_i is a compact FC-subspace of X_i such that $G(K) \subset K$. It follows from the conditions (i)-(iii) that for each $i \in I$, the restriction $G_i|_K$ of G_i onto K also satisfies the conditions (i)-(iii) of Lemma 2.4. Hence all conditions of Lemma 2.4 are satisfied. By Lemma 2.4, there exists $\hat{x} \in K$ such that $G_i(\hat{x}) = \emptyset$, for each $i \in I$.

Remark 2.2. Theorem 2.1 generalized Proposition 2 of Chebbi and Florenzano [6] and Theorem 4.3 of Lin and Ansari [19] to locally FC-uniform spaces without convexity structure.

3. Existence of Solutions for the SGQVIPs

Throughout this section, unless otherwise specified, we shall fix the following notations and assumptions. Let I be any index set. For each $i \in I$, let $(X_i, \mathcal{U}_i, \varphi_{N_i})$

and $(Y_i, \mathcal{U}_i, \varphi'_{N_i})$ be locally FC-uniform spaces, and Z_i be a topological space. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$ be the locally FC-uniform spaces defined as in Lemma 2.3. For each $i \in I$, let $A_i : X \times Y \to 2^{X_i}$, $T_i, S_i : X \times Y \to 2^{Y_i}$ and $\Phi_i, \Psi_i : Y_i \times X_i \times X \to 2^{Z_i}$ be set-valued mappings.

Definition 3.1. For each $i \in I$ and $y \in Y$, Φ_i is said to be

(i) Ψ_i-FC-quasiconvex of type (I) in the first two arguments if for each $N_i = \{u_{i,0}, \cdots, u_{i,n}\} \in < X_i >$ and for each $x \in X$ with $x_i \in FC(N_i)$, there exists $j \in \{0, \cdots, n\}$ such that $\Phi_i(v_i, u_{i,j}, x) \subseteq \Psi_i(v_i, x_i, x)\}, \ \forall \ v_i \in T_i(x, y)$,

(ii) Ψ_i-FC-quasiconvex of type (II) in the first two arguments if each $N_i = \{u_{i,0}, \cdots, u_{i,n}\} \in < X_i >$ and for each $x \in X$ with $x_i \in FC(N_i)$, there exists $j \in \{0, \cdots, n\}$ such that $\Phi_i(v_i, u_{i,j}, x) \not\subseteq \Psi_i(v_i, x_i, x)\}, \ \forall \ v_i \in T_i(x, y)$,

(iii) Ψ_i-FC-quasiconvex of type (III) in the first two arguments if for each $N_i = \{u_{i,0}, \cdots, u_{i,n}\} \in < X_i >$ and for each $x \in X$ with $x_i \in FC(N_i)$, there exists $j \in \{0, \cdots, n\}$ such that $\psi_i(v_i, u_{i.j}, x) \bigcap \Psi_i(v_i, x_i, x) = \emptyset, \ \forall \ v_i \in T_i(x, y)$,

(iv) Ψ_i-FC-quasiconvex of type (IV) in the first two arguments if for each $N_i = \{u_{i,0}, \cdots, u_{i,n}\} \in < X_i >$ and for each $x \in X$ with $x_i \in FC(N_i)$, there exists $j \in \{0, \cdots, n\}$ such that $\psi_i(v_i, u_{i.j}, x) \bigcap \Psi_i(v_i, x_i, x) \neq \emptyset, \ \forall \ v_i \in T_i(x, y)$,

Lemma 3.1. *For each $i \in I$, define a set-valued mapping $P_{i,k} : X \times Y \to 2^{X_i}$, $k = 1, 2, 3, 4$ by*

$$P_{i,1}(x,y) = \{u_i \in X_i : \Phi(v_i, u_i, x) \not\subseteq \Psi_i(v_i, x_i, x), \text{ for some } v_i \in T_i(x,y)\}$$

$$(resp., P_{i,2}(x,y) = \{u_i \in X_i : \Phi(v_i, u_i, x) \subseteq \Psi_i(v_i, x_i, x), \text{ for some } v_i \in T_i(x,y)\},$$

$$P_{i,3}(x,y) = \{u_i \in X_i : \Phi(v_i, u_i, x) \bigcap \Psi_i(v_i, x_i, x) \neq \emptyset, \text{ for some } v_i \in T_i(x,y)\}$$

$$P_{i,4}(x,y) = \{u_i \in X_i : \Phi(v_i, u_i, x) \bigcap \Psi_i(v_i, x_i, x) = \emptyset, \text{ for some } v_i \in T_i(x,y)\}).$$

Then for each $y \in Y$, Φ_i is Ψ_i-FC-quasiconvex of type (I) (resp., type (II), type (III), type (IV)) if and only if for each $(x, y) \in X \times Y$, $x_i \notin FC(P_{i,1}(x, y))(resp., x_i \notin FC(P_{i,2}(x, y))$, $x_i \notin FC(P_{i,3}(x, y))$, $x_i \notin FC(P_{i,4}(x, y)))$.

Proof. We only need to prove the case $k = 1$, since the proof for the cases $k = 2, 3, 4$ is completely similar.

Necessity. Suppose that for each $y \in Y$, Φ_i is Ψ_i-FC-quasiconvex of type (I). If there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x}_i \in FC(P_{i,1}(\bar{x}, \bar{y}))$, then by Lemma 2.1, there exists $N_i = \{u_{i,0}, \cdots, u_{i,n}\} \in < P_i(\bar{x}, \bar{y}) >$ such that $\bar{x}_i \in FC(N_i)$. By the definition of $P_{i,1}$, we have that for each $j \in \{0, \cdots, n\}$, there exists $\bar{v}_{i,j} \in T_i(\bar{x}, \bar{y})$ such that

$$\Phi(\bar{v}_{i,j}, u_{i,j}, \bar{x}) \not\subseteq \Psi_i(\bar{v}_{i,j}, \bar{x}_i, \bar{x})$$

which contradicts that the assumption for each $y \in Y$, Φ_i is Ψ_i-FC-quasiconvex of type (I).

Sufficiency. Suppose that for each $(x, y) \in X \times Y$, $x_i \notin FC(P_{i,1}(x, y))$. If for some $\bar{y} \in Y$, Φ_i is not Ψ_i-FC-quasiconvex of type (I), then there exist $N_i = \{u_{i,0}, \cdots, u_{i,n}\} \in < X_i >$ and $\bar{x} \in X$ with $\bar{x}_i \in FC(N_i)$ such that for each $j \in \{0, \cdots, n\}$, there exists $\bar{v}_{i,j} \in T_i(\bar{x}, \bar{y})$ such that

$$\Phi(\bar{v}_{i,j}, u_{i,j}, \bar{x}) \not\subseteq \Psi_i(\bar{v}_{i,j}, \bar{x}_i, \bar{x}).$$

It follows that $N_i \subseteq P_{i,1}(\bar{x}, \bar{y})$ and hence we have $\bar{x}_i \in FC(N_i) \subseteq FC(P_{i,1}(\bar{x}, \bar{y}))$ which is a contradiction.

Lemma 3.2. [1]. *Let X and Y be topological spaces and $G : X \to 2^Y$ be a set-valued mapping. Then G is lower semicontinuous in $x \in X$ if and only if for any $y \in G(x)$ and any net $\{x_\alpha\} \subset X$ satisfying $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in G(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 3.3. *For each $i \in I$, let X_i, Y_i and Z_i be topological spaces, and let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $A_i : X \times Y \to 2^{X_i}$, $T_i, S_i : X \times Y \to 2^{Y_i}$ and $\Phi_i, \Psi_i : Y_i \times X_i \times X \to 2^{Z_i}$ be set-valued mappings such that*

(*i*) *T_i is lower semicontinuous on each compact subsets of $X \times Y$,*

(*ii*) *for each $u_i \in X_i$, $(v_i, x) \mapsto \Phi_i(v_i, u_i, x)$ is lower semicontinuous on each compact subset of $Y_i \times X$,*

(*iii*) *the mapping Ψ_i (resp. $(y_i, x_i, x) \mapsto Z_i \setminus \Psi_i(y_i, x_i, x)$) is upper semicontinuous on each compact subsets of $Y_i \times X_i \times X$ with closed values.*

Then for each $i \in I$ and $u_i \in X_i$, the set $M_{i,1} = \{(x, y) \in X \times Y : \Phi(v_i, u_i, x) \not\subseteq \Psi_i(v_i, x_i, x), \text{ for some } v_i \in T_i(x, y)\}$ (resp., $M_{i,3} = \{(x, y) \in X \times Y : \Phi(v_i, u_i, x) \bigcap \Psi_i(v_i, x_i, x) \neq \emptyset, \text{ for some } v_i \in T_i(x, y)\}$) is compactly open in $X \times Y$.

Proof. For each $i \in I$ and $u_i \in X_i$, let $\Omega_{i,1} = (X \times Y) \setminus M_{i,1} = \{(x, y) \in X \times Y : \Phi(v_i, u_i, x) \subseteq \Psi_i(v_i, x_i, x), \ \forall \ v_i \in T_i(x, y)\}$. For any compact subset K of $X \times Y$, if $(x, y) \in \mathrm{cl}_K(\Omega_{i,1} \bigcap K)$, then there exists a net $((x_\lambda, y_\lambda))_{\lambda\in\Lambda} \subseteq \Omega_{i,1} \bigcap K$ such that $(x_\lambda, y_\lambda) \to (x, y) \in K$. Hence we have

$$\Phi_i(v_i, u_i, x_\lambda) \subseteq \Psi_i(v_i, x_{i,\lambda}, x_\lambda), \ \forall \ v_i \in T_i(x_\lambda, y_\lambda). \tag{3.1}$$

By (i) and Lemma 3.2, for each $v_i \in T_i(x, y)$, there exists a net $(v_{i,\lambda})_{\lambda\in\Lambda} \subseteq Y_i$ such that $v_{i,\lambda} \in T_i(x_\lambda, y_\lambda)$ and $v_{i,\lambda} \to v_i$. By (3.1), we have

$$\Phi_i(v_{i,\lambda}, u_i, x_\lambda) \subseteq \Psi_i(v_{i,\lambda}, x_{i,\lambda}, x_\lambda), \ \forall \ \lambda \in \Lambda. \tag{3.2}$$

Since the mapping $(v_i, x) \mapsto \Phi_i(v_i, u_i, x)$ is lower semicontinuous at (v_i, x), by Lemma 3.2, for each $z_i \in \Phi_i(v_i, u_i, x)$, there exists a net $(z_{i,\lambda})_{\lambda\in\Lambda}$ such that $z_{i,\lambda} \in \Phi_i(v_{i,\lambda}, u_i, x_\lambda)$ and $z_{i,\lambda} \to z_i$. By (3.2), we have

$$z_{i,\lambda} \in \Psi_i(v_{i,\lambda}, x_{i,\lambda}, x_\lambda),\ \forall\ \lambda \in \Lambda. \tag{3.3}$$

It follows from condition (iii) and (3.3) that $z_i \in \Psi(v_i, x_i, x)$ and hence

$$\Phi_i(v_i, u_i, x) \subseteq \Psi(v_i, x_i, x)$$

Therefore $(x, y) \in \Omega_{i,1} \bigcap K$ and $\Omega_{i,1}$ is compactly closed in $X \times Y$. Hence $M_{i,1}$ is compactly open in $X \times Y$. By using similar argument, we can prove that $M_{i,3}$ is also compactly open in $X \times Y$.

Lemma 3.4. *For each $i \in I$, let X_i, Y_i and Z_i be topological spaces, and let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $A_i : X \times Y \to 2^{X_i}$, $T_i, S_i : X \times Y \to 2^{Y_i}$ and $\Phi_i, \Psi_i : Y_i \times X_i \times X \to 2^{Z_i}$ be set-valued mappings such that*

- (i) *T_i is lower semicontinuous on each compact subsets of $X \times Y$,*
- (ii) *for each $u_i \in X_i$, $(v_i, x) \mapsto \Phi_i(v_i, u_i, x)$ is upper semicontinuous on each compact subset of $Y_i \times X$ with compact values,*
- (iii) *the mapping $(y_i, x_i, x) \mapsto Z_i \setminus \Psi_i(y_i, x_i, x)$ (resp., Ψ_i) is upper semicontinuous on each compact subsets of $Y_i \times X_i \times X$ with closed values.*

Then for each $i \in I$ and $u_i \in X_i$, the set $M_{i,2} = \{(x, y) \in X \times Y : \Phi(v_i, u_i, x) \subseteq \Psi_i(v_i, x_i, x), \text{ for some } v_i \in T_i(x, y)\}$ (resp., $M_{i,4} = \{(x, y) \in X\times Y : \Phi(v_i, u_i, x) \bigcap \Psi_i(v_i, x_i, x) = \emptyset, \text{ for some } v_i \in T_i(x, y)\}$) is compactly open in $X \times Y$.

Proof. For each $i \in I$ and $u_i \in X_i$, let $\Omega_{i,2} = (X \times Y) \setminus M_{i,2} = \{(x, y) \in X\times Y : \Phi(v_i, u_i, x) \not\subseteq \Psi_i(v_i, x_i, x), \forall\, v_i \in T_i(x, y)\}$. For any compact subset K of $X\times Y$, if $(x, y) \in \mathrm{cl}_K(\Omega_{i,2} \bigcap K)$, then there exists a net $((x_\lambda, y_\lambda))_{\lambda\in\Lambda} \subseteq \Omega_{i,2} \bigcap K$ such that $(x_\lambda, y_\lambda) \to (x, y) \in K$. Hence we have

$$\Phi_i(v_i, u_i, x_\lambda) \not\subseteq \Psi_i(v_i, x_{i,\lambda}, x_\lambda),\ \forall\ v_i \in T_i(x_\lambda, y_\lambda). \tag{3.4}$$

By (i) and Lemma 3.2, for each $v_i \in T_i(x, y)$, there exists a net $(v_{i,\lambda})_{\lambda\in\Lambda} \subseteq Y_i$ such that $v_{i,\lambda} \in T_i(x_\lambda, y_\lambda)$ and $v_{i,\lambda} \to v_i$. By (3.4), we have

$$\Phi_i(v_{i,\lambda}, u_i, x_\lambda) \not\subseteq \Psi_i(v_{i,\lambda}, x_{i,\lambda}, x_\lambda),\ \forall\ \lambda \in \Lambda.$$

It follows that for each $\lambda \in \Lambda$, there exists $z_{i,\lambda} \in \Phi_i(v_{i,\lambda}, u_i, x_\lambda)$ such that $z_{i,\lambda} \in Z_i \setminus \Psi_i(v_{i,\lambda}, x_{i,\lambda}, x_\lambda)$. Let $L = (x_\lambda)_{\lambda\in\Lambda} \bigcup\{x\}$ and $M_i = (v_{i,\lambda})_{\lambda\in\Lambda} \bigcup\{v_i\}$, then L and M_i are compact in X and Y_i respectively. By the condition (ii), we have

$\Phi_i(M_i, u_i, L)$ is compact in Z_i. Without loss of generality, we can assume that $z_{i,\lambda} \to z_i$ and so we have $z_i \in \Phi_i(v_i, u_i, x)$. By the condition (iii), we have $z_i \in Z_i \setminus \Psi_i(v_i, x_i, x)$. It follows that

$$\Phi_i(v_i, u_i, x) \not\subseteq \Psi(v_i, x_i, x),\ \forall\ v_i \in T_i(x, y).$$

Therefore $(x, y) \in \Omega_{i,2} \bigcap K$ and $\Omega_{i,2}$ is compactly closed in $X \times Y$. Hence $M_{i,2}$ is compactly open in $X \times Y$. By using similar argument, we can prove that $M_{i,4}$ is also compactly open in $X \times Y$.

Theorem 3.1. *Suppose that for each $i \in I$, the following conditions are satisfied:*

- (i) *for each $(x, y) \in X \times Y$, $A_i(x, y)$ and $S_i(x, y)$ are both nonempty FC-subspaces of X_i and Y_i, and for each $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $S_i^{-1}(v_i)$ are compactly open in $X \times Y$,*
- (ii) *for each $y \in Y$, Φ_i is Ψ_i-FC-quasiconvex of type (I) in the first two arguments,*
- (iii) *T_i is lower semicontinuous on each compact subsets of $X \times Y$,*
- (iv) *Φ_i is lower semicontinuous on each compact subset of $Y_i \times X_i \times X$,*
- (v) *the mapping Ψ_i is upper semicontinuous on each compact subsets of $Y_i \times X_i \times X$ with closed values.*
- (vi) *the set $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in S_i(x, y)\}$ is compactly closed in $X \times Y$,*
- (vii) *the mapping $(A \times S) : X \times Y \to 2^{X \times Y}$ defined by*

$$(A \times S)(x, y) = \Big[\prod_{i \in I} A_i(x, y)\Big] \times \Big[\prod_{i \in I} S_i(x, y)\Big],\ \forall\ (x, y) \in X \times Y$$

is Φ condensing on $X \times Y$ where Φ is the measure of noncompactness on $X \times Y$.

Then the solution set of the SGVQVIP(I)

$$\Theta_1 = \bigcap_{i \in I} \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in S_i(x, y) \text{ and } \Phi_i(v_i, u_i, x) \subseteq \Psi_i(v_i, x_i, x),\ \forall\ v_i \in T_i(x, y),\ u_i \in A_i(x, y)\}$$

is nonempty and compact in $X \times Y$.

Proof. For each $i \in I$, define a set-valued mapping $P_{i,1}, : X \times Y \to 2^{X_i}$ by

$$P_{i,1}(x, y) = \{u_i \in X_i : \Phi_i(v_i, u_i, x) \not\subseteq \Psi_i(v_i.x_i, x),\ \text{for some } v_i \in T_i(x, y)\}$$

By (ii) and Lemma 3.1, we have that for each $(x, y) \in X \times Y$,

$$x_i \notin FC(P_{i,1}(x, y)). \tag{3.5}$$

By the condition (iii)-(v) and Lemma 3.3, for each $i \in I$ and $u_i \in X_i$,

$$P_{i,1}^{-1}(u_i) = \{(x, y) \in X \times Y : \Phi_i(v_i, u_i, x) \not\subseteq \Psi_i(v_i, x_i, x), \text{ for some } v_i \in T_i(x, y)\}$$

is compactly open in $X \times Y$. It follows from Lemma 2.2 that $(FC(P_{i,1}))^{-1}(u_i)$ is also compactly open in $X \times Y$ for each $u_i \in X_i$. By Lemma 2.3, for each $i \in I$, $X_i \times Y_i$ is a locally FC-uniform space and $X \times Y$ is also a locally FC-uniform space. For each $i \in I$, define a set-valued mapping $G_i : X \times Y \to 2^{X_i \times Y_i}$ by

$$G_i(x, y) = \begin{cases} [A_i(x, y) \bigcap FC(P_{i,1}(x, y))] \times S_i(x, y), & \text{if } (x, y) \in W_i, \\ A_i(x, y) \times S_i(x, y), & \text{if } (x, y) \notin W_i, \end{cases}$$

By the condition (i), for each $i \in I$ and $(x, y) \in X \times Y$, $G_i(x, y)$ is a FC-subspace of $X_i \times Y_i$. By the definition of W_i and (3.5), for each $i \in I$ and $(x, y) \in X \times Y$, $(x_i, y_i) \notin G_i(x, y)$. For each $i \in I$ and $(u_i, v_i) \in X_i \times Y_i$, we have

$$\begin{aligned} G_i^{-1}(u_i, v_i) &= [A_i^{-1}(u_i)) \bigcap (FC(P_{i,1}))^{-1}(u_i) \bigcap S_i^{-1}(v_i)] \\ &\quad \bigcup [((X \times Y) \setminus W_i) \bigcap A_i^{-1}(u_i) \bigcap S_i^{-1}(v_i)]. \end{aligned}$$

Since $(FC(P_{i,1}))^{-1}(u_i)$ is compactly open in $X \times Y$ for each $u_i \in X_i$, by the conditions (i) and (vi), $G_i^{-1}(u_i, v_i)$ is also compactly open in $X \times Y$. Define a set-valued mapping $G : X \times Y \to 2^{X \times X}$ by

$$G(x, y) = \prod_{i \in I} G_i(x, y), \ \forall \ (x, y) \in X \times Y.$$

Then we have

$$G(x, y) \subseteq (A \times T)(x, y), \ \forall \ (x, y) \in X \times Y.$$

By the condition (vii) and Remark 2.1, G is also Φ-condensing on $X \times Y$. All conditions of Theorem 2.1 are satisfied. By Theorem 2.1, there exist a compact $K = \prod_{i \in I} K_i$ of $X \times Y$ and $(\hat{x}, \hat{y}) \in K$ such that $G_i(\hat{x}, \hat{y}) = \emptyset$ for each $i \in I$. If $(\hat{x}, \hat{y}) \notin W_j$ for some $j \in I$, then either $A_j(\hat{x}, \hat{y}) = \emptyset$ or $S_j(\hat{x}, \hat{y}) = \emptyset$ which contradicts the condition (i). Therefore $(\hat{x}, \hat{y}) \in W_i$ for each $i \in I$. This shows that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x}, \hat{y})$, $\hat{y}_i \in S_i(\hat{x}, \hat{y})$ and $A_i(\hat{x}, \hat{y}) \bigcap FC(P_{i,1}(\hat{x}, \hat{y})) = \emptyset$ and hance $A_i(\hat{x}, \hat{y}) \bigcap P_{i,1}(\hat{x}, \hat{y}) = \emptyset$. Therefore, for each $i \in I$,

$$\begin{aligned} \hat{x}_i &\in A_i(\hat{x}, \hat{y}), \ \hat{y}_i \in S_i(\hat{x}, \hat{y}) \text{ and } \ \Phi_i(v_i, u_i, \hat{x}) \\ &\subseteq \Psi(v_i, \hat{x}_i, \hat{x}), \ \forall \ v_i \in T_i(\hat{x}, \hat{y}), u_i \in A_i(\hat{x}, \hat{y}). \end{aligned}$$

Hence $\Theta_1 \subseteq K$ is nonempty. For each $i \in I$, let

$$\begin{aligned} Q_i &= \{(x,y) \in K : x_i \in A_i(x,y),\ y_i \in S_i(x,y) \text{ and} \\ &\quad \Phi_i(v_i,u_i,x) \subseteq \Psi_i(v_i,x_i,x),\ \forall\ v_i \in T_i(x,y),\ u_i \in A_i(x,y)\} \\ &= W_i \bigcap \{(x,y) \in K : \Phi_i(v_i,u_i,x) \\ &\quad \subseteq \Psi_i(v_i,x_i,x),\ \forall\ v_i \in T_i(x,y),\ u_i \in A_i(x,y)\} \\ &= W_i \bigcap B_i, \end{aligned}$$

where $B_i = \{(x,y) \in K : \Phi_i(v_i,u_i,x) \subseteq \Psi_i(v_i,x_i,x),\ \forall\ v_i \in T_i(x,y),\ u_i \in A_i(x,y)\}$. Therefore we have $\Theta_1 = \bigcap_{i\in I} Q_i = \bigcap_{i\in I}(W_i \bigcap B_i)$. Now we prove that for each $i \in I$, B_i is closed in K. Indeed, if $(x,y) \in \overline{B_i}$, then there exists a net $(x_\lambda, y_\lambda)_{\lambda\in\Lambda} \subseteq B_i$ such that $(x_\lambda, y_\lambda) \to (x,y) \in K$. Hence we have

(3.6)

$$\Phi_i(v_i,u_i,x_\lambda) \subseteq \Psi_i(v_i,x_{i,\lambda},x_\lambda),\ \forall\ v_i \in T_i(x_\lambda,y_\lambda),\ u_i \in A_i(x_\lambda,y_\lambda) \text{ and } \lambda \in \Lambda.$$

Since T_i is lower semicontinuous on K by (iii), it follows from Lemma 3.2 that for each $v_i \in T_i(x,y)$, there exists a net $(v_{i,\lambda})_{\lambda\in\Lambda} \subseteq Y_i$ such that $v_{i,\lambda} \in T_i(x_\lambda,y_\lambda)$ and $v_{i,\lambda} \to v_i$. Since for each $u_i \in X_i$, $A_i^{-1}(u_i)$ is compactly open in $X \times Y$, A_i is also lower semicontinuous on K by Takahashi [26]. It follows from Lemma 3.2 that for each $u_i \in A_i(x,y)$, there exists a net $(u_{i,\lambda})_{\lambda\in\Lambda} \subseteq X_i$ such that $u_{i,\lambda} \in A_i(x_\lambda,y_\lambda)$ and $u_{i,\lambda} \to u_i$. By (3.6), we have

$$\Phi_i(v_{i,\lambda},u_{i,\lambda},x_\lambda) \subseteq \Psi_i(v_{i,\lambda},x_{i,\lambda},x_\lambda),\ \forall\ \lambda \in \Lambda. \tag{3.7}$$

For each $z_i \in \Phi_i(v_i,u_i,x)$, since Φ_i is lower semicontinuous at (v_i,u_i,x) by (vi), there exists a net $(z_{i,\lambda})_{\lambda\in\Lambda}$ such that $z_{i,\lambda} \in \Phi_i(v_{i,\lambda},u_{i,\lambda},x_\lambda)$ and $z_{i,\lambda} \to z_i$. By (3.7), $z_{i,\lambda} \in \Psi_i(v_{i,\lambda},x_{i,\lambda},x_\lambda)$ for all $\lambda \in \Lambda$. By (v), ψ_i is upper semicontinuous with closed values and so we have $z_i \in \Psi_i(v_i,x_i,x)$. Hence,

$$\Phi(v_i,u_i,x) \subseteq \Psi(v_i,x_i,x),\ \forall\ v_i \in T_i(x,y) \text{ and } u_i \in A_i(x,y).$$

So $(x,y) \in B_i$ and B_i is closed in K. By (vi), W_i is also closed in K and hence $\Theta_1 = \bigcap_{i\in I}(W_i \bigcap B_i)$ is closed in K and hence it is also compact. This completes the proof.

Remark 3.1. Theorem 3.1 improves and generalizes Theorem 2.3 of Ding [12] under weaker conditions, and the conclusion of Theorem 3.1 is better than that of Theorem 2.3 in [12]. Theorem 3.1 is also a new existence result of solutions for the SGVQVIP(I) which is different from Theorem 3.2 of Hai and Khanh [17] in the following ways: (1) the mathematical model of SGQVIP (I) is more general than the mathematical model of (SQIP2) in [17]; (2) for each $i \in I$, X_i and Y_i

may be locally FC-uniform spaces without convexity structure and Z_i may be any topological space; (3) the conclusion of Theorem 3.1 is better than that of Theorem 3.2 in [17].

Theorem 3.2. *Suppose that for each $i \in I$, the following conditions are satisfied:*

(i) *for each $(x, y) \in X \times Y$, $A_i(x, y)$ and $S_i(x, y)$ are both nonempty FC-subspaces of X_i and Y_i, and for each $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $S_i^{-1}(v_i)$ are compactly open in $X \times Y$,*

(ii) *for each $y \in Y$, Φ_i is Ψ_i-FC-quasiconvex of type (II) in the first two arguments,*

(iii) *T_i is lower semicontinuous on each compact subsets of $X \times Y$,*

(iv) *Φ_i is upper semicontinuous on each compact subset of $Y_i \times X_i \times X$ with compact values,*

(v) *the mapping $(y_i, x_i, x) \mapsto Z_i \setminus \Psi_i(y_i, x_i, x)$ is upper semicontinuous on each compact subsets of $Y_i \times X_i \times X$ with closed values.*

(vi) *the set $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in S_i(x, y)\}$ is compactly closed in $X \times Y$,*

(vii) *the mapping $(A \times S) : X \times Y \to 2^{X \times Y}$ defined by*

$$(A \times S)(x, y) = \Big[\prod_{i \in I} A_i(x, y)\Big] \times \Big[\prod_{i \in I} S_i(x, y)\Big],\ \forall\ (x, y) \in X \times Y$$

is Φ condensing on $X \times Y$ where Φ is the measure of noncompactness on $X \times Y$.

Then the solution set of the SGVQVIP(II)

$$\begin{aligned}\Theta_2 = \bigcap_{i \in I} &\{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in S_i(x, y) \text{ and} \\ &\Phi_i(v_i, u_i, x) \nsubseteq \Psi_i(v_i, x_i, x),\ \forall\ v_i \in T_i(x, y),\ u_i \in A_i(x, y)\}\end{aligned}$$

is nonempty and compact in $X \times Y$.

Proof. For each $i \in I$, define a set-valued mapping $P_{i,2}, : X \times Y \to 2^{X_i}$ by

$$P_{i,2}(x, y) = \{u_i \in X_i : \Phi_i(v_i, u_i, x) \subseteq \Psi_i(v_i.x_i, x),\ \text{for some } v_i \in T_i(x, y)\}$$

By (ii) and Lemma 3.1, we have that for each $(x, y) \in X \times Y$,

$$x_i \notin FC(P_{i,2}(x, y)). \tag{3.8}$$

By the condition (iii)-(v) and Lemma 3.4, for each $i \in I$ and $u_i \in X_i$,

$$P_{i,2}^{-1}(u_i) = \{(x,y) \in X \times Y : \Phi_i(v_i, u_i, x) \subseteq \Psi_i(v_i, x_i, x), \text{ for some } v_i \in T_i(x,y)\}$$

is compactly open in $X \times Y$. It follows from Lemma 2.2 that $(FC(P_{i,1}))^{-1}(u_i)$ is also compactly open in $X \times Y$ for each $u_i \in X_i$. By Lemma 2.3, for each $i \in I$, $X_i \times Y_i$ is a locally FC-uniform space and $X \times Y$ is also a locally FC-uniform space. For each $i \in I$, define a set-valued mapping $G_i : X \times Y \to 2^{X_i \times Y_i}$ by

$$G_i(x,y) = \begin{cases} [A_i(x,y) \bigcap FC(P_{i,2}(x,y))] \times S_i(x,y), & \text{if } (x,y) \in W_i, \\ A_i(x,y) \times S_i(x,y), & \text{if } (x,y) \notin W_i, \end{cases}$$

By the condition (i), for each $i \in I$ and $(x,y) \in X \times Y$, $G_i(x,y)$ is a FC-subspace of $X_i \times Y_i$. By the definition of W_i and (3.8), for each $i \in I$ and $(x,y) \in X \times Y$, $(x_i, y_i) \notin G_i(x,y)$. For each $i \in I$ and $(u_i, v_i) \in X_i \times Y_i$, we have

$$\begin{aligned} G_i^{-1}(u_i, v_i) = {} & [A_i^{-1}(u_i)) \bigcap (FC(P_{i,2}))^{-1}(u_i) \bigcap S_i^{-1}(v_i)] \\ & \bigcup [((X \times Y) \setminus W_i) \bigcap A_i^{-1}(u_i) \bigcap S_i^{-1}(v_i)]. \end{aligned}$$

Since $(FC(P_{i,2}))^{-1}(u_i)$ is compactly open in $X \times Y$ for each $u_i \in X_i$, by the conditions (i) and (vi), $G_i^{-1}(u_i, v_i)$ is also compactly open in $X \times Y$. Define a set-valued mapping $G : X \times Y \to 2^{X \times X}$ by

$$G(x,y) = \prod_{i \in I} G_i(x,y), \ \forall\ (x,y) \in X \times Y.$$

Then we have

$$G(x,y) \subseteq (A \times T)(x,y), \ \forall\ (x,y) \in X \times Y.$$

By the condition (vii) and Remark 2.1, G is also Φ-condensing on $X \times Y$. All conditions of Theorem 2.1 are satisfied. By Theorem 2.1, there exist a compact $K = \prod_{i \in I} K_i$ of $X \times Y$ and $(\hat{x}, \hat{y}) \in K$ such that $G_i(\hat{x}, \hat{y}) = \emptyset$ for each $i \in I$. If $(\hat{x}, \hat{y}) \notin W_j$ for some $j \in I$, then either $A_j(\hat{x}, \hat{y}) = \emptyset$ or $S_j(\hat{x}, \hat{y}) = \emptyset$ which contradicts the condition (i). Therefore $(\hat{x}, \hat{y}) \in W_i$ for each $i \in I$. This shows that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x}, \hat{y})$, $\hat{y}_i \in S_i(\hat{x}, \hat{y})$ and $A_i(\hat{x}, \hat{y}) \bigcap FC(P_{i,2}(\hat{x}, \hat{y})) = \emptyset$ and hance $A_i(\hat{x}, \hat{y}) \bigcap P_{i,2}(\hat{x}, \hat{y}) = \emptyset$. Therefore, for each $i \in I$,

$$\begin{aligned} & \hat{x}_i \in A_i(\hat{x}, \hat{y}), \ \hat{y}_i \in S_i(\hat{x}, \hat{y}) \text{ and } \ \Phi_i(v_i, u_i, \hat{x}) \\ & \quad \not\subseteq \Psi(v_i, \hat{x}_i, \hat{x}), \ \forall\ v_i \in T_i(\hat{x}, \hat{y}), u_i \in A_i(\hat{x}, \hat{y}). \end{aligned}$$

Hence $\Theta_2 \subseteq K$ is nonempty. For each $i \in I$, let

$$\begin{aligned} Q_i &= \{(x,y)\in K : x_i\in A_i(x,y),\ y_i\in S_i(x,y) \text{ and} \\ &\quad \Phi_i(v_i,u_i,x)\not\subseteq \Psi_i(v_i,x_i,x),\ \forall\ v_i\in T_i(x,y),\ u_i\in A_i(x,y)\} \\ &= W_i\bigcap\{(x,y)\in K : \Phi_i(v_i,u_i,x) \\ &\quad \not\subseteq \Psi_i(v_i,x_i,x),\ \forall\ v_i\in T_i(x,y),\ u_i\in A_i(x,y)\} \\ &= W_i\bigcap B_i, \end{aligned}$$

where $B_i = \{(x,y)\in K : \Phi_i(v_i,u_i,x)\not\subseteq \Psi_i(v_i,x_i,x),\ \forall\ v_i\in T_i(x,y),\ u_i\in A_i(x,y)\}$. Therefore we have $\Theta_2=\bigcap_{i\in I}Q_i=\bigcap_{i\in I}(W_i\bigcap B_i)$. Now we prove that for each $i\in I$, B_i is closed in K. Indeed, if $(x,y)\in\overline{B_i}$, then there exists a net $(x_\lambda,y_\lambda)_{\lambda\in\Lambda}\subseteq B_i$ such that $(x_\lambda,y_\lambda)\to(x,y)\in K$. Hence we have

(3.9)
$$\Phi_i(v_i,u_i,x_\lambda)\not\subseteq \Psi_i(v_i,x_{i,\lambda},x_\lambda),\ \forall\ v_i\in T_i(x_\lambda,y_\lambda),\ u_i\in A_i(x_\lambda,y_\lambda) \text{ and } \lambda\in\Lambda.$$

Since T_i is lower semicontinuous on K by (iii), it follows from Lemma 3.2 that for each $v_i\in T_i(x,y)$, there exists a net $(v_{i,\lambda})_{\lambda\in\Lambda}\subseteq Y_i$ such that $v_{i,\lambda}\in T_i(x_\lambda,y_\lambda)$ and $v_{i,\lambda}\to v_i$. Since for each $u_i\in X_i$, $A_i^{-1}(u_i)$ is compactly open in $X\times Y$, A_i is also lower semicontinuous on K by Takahashi [26]. it follows from Lemma 3.2 that for each $u_i\in A_i(x,y)$, there exists a net $(u_{i,\lambda})_{\lambda\in\Lambda}\subseteq X_i$ such that $u_{i,\lambda}\in A_i(x_\lambda,y_\lambda)$ and $u_{i,\lambda}\to u_i$. By (3.9), we have

$$\Phi_i(v_{i,\lambda},u_{i,\lambda},x_\lambda)\not\subseteq \Psi_i(v_{i,\lambda},x_{i,\lambda},x_\lambda),\ \forall\ \lambda\in\Lambda. \tag{3.10}$$

It follows that for each $\lambda\in\Lambda$, there exists $z_{i,\lambda}\in\Phi_i(v_{i,\lambda},u_{i,\lambda},x_\lambda)$ such that $z_{i,\lambda}\in Z_i\setminus\Psi_i(v_{i,\lambda},x_{i,\lambda},x_\lambda)$. Let $L_i=(v_{i,\lambda})_{\lambda\in\Lambda}\bigcup\{v_i\}$, $M_i=(u_{i,\lambda})_{\lambda\in\Lambda}\bigcup\{u_i\}$ and $N=(x_\lambda)_{\lambda\in\Lambda}\bigcup\{x\}$, then $L_i\times M_i\times N$ is compact in $Y_i\times X_i\times X$. By (iv), $\Phi_i(L_i,M_i,N)$ is compact in Z_i. Without loss of generality, we can assume $z_{i,\lambda}\to z_i$. By (iv), we have $z_i\in\Phi_i(v_i,u_i,x)$. It follows from the condition (v) that $z_i\in Z_i\setminus\Psi_i(v_i,x_i,x)$. Therefore,

$$\Phi_i(v_i,u_i,x)\not\subseteq \Psi(v_i,x_i,x),\ \forall\ v_i\in T_i(x,y),\ u_i\in A_i(x,y).$$

So $(x,y)\in B_i$ and B_i is closed in K. By (vi), W_i is also closed in K and hence $\Theta_2=\bigcap_{i\in I}(W_i\bigcap B_i)$ is closed in K and hence it is also compact. This completes the proof.

Remark 3.2. Theorem 3.2 improves and generalizes Theorem 2.2 of Ding [12] to more general mathematical model under weaker conditions, and the conclusion of Theorem 3.2 is better than that of Theorem 2.2 in [12].

By using the similar argument as in the proof of Theorem 3.1 and Theorem 3.2, it is easy to prove the following results.

Theorem 3.3. *Suppose that for each $i \in I$, the following conditions are satisfied:*

(i) *for each $(x, y) \in X \times Y$, $A_i(x, y)$ and $S_i(x, y)$ are both nonempty FC-subspaces of X_i and Y_i, and for each $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $S_i^{-1}(v_i)$ are compactly open in $X \times Y$,*

(ii) *for each $y \in Y$, Φ_i is Ψ_i-FC-quasiconvex of type (III) in the first two arguments,*

(iii) *T_i is lower semicontinuous on each compact subsets of $X \times Y$,*

(iv) *Φ_i is lower semicontinuous on each compact subset of $Y_i \times X_i \times X$,*

(v) *the mapping $(y_i, x_i, x) \mapsto Z_i \setminus \Psi_i(y_i, x_i, x)$ is upper semicontinuous on each compact subsets of $Y_i \times X_i \times X$ with closed values.*

(vi) *the set $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in S_i(x, y)\}$ is compactly closed in $X \times Y$,*

(vii) *the mapping $(A \times S) : X \times Y \to 2^{X \times Y}$ defined by*

$$(A \times S)(x, y) = \Big[\prod_{i \in I} A_i(x, y)\Big] \times \Big[\prod_{i \in I} S_i(x, y)\Big],\ \forall\ (x, y) \in X \times Y$$

is Φ condensing on $X \times Y$ where Φ is the measure of noncompactness on $X \times Y$.

Then the solution set of the SGVQVIP(III)

$$\begin{aligned}\Theta_3 = \bigcap_{i \in I} \{&(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in S_i(x, y) \text{ and} \\ &\Phi_i(v_i, u_i, x) \bigcap \Psi_i(v_i, x_i, x) = \emptyset,\ \forall\ v_i \in T_i(x, y),\ u_i \in A_i(x, y)\}\end{aligned}$$

is nonempty and compact in $X \times Y$.

Remark 3.3. Theorem 3.3 improves and generalizes Theorem 2.4 of Ding [12] to more general mathematical model under weaker conditions, and the conclusion of Theorem 3.3 is better than that of Theorem 2.4 in [12]. obtains .

Theorem 3.4. *Suppose that for each $i \in I$, the following conditions are satisfied:*

(i) *for each $(x, y) \in X \times Y$, $A_i(x, y)$ and $S_i(x, y)$ are both nonempty FC-subspaces of X_i and Y_i, and for each $(u_i, v_i) \in X_i \times Y_i$, $A_i^{-1}(u_i)$, $S_i^{-1}(v_i)$ are compactly open in $X \times Y$,*

(ii) *for each $y \in Y$, Φ_i is Ψ_i-FC-quasiconvex of type (IV) in the first two arguments,* (*iii*) *T_i is lower semicontinuous on each compact subsets of $X \times Y$,*

(*iv*) *Φ_i is upper semicontinuous on each compact subset of $Y_i \times X_i \times X$ with compact values,*

(*v*) *the mapping the mapping Ψ_i is upper semicontinuous on each compact subsets of $Y_i \times X_i \times X$ with closed values.*

(*vi*) *the set $W_i = \{(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in S_i(x, y)\}$ is compactly closed in $X \times Y$,*

(*vii*) *the mapping $(A \times S) : X \times Y \to 2^{X \times Y}$ defined by*

$$(A \times S)(x, y) = \Big[\prod_{i \in I} A_i(x, y)\Big] \times \Big[\prod_{i \in I} S_i(x, y)\Big],\ \forall\ (x, y) \in X \times Y$$

is Φ condensing on $X \times Y$ where Φ is the measure of noncompactness on $X \times Y$.

Then the solution set of the SGVQVIP(IV)

$$\begin{aligned}\Theta_4 = \bigcap_{i \in I} \{&(x, y) \in X \times Y : x_i \in A_i(x, y),\ y_i \in S_i(x, y) \text{ and} \\ &\Phi_i(v_i, u_i, x) \bigcap \Psi_i(v_i, x_i, x) \neq \emptyset,\ \forall\ v_i \in T_i(x, y),\ u_i \in A_i(x, y)\}\end{aligned}$$

is nonempty and compact in $X \times Y$.

Remark 3.4. Theorem 3.4 improves and generalizes Theorem 2.5 of Ding [12] to more general mathematical model under weaker conditions, and the conclusion of Theorem 3.4 is better than that of Theorem 2.5 in [12]. Theorem 3.1 is also a new existence result of solutions for the SGVQVIP(IV) which is different from Theorem 3.4 of Hai and Khanh [17] in the following ways: (1) the mathematical model of SGQVIP (IV) is more general than the mathematical model of (SQIP4) in [17]; (2) for each $i \in I$, X_i and Y_i may be FC-spaces without convexity structure and Z_i may be any topological space; (3) the conclusion of Theorem 3.1 is better than that of Theorem 3.4 in [17].

4. Mathematical Programs with Systems of Generalized Vector Quasi-variational Inclusion Constraints

In this section, by applying the results in above section, we shall establish some existence results of solutions for mathematical programs with systems of generalized vector quasi-variational inclusion constraints.

Definition 4.1. [24]. Let V be a topological vector space ordered by a closed convex cone D in V and $M \subseteq V$ be a nonempty set. A point $v_0 \in M$ is said to be an efficient point of M if there is no $v \in M$ such that $v_0 \in v + D \setminus \{0\}$, where 0 is the zero element of V. The set of all efficient points of M is denoted by $\mathrm{Min}_D(M)$.

Lemma 4.1 [24]. *Let V be a topological vector space ordered by a closed convex cone D in V. If M is a nonempty compact subset of V, then $\mathrm{Min}_D(M) \neq \emptyset$.*

Let $h : X \times Y \to 2^V$ be a set-valued mapping. We consider the following Mathematical programs with systems of generalized vector quasi-variational inclusion constraints:

$$\mathrm{MP(I)} \begin{cases} \text{Find } (\hat{x},\hat{y}) \in \Theta_1 \text{ such that } h(\hat{x},\hat{y}) \bigcap \mathrm{Min}_D(\Theta_1) \neq \emptyset \\ \text{where } \Theta_1 = \bigcap_{i\in I} \{(x,y) \in X \times Y : x_i \in A_i(x,y),\ y_i \in S_i(x,y) \text{ and} \\ \Phi_i(v_i,u_i,x) \subseteq \Psi_i(v_i,x_i,x),\ \forall\ v_i \in T_i(x,y),\ u_i \in A_i(x,y)\} \end{cases}$$

$$\mathrm{MP(II)} \begin{cases} \text{Find } (\hat{x},\hat{y}) \in \Theta_2 \text{ such that } h(\hat{x},\hat{y}) \bigcap \mathrm{Min}_D(\Theta_2) \neq \emptyset \\ \text{where } \Theta_2 = \bigcap_{i\in I} \{(x,y) \in X \times Y : x_i \in A_i(x,y),\ y_i \in S_i(x,y) \text{ and} \\ \Phi_i(v_i,u_i,x) \not\subseteq \Psi_i(v_i,x_i,x),\ \forall\ v_i \in T_i(x,y),\ u_i \in A_i(x,y)\} \end{cases}$$

$$\mathrm{MP(III)} \begin{cases} \text{Find } (\hat{x},\hat{y}) \in \Theta_3 \text{ such that } h(\hat{x},\hat{y}) \bigcap \mathrm{Min}_D(\Theta_3) \neq \emptyset \\ \text{where } \Theta_1 = \bigcap_{i\in I} \{(x,y) \in X \times Y : x_i \in A_i(x,y),\ y_i \in S_i(x,y) \text{ and} \\ \Phi_i(v_i,u_i,x) \bigcap \Psi_i(v_i,x_i,x) = \emptyset,\ \forall\ v_i \in T_i(x,y),\ u_i \in A_i(x,y)\} \end{cases}$$

$$\mathrm{MP(IV)} \begin{cases} \text{Find } (\hat{x},\hat{y}) \in \Theta_4 \text{ such that } h(\hat{x},\hat{y}) \bigcap \mathrm{Min}_D(\Theta_4) \neq \emptyset \\ \text{where } \Theta_4 = \bigcap_{i\in I} \{(x,y) \in X \times Y : x_i \in A_i(x,y),\ y_i \in S_i(x,y) \text{ and} \\ \Phi_i(v_i,u_i,x) \bigcap \Psi_i(v_i,x_i,x) \neq \emptyset,\ \forall\ v_i \in T_i(x,y),\ u_i \in A_i(x,y)\} \end{cases}$$

Theorem 4.1. *Assume that all conditions of Theorem 3.1 are satisfied. Let V be a topological vector space ordered by a closed convex cone D in V and $h : X \times Y \to 2^V$ be a upper semicontinuous set-valued mapping with nonempty compact values. Then there exists a solution of MP (I).*

Proof. It follows from Theorem 3.1 that the solution set Θ_1 of the SGVQVIP(I) is nonempty compact on $X \times Y$. Since h be a upper semicontinuous set-valued mapping with nonempty compact values, $h(\Theta_1)$ is compact in V. By Lemma 4.1, $\mathrm{Min}_D(h(\Theta_1)) \neq \emptyset$. Hence there exists $(\hat{x},\hat{y}) \in \Theta_1$ and $\bar{u} \in h(\hat{x},\hat{y})$ such that $\hat{u} \in \mathrm{Min}_D(h(\Theta_1))$ and so $h(\hat{x},\hat{y}) \bigcap \mathrm{Min}_D(h(\Theta_1)) \neq \emptyset$, where $\Theta_1 = \bigcap_{i\in I} \{(x,y) \in X \times Y : x_i \in A_i(x,y),\ y_i \in S_i(x,y)$ and $\Phi_i(v_i,u_i,x) \subseteq \Psi_i(v_i,x_i,x),\ \forall\ v_i \in$

$T_i(x, y),\ u_i \in A_i(x, y)\}$.

By using Theorems 3.2-3.4, and the similar argument as in the proof of Theorem 4.1, it is easy to prove the following results.

Theorem 4.2. *Assume that all conditions of Theorem 3.2 are satisfied. Let V be a topological vector space ordered by a closed convex cone D in V and $h : X \times Y \to 2^V$ be a upper semicontinuous set-valued mapping with nonempty compact values. Then there exists a solution of MP (II).*

Theorem 4.3. *Assume that all conditions of Theorem 3.3 are satisfied. Let V be a topological vector space ordered by a closed convex cone D in V and $h : X \times Y \to 2^V$ be a upper semicontinuous set-valued mapping with nonempty compact values. Then there exists a solution of MP (III).*

Theorem 4.4. *Assume that all conditions of Theorem 3.4 are satisfied. Let V be a topological vector space ordered by a closed convex cone D in V and $h : X \times Y \to 2^V$ be a upper semicontinuous set-valued mapping with nonempty compact values. Then there exists a solution of MP (IV).*

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