TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 5, pp. 1411-1432, October 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

SIMULTANEOUS METRIC PROJECTIONS IN $C(Q, Y)$ WITH APPLICATIONS

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Abstract. We develop a theory of simultaneous metric projection in a normed linear space X and present various characterizations of simultaneous metric projection onto closed convex sets in terms of the elements of X^*. Also, we characterize the elements of simultaneous metric projection onto closed convex sets in terms of extreme points of the closed unit ball B_{X^*}. Finally, as an application, we give various characterizations of simultaneous metric projection onto subspaces of the Banach space $C(Q, Y)$.

1. Introduction

The theory of simultaneous metric projection onto closed convex sets (in particular, subspaces) has been studied by many authors, e.g., [1, 2, 6, 8, 9, 10, 11, 13, 14, 16, 17, 19]. In this paper, we use totally bounded sets to give various characterizations of simultaneous metric projection onto closed convex sets in a normed linear space X in terms of the elements of X^*, and the extreme points of the closed unit ball B_{X^*}. Also, we present various characterizations of simultaneous metric projection onto subspaces of the Banach space $C(Q, Y)$.

The structure of the paper is as follows: In section 2, we give some preliminary definitions on simultaneous metric projection. Various characterizations of simultaneous metric projection in terms of the elements of X^* are given in section 3. In section 4, we present characterizations of simultaneous metric projection in terms of the extreme points of the closed unit ball B_{X^*}. Applications and characterizations of simultaneous metric projection onto subspaces of the Banach space $C(Q, Y)$ are given in section 5.

Received June 23, 2007, accepted January 11, 2008.
Communicated by Boris S. Mordukhovich.
2000 *Mathematics Subject Classification*: 41A28, 46B25, 46E15.
Key words and phrases: Simultaneous metric projection, Extreme point, Totaly bounded set, Normed linear space.

2. Preliminaries

Let X be a normed linear space and W a subset of X. If S is a bounded set in X, we define

$$d(S,W) := \inf_{\omega \in W} \sup_{s \in S} \|s - w\|. \tag{2.1}$$

We recall (see [13]) that a point $\omega_0 \in W$ is called *a simultaneous metric projection of S onto W or a best simultaneous approximation to S from W* if

$$\sup_{s \in S} \|s - \omega_0\| = d(S,W).$$

The set of all simultaneous metric projections of S onto W will be denoted by $\mathbf{S}_W(S)$:

$$\mathbf{S}_W(S) := \{w \in W : \sup_{s \in S} \|s - w\| = d(S,W)\}. \tag{2.2}$$

It is well-known that $\mathbf{S}_W(S)$ is a bounded subset of X and if W is a closed and convex subset of X, then $\mathbf{S}_W(S)$ is closed and convex.

For any subset W of a (real) normed linear space X, the *polar set* of W is defined by

$$W^\circ := \{f \in X^* : f(w) \leq 0 \quad \forall\, w \in W\},$$

where X^* is the dual space of X.

We recall (see [7]) that for an arbitrary compact Hausdorff space Q, we denote by $C_{\mathbb{R}}(Q)$ the Banach space of all real valued continuous functions defined on Q, and $C(Q,Y)$ denotes the Banach space of all continuous functions f from Q to the Banach space Y equipped with the norm defined by

$$\|f\| = \sup_{s \in S} \|f(s)\|.$$

A set M in X is called an *extremal subset* of a closed and convex set W if:

(i) M is a closed convex subset of W.

(ii) Together with every interior point of a segment in W it contains the whole segment, that is, the relations $x, y \in W$, $\lambda x + (1-\lambda)y \in M$ and $0 < \lambda < 1$ imply $x, y \in$M.

An extremal subset of W consisting of a single point (i.e. a point of W which is not an interior point of any segment in W) is called an *extreme point* of W. We denote by $\mathcal{E}(W)$ the set of all extreme points of W (for more details see [15]).

For a normed linear space X and $n \in \mathbb{N}$, we define X^n to be the $n-$fold direct sum of X equipped with the norm:

$$\|(x_1, x_2, ..., x_n)\| = \max_{1\leq i\leq n} \|x_i\|. \tag{2.3}$$

Throughout this paper, we assume that X is a (real) normed linear space.

3. Characterizations of Simultaneous Metric Projection in Terms of the Elements of X^*

In this section, we give various characterizations of simultaneous metric projection onto closed convex sets in terms of the elements of X^*. We start with the following theorem.

Theorem 3.1. *Let W be a closed and convex set in a real normed linear space X, S be a totally bounded set in X with $S \cap W = \emptyset$, and $\omega_0 \in W$. Assume that $W \cap \overline{co}(\{\omega_0\} \cup S) = \{\omega_0\}$. Then the following assertions are equivalent:*

- (i) $\omega_0 \in \boldsymbol{S}_W(S)$,
- (ii) *For each $\epsilon > 0$ there exists a finite subset $\{s_1, s_2, ..., s_n\}$ of S and bounded linear functionals $f_i \in X^*$ $(i = 1, 2, ..., n)$ such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$,*

$$\sum_{i=1}^{n} \|f_i\| = 1, \tag{3.1}$$

$$\sum_{i=1}^{n} f_i(\omega - \omega_0) \leq 0 \qquad (\omega \in W), \tag{3.2}$$

and

$$\sum_{i=1}^{n} f_i(s_i - \omega_0) = \max_{1\leq i\leq n} \|s_i - \omega_0\|. \tag{3.3}$$

Proof. $(i) \Rightarrow (ii)$. Let $\epsilon > 0$ be given. Since S is a totally bounded set, it follows that there exists a finite subset $\{s_1, s_2, ..., s_n\}$ of S such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$. Since $\omega_0 \in \mathbf{S}_W(S)$, we conclude that for each $s \in S$, we have

$$\|s - \omega_0\| \leq \max_{1\leq i\leq n} \|s_i - \omega_0\| + \epsilon \leq \sup_{s\in S} \|s - \omega_0\| + \epsilon = d(S, W) + \epsilon, \tag{3.4}$$

where $\mathcal{N}(x, \epsilon) := \{y \in X : \|y - x\| < \epsilon\}$ $(x \in X)$. Now, let $r = \max_{1\leq i\leq n} \|s_i - \omega_0\|$. Then, $r > 0$ because $S \cap W = \emptyset$. We define

$$B_i := \{y \in \overline{co}(\{\omega_0\} \cup S) : \|s_i - y\| \leq r\} \qquad (i = 1, 2, ..., n). \tag{3.5}$$

It follows that $\omega_0 \in B_i$ for all $i = 1, 2, ..., n$, and for each $i = 1, 2, ..., n$, we have $s_i \in B_i$.

It is clear that each B_i is a closed and convex subset of X. Moreover, in view of (3.4) and that $W \cap \overline{co}(\{\omega_0\} \cup S) = \{\omega_0\}$, we get $intB_i \cap W = \emptyset$, $i = 1, 2, ..., n$. Therefore, by Hahn-Banach Theorem, for each $1 \leq i \leq n$, there exist bounded linear functionals $g_i \in X^*$ and $\lambda_i \in \mathbb{R}$ such that,

$$g_i(s_i - \omega) \geq \lambda_i \quad (\forall\, \omega \in W),$$

and

$$g_i(s_i - y) \leq \lambda_i \quad (\forall\, y \in B_i).$$

Thus, we have $g_i(s_i - \omega_0) = \lambda_i \neq 0$, $i = 1, 2, ..., n$. Since $s_i \in B_i$, it follows that $\lambda_i > 0$ for all $1 \leq i \leq n$. Let $f_i = \frac{r}{n}\lambda_i^{-1} g_i$, $i = 1, 2, ..., n$. Therefore, $f_i \in X^*$, $i = 1, 2, ..., n$. Then, we get

$$f_i(s_i - \omega) \geq \frac{r}{n} \qquad (\forall\, \omega \in W;\ i = 1, 2, ..., n), \tag{3.6}$$

$$f_i(s_i - y) \leq \frac{r}{n} \qquad (\forall\, y \in B_i;\ i = 1, 2, ..., n), \tag{3.7}$$

and

$$\sum_{i=1}^{n} f_i(s_i - \omega_0) = r. \tag{3.8}$$

We prove that $\sum_{i=1}^{n} \|f_i\| = 1$. Indeed, for each $1 \leq i \leq n$, we have

$$\frac{r}{n} = f_i(s_i - \omega_0) \leq \|f_i\|\|s_i - \omega_0\| \leq \|f_i\| \max_{1 \leq i \leq n} \|s_i - \omega_0\| = r\|f_i\|.$$

Thus, $\|f_i\| \geq \frac{1}{n}$ $(i = 1, ..., n)$. We claim that $\|f_i\| = \frac{1}{n}$ $(1 \leq i \leq n)$. If not, then for each $1 \leq i \leq n$, there exists $z_i \in X$ such that $\|z_i\| = 1$ and $f_i(z_i) > \frac{1}{n}$. Let $t_i = s_i - rz_i \in X$, $i = 1, 2, ..., n$. Since for each $i = 1, 2, ..., n$, we have $\|t_i - s_i\| = r$, it follows that $t_i \in B_i$, $i = 1, 2, ..., n$. But, we have $f_i(s_i - t_i) > \frac{r}{n}$. This is a contradiction because $f_i(s_i - y) \leq \frac{r}{n}$ for each $y \in B_i$, $i = 1, 2, ..., n$. Hence, for each $i = 1, ..., n$, we have $\|f_i\| = \frac{1}{n}$, and hence $\sum_{i=1}^{n} \|f_i\| = 1$. Also, in view of (3.6) and (3.8), we have

$$\sum_{i=1}^{n} f_i(\omega - \omega_0) = \sum_{i=1}^{n} f_i(s_i - \omega_0) - \sum_{i=1}^{n} f_i(s_i - \omega) \leq r - r = 0,$$

for all $\omega \in W$. Thus, (ii) holds.

$(ii) \Rightarrow (i)$. Assume that (ii) holds. For each $\omega \in W$, we have

$$\begin{aligned}\max_{1\le i\le n}\|s_i-\omega_0\| &= \sum_{i=1}^{n} f_i(s_i-\omega_0)\\ &\le \sum_{i=1}^{n} f_i(s_i-\omega)+\sum_{i=1}^{n} f_i(\omega-\omega_0)\\ &\le \sum_{i=1}^{n} f_i(s_i-\omega)\le \max_{1\le i\le n}\|s_i-\omega\|\sum_{i=1}^{n}\|f_i\|\\ &= \max_{1\le i\le n}\|s_i-\omega\|.\end{aligned}$$

Also, since $S \subset \bigcup_{i=1}^{n}\mathcal{N}(s_i,\epsilon)$, we conclude that for each $s \in S$ there exists $1 \le i_0 \le n$ such that

$$\begin{aligned}\|s-\omega_0\| &\le \|s_{i_0}-\omega_0\|+\epsilon\\ &\le \max_{1\le i\le n}\|s_i-\omega_0\|+\epsilon\\ &\le \max_{1\le i\le n}\|s_i-w\|+\epsilon \qquad (\omega\in W).\end{aligned}$$

This implies that

$$\sup_{s\in S}\|s-\omega_0\| \le \sup_{s\in S}\|s-\omega\|+\epsilon,$$

for each $\omega \in W$. Since $\epsilon > 0$ was arbitrary, we have (i), and the proof is complete. ■

In the following, we give a characterization of simultaneous metric projection for a subset M of $\mathbf{S}_W(S)$.

Theorem 3.2. *Let W be a closed and convex set in a real normed linear space X, S be a totally bounded set in X with $S \cap W = \emptyset$, and $M \subset W$. Assume that $W \cap \overline{co}(\{\omega\} \cup S) = \{\omega\}$ for each $\omega \in M$. Then the following assertions are equivalent:*

- (i) $M \subseteq \mathbf{S}_W(S)$,
- (ii) *For each $\epsilon > 0$ there exist a finite subset $\{s_1, s_2, ..., s_n\}$ of S and bounded linear functionals $f_i \in X^*$ $(i = 1, ..., n)$ such that $S \subset \bigcup_{i=1}^{n}\mathcal{N}(s_i,\epsilon)$,*

$$\sum_{i=1}^{n}\|f_i\| = 1, \tag{3.9}$$

$$\sum_{i=1}^{n} f_i \in (W-\omega)^{\circ} \qquad (\omega\in M), \tag{3.10}$$

and

$$\max_{1\leq i\leq n}\|s_i-\omega\|=\sum_{i=1}^{n} f_i(s_i-\omega) \qquad (\omega\in M). \tag{3.11}$$

Proof. $(i)\Rightarrow(ii)$. Suppose that (i) holds. Let $\omega_0\in M\subset\mathcal{S}_W(S)$ be fixed. By Theorem 3.1, for each $\epsilon>0$ there exist a finite subset $\{s_1,s_2,...,s_n\}$ of S and linear functionals $f_i\in X^*$ $(i=1,...,n)$ such that $\sum_{i=1}^{n}\|f_i\|=1$,

$$\sum_{i=1}^{n} f_i(\omega-\omega_0)\leq 0 \quad (\omega\in W) \tag{3.12}$$

and

$$\max_{1\leq i\leq n}\|s_i-\omega_0\|=\sum_{i=1}^{n} f_i(s_i-\omega_0). \tag{3.13}$$

Assume now that $\omega\in M\subset\mathcal{S}_W(S)$ is arbitrary. Then, by Theorem 3.1, there exist linear functionals $h_i\in X^*$ $(i=1,2,...,n)$ such that $\sum_{i=1}^{n}\|h_i\|=1$,

$$\sum_{i=1}^{n} h_i(\omega'-\omega)\leq 0 \qquad (\omega'\in W), \tag{3.14}$$

and

$$\max_{1\leq i\leq n}\|s_i-\omega\|=\sum_{i=1}^{n} h_i(s_i-\omega). \tag{3.15}$$

Then, in view of (3.14) and (3.15), for each $\omega'\in W$, we get

$$\begin{aligned}
\sum_{i=1}^{n} f_i(s_i-\omega) &\leq \sum_{i=1}^{n}\|f_i\|\max_{1\leq i\leq n}\|s_i-\omega\| \\
&= \max_{1\leq i\leq n}\|s_i-\omega\| \\
&= \sum_{i=1}^{n} h_i(s_i-\omega) \\
&= \sum_{i=1}^{n} h_i(s_i-\omega')+\sum_{i=1}^{n} h_i(\omega'-\omega) \\
&\leq \sum_{i=1}^{n} h_i(s_i-\omega') \\
&\leq \sum_{i=1}^{n}\|h_i\|\max_{1\leq i\leq n}\|s_i-\omega'\| \\
&= \max_{1\leq i\leq n}\|s_i-\omega'\|.
\end{aligned}$$

Consequently, we have

$$\max_{1\le i\le n}\|s_i-\omega\|\le\max_{1\le i\le n}\|s_i-\omega_0\|\qquad(\omega\in M),\tag{3.16}$$

and

$$\sum_{i=1}^{n}f_i(s_i-\omega)\le\max_{1\le i\le n}\|s_i-\omega_0\|\qquad(\omega\in M).\tag{3.17}$$

Therefore, by (3.12), (3.13) and (3.17), we obtain

$$\begin{aligned}\sum_{i=1}^{n}f_i(s_i-\omega) &\le \max_{1\le i\le n}\|s_i-\omega_0\|\\ &=\sum_{i=1}^{n}f_i(s_i-\omega_0)\\ &=\sum_{i=1}^{n}f_i(s_i-\omega)+\sum_{i=1}^{n}f_i(\omega-\omega_0)\\ &\le\sum_{i=1}^{n}f_i(s_i-\omega)\qquad(\omega\in M).\end{aligned}$$

This implies that

$$\sum_{i=1}^{n}f_i(s_i-\omega)=\max_{1\le i\le n}\|s_i-\omega_0\|\qquad(\omega\in M).\tag{3.18}$$

Thus, we have

$$\begin{aligned}\max_{1\le i\le n}\|s_i-\omega_0\| &=\sum_{i=1}^{n}f_i(s_i-\omega)\\ &\le\sum_{i=1}^{n}\|f_i\|\max_{1\le i\le n}\|s_i-\omega\|=\max_{1\le i\le n}\|s_i-\omega\|\qquad(\omega\in M).\end{aligned}$$

Hence, it follows from (3.16) and (3.18) that

$$\max_{1\le i\le n}\|s_i-\omega\|=\max_{1\le i\le n}\|s_i-\omega_0\|=\sum_{i=1}^{n}f_i(s_i-\omega)\qquad(\omega\in M).$$

Now, we show that $\sum_{i=1}^{n}f_i\in(W-\omega)^{\circ}$ for each $\omega\in M$. To see this, let $\omega\in M$ and $\omega'\in W$ be arbitrary. Then, by (3.11), (3.12) and (3.17), we obtain

$$\begin{aligned}\sum_{i=1}^{n}f_i(\omega'-\omega) &=\sum_{i=1}^{n}f_i(\omega'-\omega_0)+\sum_{i=1}^{n}f_i(\omega_0-s_i)+\sum_{i=1}^{n}f_i(s_i-\omega)\\ &\le 0-\max_{1\le i\le n}\|s_i-\omega_0\|+\max_{1\le i\le n}\|s_i-\omega_0\|=0.\end{aligned}$$

$(ii) \Rightarrow (i)$. This is an immediate consequence of Theorem 3.1, which completes the proof. ■

4. Characterizations of Simultaneous Metric Projection in Terms of the Elements of $\mathcal{E}\ (B_{X^*})$

In this section, we present characterizations of simultaneous metric projection onto closed convex sets in terms of the elements of $\mathcal{E}(B_{X^*})$. Moreover, we characterize uniqueness of simultaneous metric projection onto closed convex sets.

It is easily seen that $F \in (X^n)^*$ if and only if there exist functionals $f_1, f_2, ..., f_n$ in X^* such that $F(x_1, x_2, ..., x_n) = \sum_{i=1}^n f_i(x_i)$, where $x_i \in X$ and $\|F\| = \sum_{i=1}^n \|f_i\|$.

The following lemma shows that if F is an extreme point of $B_{(X^n)^*}$, then, $nf_i\ (i = 1, 2, ..., n)$ is an extreme point of B_{X^*}.

Lemma 4.1. *Let $F \in (X^n)^*$ be an extreme point of $B_{(X^n)^*}$ and $f_i \in X^*$ be such that $F(x_1, ..., x_n) = \sum_{i=1}^n f_i(x_i)$ and $\|F\| = \sum_{i=1}^n \|f_i\|$, where $x_i \in X\ (i = 1, 2, ..., n)$. Then, $nf_i \in \mathcal{E}(B_{X^*})\ i = 1, 2, ..., n$.*

Proof. Assume that

$$nf_i = \lambda g_i + (1 - \lambda)h_i, \tag{4.1}$$

where $g_i, h_i \in B_{X^*}$. Therefore, $\sum_{i=1}^n f_i = \lambda \sum_{i=1}^n \frac{1}{n} g_i + (1 - \lambda) \sum_{i=1}^n \frac{1}{n} h_i$. Consider the functionals $F_1, F_2 \in (X^n)^*$ defined by

$$F_1(x_1, x_2, ..., x_n) = \sum_{i=1}^n \frac{1}{n} g_i(x_i) \quad \text{and} \quad F_2(x_1, x_2, ..., x_n) = \sum_{i=1}^n \frac{1}{n} h_i(x_i).$$

It is clear that $F_1,\ F_2 \in B_{(X^n)^*}$. Now, since $F = \lambda F_1 + (1 - \lambda)F_2$ and $F \in \mathcal{E}(B_{(X^n)^*})$, it follows that $\lambda = 0,\ or\ \lambda = 1$. In view of 4.1, we conclude that $nf_i = g_i,\ or\ nf_i = h_i$. Therefore, we have $nf_i \in \mathcal{E}(B_{X^*})\ (i = 1, 2, .., n)$, which completes the proof. ■

Theorem 4.1. *Under the hypotheses of Theorem 3.1 the assertions (i) and (ii) are equivalent. Moreover, $nf_i \in \mathcal{E}(B_{X^*})$ for all $i = 1, 2, ..., n$.*

Proof. $(i) \Rightarrow (ii)$. Assume that (i) holds and $\epsilon > 0$ is given. By Theorem 3.1, there exist a finite subset $\{s_1, s_2, ..., s_n\}$ of S and linear functionals $g_i \in X^*\ (i = 1, 2, ..., n)$ such that

$$\sum_{i=1}^n \|g_i\| = 1,$$

$$\sum_{i=1}^{n} g_i(\omega-\omega_0) \leq 0 \qquad (\omega \in W),$$

and

$$\max_{1\leq i\leq n} \|s_i-\omega_0\| = \sum_{i=1}^{n} g_i(s_i-\omega_0).$$

Let

$$\mathcal{M}_1 := \{F \in (X^n)^* : \|F\| = 1, \max_{1\leq i\leq n} \|s_i-\omega_0\| = F(s_1-\omega_0, ..., s_n-\omega_0)\}.$$

Since there exists a linear functional $F_0 \in (X^n)^*$ such that

$$F_0(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} g_i(x_i), \ \textit{and} \ \|F_0\| = \sum_{i=1}^{n} \|g_i\|, \tag{4.2}$$

we conclude that $F_0 \in \mathcal{M}_1$, and hence $\mathcal{M}_1 \neq \emptyset$. It is clear that $\mathcal{M}_1$ is closed and convex. We show that $\mathcal{M}_1$ is an extremal subset of $B_{(X^n)^*}$. To do this, assume that for an $F \in \mathcal{M}_1$, *and a* λ with $0 < \lambda < 1$, we have $F = \lambda F_1 + (1-\lambda)F_2$ for some $F_1, F_2 \in B_{(X^n)^*}$. Since $F \in \mathcal{M}_1$, we get

$$\begin{aligned}\max\nolimits_{1\leq i\leq n} \|s_i-\omega_0\| &= F(s_1-\omega_0, ..., s_n-\omega_0)\\ &= \lambda F_1(s_1-\omega_0, ..., s_n-\omega_0)\\ &\quad +(1-\lambda)F_2(s_1-\omega_0, ..., s_n-\omega_0).\end{aligned} \tag{4.3}$$

On the other hand, we have

$$F_i(s_1-\omega_0, ..., s_n-\omega_0) \leq \|F_i\| \max_{1\leq i\leq n} \|s_i-\omega_0\| \leq \max_{1\leq i\leq n} \|s_i-\omega_0\| \quad (i = 1, 2).$$

We show that

$$F_1(s_1-\omega_0, ..., s_n-\omega_0) = \max_{1\leq i\leq n} \|s_i-\omega_0\| = F_2(s_1-\omega_0, ..., s_n-\omega_0).$$

Indeed, assume on the contrary that $F_1(s_1-\omega_0, ..., s_n-\omega_0) \neq \max_{1\leq i\leq n} \|s_i-\omega_0\|$. It follows that

$$F_1(s_1-\omega_0, ..., s_n-\omega_0) < \max_{1\leq i\leq n} \|s_i-\omega_0\|. \tag{4.4}$$

By (4.3) and (4.4), we have

$$\begin{aligned}\max_{1\leq i\leq n} \|s_i-\omega_0\| &= \lambda F_1(s_1-\omega_0, ..., s_n-\omega_0) + (1-\lambda)F_2(s_1-\omega_0, ..., s_n-\omega_0)\\ &< \lambda \max_{1\leq i\leq n} \|s_i-\omega_0\| + (1-\lambda) \max_{1\leq i\leq n} \|s_i-\omega_0\|\\ &= \max_{1\leq i\leq n} \|s_i-\omega_0\|.\end{aligned}$$

This is a contradiction. It is easy to show that

$$\|F_1\| = \|F_2\| = 1. \tag{4.5}$$

Therefore, $F_1, F_2 \in \mathcal{M}_1$. Thus, we conclude that $\mathcal{M}_1$ is an extremal subset of $B_{(X^n)^*}$, and hence $\mathcal{M}_1$ is weak*-compact.

Now, consider $\mathcal{M}_2 := \mathcal{M}_1 \cap [(W-\omega_0)^n]^\circ$. Clearly, $\mathcal{M}_2$ is convex and it is also weak*$-$compact because $[(W-\omega_0)^n]^\circ$ is weak*$-$closed. Consequently, by a virtue of Krein-Milman Theorem [[4], p. 440; Theorem 4], we get $\mathcal{E}(\mathcal{M}_2) \neq \varnothing$. Also, note that $\mathcal{M}_2$ is an extremal subset of $B_{(X^n)^*}$. Taking into account [[15], p. 58,; Lemma 1.7], we get $\mathcal{E}(\mathcal{M}_2) = \mathcal{E}(B_{(X^n)^*}) \cap \mathcal{M}_2 \neq \varnothing$. This implies that there exists a linear functional $F \in \mathcal{E}(B_{(X^n)^*})$ such that $\|F\| = 1$,

$$F(\omega-\omega_0, ..., \omega-\omega_0) \leq 0 \qquad (\omega \in W), \tag{4.6}$$

and

$$F(s_1-\omega_0, ..., s_n-\omega_0) = \max_{1\leq i\leq n} \|s_i-\omega_0\|. \tag{4.7}$$

Now, choose linear functionals $f_i \in X^*$ $(i=1,2,..,n)$ such that $F(x_1,x_2,...,x_n) = \sum_{i=1}^n f_i(x_i)$ and $\|F\| = \sum_{i=1}^n \|f_i\|$. By Lemma 4.1, we have $nf_i \in \mathcal{E}(B_{X^*})$ $(i=1,2,...,n)$. In view of (4.6) and (4.7), we conclude that (3.1), (3.2) and (3.3) hold.

$(ii) \Rightarrow (i)$. This is an immediate consequence of Theorem 3.1 (the implication $(ii) \Rightarrow (i)$). ∎

Theorem 4.2. *Under the hypotheses of Theorem 3.1 the following assertions are equivalent:*

(i) $\omega_0 \in \boldsymbol{S}_W(S)$,

(ii) *For each* $\epsilon > 0$ *there exist a finite subset* $\{s_1, s_2, ..., s_n\}$ *of* S *and bounded linear functionals* $f_i \in X^*$ $(i=1,...,n)$ *such that* $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i,\epsilon)$, $nf_i \in \mathcal{E}(B_{X^*})$ *with the following properties:*

$$\sum_{i=1}^n \|f_i\| = 1, \tag{4.8}$$

$$|\sum_{i=1}^n f_i(s_i-\omega_0)| = \max_{1\leq i\leq n} \|s_i-\omega_0\|, \tag{4.9}$$

and

$$|\sum_{i=1}^n f_i(s_i-\omega_0)| \leq |\sum_{i=1}^n f_i(s_i-\omega)| \qquad (\omega \in W). \tag{4.10}$$

(iii) *For each* $\epsilon > 0$ *there exist a finite subset* $\{s_1, s_2, ..., s_n\}$ *of* S *and bounded linear functionals* $f_i \in X^*$ $(i = 1, ..., n)$ *such that* $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$, $nf_i \in \mathcal{E}(B_{X^*})$ *satisfying* (3.2), (4.9) *and*

$$\sum_{i=1}^n f_i(\omega - \omega_0) \sum_{i=1}^n f_i(s_i - \omega_0) \leq 0 \qquad (\omega \in W). \tag{4.11}$$

Proof. $(i) \Rightarrow (ii)$. Assume that (i) holds and $\epsilon > 0$ is arbitrary. Then, by Theorem 4.1, there exist a finite subset $\{s_1, s_2, ..., s_n\}$ of S and bounded linear functionals $f_i \in X^*$ such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$, $nf_i \in \mathcal{E}(B_{X^*})$ $(i = 1, 2, ...n)$ and (3.1), (3.2) and (3.3) hold. Therefore, by (3.2) and (3.3), we have

$$|\sum_{i=1}^n f_i(s_i - \omega_0)| \leq |\sum_{i=1}^n f_i(s_i - \omega)| \qquad (\forall\, \omega \in W).$$

Hence, (i) implies (ii).

$(ii) \Rightarrow (i)$. Assume that (ii) holds. Since $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$, by a similar argument as in the proof of Theorem 3.1 (the implication $(ii) \Rightarrow (i)$) and using (4.10), we get

$$\begin{aligned}
\sup_{s \in S} \|s - \omega_0\| &\leq \max_{1 \leq i \leq n} \|s_i - \omega_0\| + \epsilon \\
&\leq \max_{1 \leq i \leq n} \|s_i - \omega\| + \epsilon \\
&\leq \sup_{s \in S} \|s - \omega\| + \epsilon,
\end{aligned}$$

for each $\omega \in W$. Since $\epsilon > 0$ was arbitrary, this implies that $\omega_0 \in \mathbf{S}_W(S)$.

$(i) \Rightarrow (iii)$. Assume now that (i) holds and $\epsilon > 0$ is arbitrary. Then, by Theorem 4.1, there exist a finite subset $\{s_1, s_2, ..., s_n\}$ of S and bounded linear functionals $f_i \in X^*$ such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$, $nf_i \in \mathcal{E}(B_{X^*})$ $(i = 1, ..., n)$ and that (3.1), (3.2) and (3.3) hold. Thus, we conclude that (4.9) holds and we have

$$\sum_{i=1}^n f_i(\omega - \omega_0) \sum_{i=1}^n f_i(s_i - \omega_0) \leq 0 \qquad (\omega \in W).$$

Therefore, (i) implies (iii).

$(iii) \Rightarrow (i)$. If (iii) holds, then for each $\epsilon > 0$ there exist a finite subset $\{s_1, s_2, ..., s_n\}$ of S and bounded linear functionals $f_i \in X^*$ such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$, $nf_i \in \mathcal{E}(B_{X^*})$ $(i = 1, ..., n)$ satisfying (3.2), (4.9) and

$$\sum_{i=1}^n f_i(\omega - \omega_0) \sum_{i=1}^n f_i(s_i - \omega_0) \leq 0 \qquad (\omega \in W).$$

Now, for each $1 \leq i \leq n$, put

$$\psi_i = sign[\sum_{i=1}^{n} f_i(s_i - \omega_0)] f_i. \tag{4.12}$$

Then, we have $n\psi_i \in \mathcal{E}(B_{X^*})$, and

$$\begin{aligned}
\sum_{i=1}^{n} \psi_i(s_i - \omega_0) &= \text{sign}\,[\sum_{i=1}^{n} f_i(s_i - \omega_0)] \sum_{i=1}^{n} f_i(s_i - \omega_0) \\
&= \frac{\sum_{i=1}^{n} f_i(s_i - \omega_0)}{|\sum_{i=1}^{n} f_i(s_i - \omega_0)|} \sum_{i=1}^{n} f_i(s_i - \omega_0) \\
&= |\sum_{i=1}^{n} f_i(s_i - \omega_0)| = \max_{1 \leq i \leq n} \|s_i - \omega_0\|.
\end{aligned}$$

Also, by (4.11), we conclude that

$$\begin{aligned}
\sum_{i=1}^{n} \psi_i(\omega - \omega_0) &= \text{sign}\,[\sum_{i=1}^{n} f_i(s_i - \omega_0))] \sum_{i=1}^{n} f_i(\omega - \omega_0) \\
&= \frac{\sum_{i=1}^{n} f_i(s_i - \omega_0)}{|\sum_{i=1}^{n} f_i(s_i - \omega_0)|} \sum_{i=1}^{n} f_i(\omega - \omega_0) \leq 0 \quad (\omega \in W).
\end{aligned}$$

Note that in view of (4.9) and that $nf_i \in \mathcal{E}(B_{X^*})$ $(i = 1, 2, ..., n)$, we conclude that $\sum_{i=1}^{n} \|f_i\| = 1$, and hence by (4.12) we have $\sum_{i=1}^{n} \|\psi_i\| = 1$. Whence, the functionals ψ_i defined by (4.12) satisfy (3.1), (3.2) and (3.3), and therefore by Theorem 3.1, we have $\omega_0 \in \mathbf{S}_W(S)$. Thus, (iii) implies (i), which completes the proof. ■

Remark 4.1. It is worth noting that under the hypotheses of Theorem 3.1, in the following we obtain results of a different nature. In fact, we give a characterization for uniqueness of simultaneous metric projection onto closed convex sets.

Theorem 4.3. *Under the hypotheses of Theorem 3.1 the following assertions are equivalent:*

(i) $\mathbf{S}_W(S) = \{\omega_0\}$,

(ii) *$\omega_0 \in \boldsymbol{S}_W(S)$ and for each $\epsilon > 0$ there do not exist $\omega \in W \setminus \{\omega_0\}$, a finite subset $\{s_1, s_2, ..., s_n\}$ of S such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$ and $f_i \in X^*$ $(i = 1, 2, ..., n)$ with properties*

$$\sum_{i=1}^n \|f_i\| = 1, \tag{4.13}$$

$$\sum_{i=1}^n f_i(\omega) = \sum_{i=1}^n f_i(\omega_0), \tag{4.14}$$

and

$$\sum_{i=1}^n f_i(s_i - \omega) = \max_{1\le i\le n} \|s_i - \omega\|. \tag{4.15}$$

(iii) *$\omega_0 \in \boldsymbol{S}_W(S)$ and for each $\epsilon > 0$ there do not exist $\omega \in W \setminus \{\omega_0\}$, a finite subset $\{s_1, s_2, ..., s_n\}$ of S such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$ and $f_i \in X^*$ (i=1,2,...,n) with properties (4.14), (4.15) and*

$$nf_i \in \mathcal{E}(B_{X^*}) \quad (i = 1, 2, ..., n). \tag{4.16}$$

(iv) *$\omega_0 \in \boldsymbol{S}_W(S)$ and for each $\epsilon > 0$ there do not exist $\omega \in W \setminus \{\omega_0\}$, a finite subset $\{s_1, s_2, ..., s_n\}$ of S such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$ and $f_i \in X^*$ $(i = 1, 2, ..., n)$ with properties (4.14), (4.16) and*

$$|\sum_{i=1}^n f_i(s_i - \omega)| = \max_{1\le i\le n} \|s_i - \omega\|. \tag{4.17}$$

(v) *$\omega_0 \in \boldsymbol{S}_W(S)$ and for each $\epsilon > 0$ there do not exist $\omega \in W \setminus \{\omega_0\}$, a finite subset $\{s_1, s_2, ..., s_n\}$ of S such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$ and $f_i \in X^*$ $(i = 1, 2, ..., n)$ with properties (4.16), (4.17) and*

$$\sum_{i=1}^n f_i(\omega - \omega_0) \sum_{i=1}^n f_i(s_i - \omega) \ge 0. \tag{4.18}$$

Proof. $(i) \Rightarrow (ii)$. Assume that we have (i). Suppose that (ii) does not hold. Then for each $\epsilon > 0$ there exist $\omega \in W \setminus \{\omega_0\}$, a finite subset $\{s_1, s_2, ..., s_n\}$ of S such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$ and $f_i \in X^*$ $(i = 1, 2, ..., n)$ satisfying (4.13), (4.14) and (4.15). Therefore, since $\omega_0 \in \mathcal{S}_W(S)$, we have

$$\begin{aligned}\max_{1\le i\le n} \|s_i - \omega\| &= |\sum_{i=1}^n f_i(s_i - \omega)| = |\sum_{i=1}^n f_i(s_i - \omega_0) - \sum_{i=1}^n f_i(\omega - \omega_0)| \\ &= |\sum_{i=1}^n f_i(s_i - \omega_0)| \le \max_{1\le i\le n} \|s_i - \omega_0\| \sum_{i=1}^n \|f_i\| \\ &= \max_{1\le i\le n} \|s_i - \omega_0\| \le \sup_{s\in S} \|s - \omega_0\| = d(S, W).\end{aligned}$$

It follows that $\omega \in \mathbf{S}_W(S)$, which contradicts (i). Thus, (i) implies (ii).

$(ii) \Rightarrow (iii)$. Assume that (iii) does not hold. Then for each $\epsilon > 0$ there exist $\omega \in W \setminus \{\omega_0\}$, a finite subset $\{s_1, s_2, ..., s_n\}$ of S such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$ and $f_i \in X^*$ $(i = 1, ..., n)$ with $nf_i \in \mathcal{E}(B_{X^*})$ $(i = 1, 2, ..., n)$ and (4.14), (4.15) hold. Therefore, $\|nf_i\| \leq 1$ and thus $\sum_{i=1}^n \|f_i\| \leq 1$. For the reverse inequality, by (4.15), we get $\sum_{i=1}^n \|f_i\| \geq 1$, and hence (ii) does not hold. Therefore, (ii) implies (iii).

The implication $(iii) \Rightarrow (iv)$ is obvious.

Now, assume that we have (iv). Let $\{s_1, s_2, ..., s_n\}$ be a finite subset of S such that $S \subset \bigcup_{i=1}^n \mathcal{N}(s_i, \epsilon)$. Then, for every $\omega \in W \setminus \{\omega_0\}$ and $f_i \in X^*$ $(i = 1, 2, ..., n)$ with properties (4.16) and (4.17), we conclude that

$$\sum_{i=1}^n f_i(\omega) \neq \sum_{i=1}^n f_i(\omega_0). \tag{4.19}$$

In view of (4.16) and (4.17), we obtain $\sum_{i=1}^n \|f_i\| = 1$. Consequently, by (4.19), for any such $\omega \in W \setminus \{\omega_0\}$ and $f_i \in X^*$ $(i = 1, 2, ..., n)$, we get

$$\begin{aligned}
(\max_{1\leq i\leq n} \|s_i - \omega_0\|)^2 &\geq |\sum_{i=1}^n f_i(s_i - \omega_0)|^2 \\
&= |\sum_{i=1}^n f_i(s_i - \omega) + \sum_{i=1}^n f_i(\omega - \omega_0)|^2 \\
&= |\sum_{i=1}^n f_i(s_i - \omega)|^2 + |\sum_{i=1}^n f_i(\omega - \omega_0)|^2 \\
&\quad +2\sum_{i=1}^n f_i(\omega - \omega_0)\sum_{i=1}^n f_i(s_i - \omega) \\
&> (\max_{1\leq i\leq n} \|s_i - \omega\|)^2 + 2\sum_{i=1}^n f_i(\omega - \omega_0)\sum_{i=1}^n f_i(s_i - \omega).
\end{aligned}$$

Taking into account that $\omega_0 \in \mathbf{S}_W(S)$, it follows that for any such $\omega \in W \setminus \{\omega_0\}$ and functionals $f_i \in X^*$ $(i = 1, ..., n)$, we obtain

$$\sum_{i=1}^n f_i(\omega - \omega_0)\sum_{i=1}^n f_i(s_i - \omega) < 0.$$

Thus, (iv) implies (v).

Finally, assume that we have (v), and let $\omega \in W \setminus \{\omega_0\}$ be arbitrary. Then, by Theorem 4.2 (the implication $(i) \Rightarrow (iii)$), it follows that $\omega \in W \setminus \mathbf{S}_W(S)$. Thus , (v) implies (i), and the proof is complete. ∎

5. Characterizations of Simultaneous Metric Projection in $C(Q, Y)$

Let Q be a compact Hausdorff space, Y be a Banach space and G be a proximinal subspace of Y. Let $W = C(Q, G)$ and $S = \{f_1, f_2, ..., f_n\}$ be a finite set in $X = C(Q, Y)$ such that $S \cap W = \emptyset$. As an application of the results obtained, we characterize simultaneous metric projection onto W, which is considered as a subspace of X. We start with the following theorem.

Theorem 5.1. *Let Q be a compact Hausdorff space and G be a proximinal subspace of a Banach space Y. Assume that $W = C(Q, G)$ is considered as a subspace of $X = C(Q, Y)$. Then for each $\epsilon > 0$ and each finite set $S = \{f_1, f_2, ..., f_n\}$ in X such that $S \cap W = \emptyset$ and $\max_{1\le j\le n} d(f_j(Q), G) < \frac{\epsilon}{2}$, there exist elements $x_{1j}, x_{2j}, ...x_{m_j j} \in C(Q)$ and $g_{1j}, g_{2j}, ..., g_{m_j j} \in G$ $(j = 1, 2, ..., n)$ with the following properties:*

- (i) $0 \le x_{ij} \le 1$ $(i = 1, 2, ..., m_j;\ j = 1, 2, ..., n)$,
- (ii) $\sum_{i=1}^{m_j} x_{ij} = 1$ $(j = 1, 2, ..., n)$, *and*
- (iii) $\max_{1\le r\le n} \|f_r - \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} \otimes g_{ij}\| \le \epsilon$. *Moreover,* $\boldsymbol{S}_{C(Q,G)}(S) \ne \emptyset$, *where $d(f_j(Q), G)$ $(1 \le j \le n)$ is defined by (2.1).*

Proof. Let $\epsilon > 0$ be given and let $f_j \in S$ $(j = 1, 2, ..., n)$ be fixed. Put $K_j := f_j(Q)$ $(j = 1, 2, ..., n)$. Since K_j is a compact subset of Y, it follows that K_j is a totally bounded set. Thus, for each $j = 1, 2, ..., n$, there exist elements $y_{1j}, y_{2j}, ..., y_{m_j j} \in K_j$ such that $K_j \subset \cup_{i=1}^{m_j}\mathcal{N}(y_{ij}, \frac{\epsilon}{2})$ $(j = 1, 2, ..., n)$. Then, by ([12]; Theorem 2.13), for each $j = 1, 2, ..., n$, there exist functions $h_{ij} \in C(Y)$ such that $h_{ij}(x) = 0$ for each $x \notin \mathcal{N}(y_{ij}, \frac{\epsilon}{2})$, $0 \le h_{ij} \le 1$ $(i = 1, 2, ..., m_j)$, and $\sum_{i=1}^{m_j} h_{ij}(q) = 1$ for all $q \in Q$ and all $j = 1, 2, ..., n$. Put $x_{ij} = h_{ij} \circ f_j$ $(i = 1, 2, ..., m_j;\ j = 1, 2, ..., n)$. Then, $x_{ij} \in C(Q)$, $0 \le x_{ij} \le 1$ $(i = 1, 2, ..., m_j;\ j = 1, 2, ..., n)$, and $\sum_{i=1}^{m_j} x_{ij} = 1$ $(j = 1, 2, ..., n)$. Now, let $q \in Q$ be arbitrary. Since $f_j(q) \in K_j$ and $K_j \subset \cup_{i=1}^{m_j}\mathcal{N}(y_{ij}, \frac{\epsilon}{2})$ $(j = 1, 2, ..., n)$, it follows that

$$\|f_j(q) - y_{ij}\| < \frac{\epsilon}{2} \quad \text{for some } i = 1, 2, ..., m_j. \tag{5.1}$$

Since G is a proximinal subspace of Y and $y_{ij} \in Y$ $(i = 1, 2, ..., m_j;\ j = 1, 2, ..., n)$, we conclude that there exists $g_{ij} \in G$ $(i = 1, 2, ..., m_j;\ j = 1, 2, ..., n)$ such that

$$\|y_{ij} - g_{ij}\| = d(y_{ij}, G), \quad (i = 1, 2, ..., m_j;\ j = 1, 2, ..., n). \tag{5.2}$$

In view of (5.1) and (5.2) and that $y_{ij} \in K_j$ $(i = 1, 2, ..., m_j;\ j = 1, 2, ..., n)$, we obtain

$$
\begin{aligned}
\|f_j(q) - g_{ij}\| &\leq \|f_j(q) - y_{ij}\| + \|y_{ij} - g_{ij}\| \\
&< \frac{\epsilon}{2} + d(y_{ij}, G) \\
&\leq \frac{\epsilon}{2} + \max_{1 \leq j \leq n} d(f_j(Q), G) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned} \tag{5.3}
$$

for all $q \in Q$ and some $i = 1, 2, ..., m_j$ $(j = 1, 2, ..., n)$.

On the other hand, we have $x_{ij}(q) = h_{ij}(f_j(q)) = 0$, if $f_j(q) \notin \mathcal{N}(y_{ij}, \frac{\epsilon}{2})$ $(i = 1, 2, ..., m_j;\ j = 1, 2, ..., n)$. This, together with (5.3) and that $\sum_{i=1}^{m_j} x_{ij} = 1$ $(j = 1, 2, ..., n)$ imply that

$$
\begin{aligned}
\|f_r(q) - \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij}(q) g_{ij}\| &= \frac{1}{n}\|n f_r(q) - \sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij}(q) g_{ij}\| \\
&= \frac{1}{n}\|\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij}(q)[f_r(q) - g_{ij}]\| \\
&\leq \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij}(q)\|f_r(q) - g_{ij}\| \\
&< \frac{1}{n}\epsilon\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij}(q) = \epsilon,
\end{aligned} \tag{5.4}
$$

for all $r = 1, 2, ..., n$ and all $q \in Q$.

Now, Consider the isometry

$$\rho : C(Q) \otimes G \to C(Q, G)$$

defined by $\rho(z) = \rho_z$, where $z = \sum_{r=1}^{k} u_r \otimes v_r \in C(Q) \otimes G$ $(k \in \mathbb{N})$ and $\rho_z(q) := \sum_{r=1}^{k} u_r(q) v_r$, for each $q \in Q$. Therefore, it follows from (5.4) that

$$\max_{1 \leq r \leq n} \|f_r - \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} \otimes g_{ij}\| \leq \epsilon.$$

Now, let $r_0 = d(S, W)$. It is obvious that $r_0 > 0$, since $S \cap W = \emptyset$. Then, by the above, we conclude that for $\epsilon = r_0 > 0$ there exist elements $x_{1j}, x_{2j}, ...x_{m_j j} \in C(Q)$ and $g_{1j}, g_{2j}, ..., g_{m_j j} \in G$ $(j = 1, 2, ..., n)$ such that $0 \leq x_{ij} \leq 1$ $(i = 1, 2, ..., m_j;\ j = 1, 2, ..., n)$, $\sum_{i=1}^{m_j} x_{ij} = 1$ $(j = 1, 2, ..., n)$, and

$$\max_{1 \leq r \leq n} \|f_r - \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} \otimes g_{ij}\| \leq r_0. \tag{5.5}$$

Let $z_0 = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} \otimes g_{ij}$. Thus, $\rho_{z_0} \in C(Q, G)$, and by (5.5), we have $\max_{1\leq r\leq n} \|f_r - \rho_{z_0}\| < r_0$. This implies that $\max_{1\leq r\leq n} \|f_r - \rho_{z_0}\| = d(S, W)$, and hence $\rho_{z_0} \in \mathbf{S}_{C(Q,G)}(S)$, which completes the proof. ■

In the sequel, let $S = \{f_1, f_2, ..., f_n\}$ be a finite subset in $C(Q, Y)$ and, For simplicity, we denote $S_q := \{f_1(q), f_2(q), ..., f_n(q)\}$ for each $q \in Q$.

Theorem 5.2. *Under the hypotheses of Theorem 5.1, for each $\epsilon > 0$ and each finite set $S = \{f_1, f_2, ..., f_n\}$ in X such that $S \cap W = \emptyset$ and $\max_{1\leq j\leq n} d(f_j(Q), G) < \frac{\epsilon}{2}$, then there exists $q_0 \in Q$ such that*

$$d(S, C(Q, G)) = \sup_{q\in Q} d(S_q, G) = d(S_{q_0}, G). \tag{5.6}$$

Proof. If $\omega \in W$ and $q \in Q$, then we have

$$d(S_q, G) \leq \max_{1\leq i\leq n} \|f_i(q) - \omega(q)\| \leq \max_{1\leq i\leq n} \|f_i - \omega\|. \tag{5.7}$$

By taking infimum on $\omega \in W$, and then supremum on $q \in Q$, we get

$$\sup_{q\in Q} d(S_q, G) \leq d(S, W). \tag{5.8}$$

For the reverse inequality, by Theorem 5.1 there exist elements $x_{1j}, x_{2j}, ...x_{m_j j} \in C(Q)$, $y_{1j}, y_{2j}, ..., y_{m_j j} \in f_j(Q)$ and $g_{1j}, g_{2j}, ..., g_{m_j j} \in G$ $(j = 1, 2, ..., n)$ such that $0 \leq x_{ij} \leq 1$, $\|g_{ij} - y_{ij}\| < \epsilon$ $(i = 1, 2, ..., m_j;\ j = 1, 2, ..., n)$, $\sum_{i=1}^{m_j} x_{ij} = 1$ $(j = 1, 2, ..., n)$, and

$$\max_{1\leq r\leq n} \|f_r - \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} \otimes g_{ij}\| \leq \epsilon. \tag{5.9}$$

Now, for each $i = 1, 2, ..., m_j$, choose $q_i \in Q$ such that $y_{ij} = f_j(q_i)$ $(j = 1, 2, ..., n)$. Choose $g_0 \in G$ such that

$$\begin{aligned}
\|y_{ij} - g_0\| &\leq \max_{1\leq j\leq n} \|y_{ij} - g_0\| \\
&= \max_{1\leq j\leq n} \|f_j(q_i) - g_0\| \\
&\leq \inf_{g\in G} \max_{1\leq j\leq n} \|f_j(q_i) - g\| + \epsilon \\
&= d(S_{q_i}, G) + \epsilon \\
&\leq \sup_{q\in Q} d(S_q, G) + \epsilon, \quad \forall\, i = 1, 2, ..., m_j;\ j = 1, 2, ..., n.
\end{aligned}$$

This implies that

$$\begin{aligned}\|g_{ij} - g_0\| &\leq \|g_{ij} - y_{ij}\| + \|y_{ij} - g_0\| \\ &< \sup_{q\in Q} d(S_q, G) + 2\epsilon, \quad \forall\, i = 1, 2, ..., m_j;\; j = 1, 2, ..., n.\end{aligned} \tag{5.10}$$

Let $z_0 = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} \otimes g_{ij}$. Thus, by a similar argument as in the proof of Theorem 5.1, we have $\rho_{z_0} \in W$ and $\rho_{z_0}(q) := \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij}(q) g_{ij}$ for all $q \in Q$. Therefore, in view of (5.9), (5.10) and that $\sum_{i=1}^{m_j} x_{ij} = 1$ for each $j = 1, 2, ..., n$, we conclude that

$$\begin{aligned} d(S, W) &\leq \|f_r - \rho_{z_0}\| = \sup_{q\in Q}\|f_r(q) - \rho_{z_0}(q)\| \\ &\leq \sup_{q\in Q}\|f_r(q) - \rho_{z_0}(q)\| + \|\frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij} \otimes (g_{ij} - g_0)\| \\ &< \epsilon + \frac{1}{n}\sup_{q\in Q}\|\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij}(q)(g_{ij} - g_0)\| \\ &\leq \epsilon + \frac{1}{n}\sup_{q\in Q}\sum_{j=1}^{n}\sum_{i=1}^{m_j} x_{ij}(q)\|g_{ij} - g_0\| \\ &\leq 3\epsilon + \sup_{q\in Q} d(S_q, G). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $d(S, C(Q, G)) = \sup_{q\in Q} d(S_q, G)$.

Finally, we define $F(q) := d(S_q, G)$ for each $q \in Q$. Now, for each $g \in G$ and each $q,\; q' \in Q$, we have

$$\|f_i(q) - g\| \leq \|f_i(q) - f_i(q')\| + \|f_i(q') - g\|,$$

and

$$\|f_i(q') - g\| \leq \|f_i(q) - f_i(q')\| + \|f_i(q) - g\|.$$

From these relations, we obtain

$$|F(q) - F(q')| \leq \max_{1\leq i\leq n}\|f_i(q) - f_i(q')\| \quad (q,\; q' \in Q).$$

This implies that F is a continuous function on Q. Since Q is compact, it follows that there exists $q_0 \in Q$ such that $\sup_{q\in Q} d(S_q, G) = d(S_{q_0}, G)$, which completes the proof. ∎

Theorem 5.3. *Under the hypotheses of Theorem 5.1, for each* $\epsilon > 0$ *and each finite set* $S = \{f_1, f_2, ..., f_n\}$ *in* X *such that* $S \cap W = \emptyset$, $\max_{1\leq j\leq n} d(f_j(Q), G) < \frac{\epsilon}{2}$ *and* $\omega_0 \in W$, *then the following assertions are equivalent:*

(i) $\omega_0 \in \mathbf{S}_W(S)$,

(ii) *There exists* $q_0 \in Q$ *such that* $\omega_0(q_0) \in \mathcal{S}_G(S_{q_0})$, *and*

$$\max_{1\leq i\leq n} \|f_i - \omega_0\| = \max_{1\leq i\leq n} \|f_i(q_0) - \omega_0(q_0)\| = d(S_{q_0}, G). \tag{5.11}$$

Proof. $(i) \Rightarrow (ii)$. Suppose (i) holds. In view of Theorem 5.2, there exists $q_0 \in Q$ such that

$$d(S_{q_0}, G) = d(S, C(Q, G)).$$

Since $\omega_0 \in \mathbf{S}_W(S)$, we get

$$\begin{aligned} \max_{1\leq i\leq n} \|f_i - \omega_0\| &= d(S, C(Q, G)) = d(S_{q_0}, G)) \\ &\leq \max_{1\leq i\leq n} \|f_i(q_0) - \omega_0(q_0)\| \\ &\leq \max_{1\leq i\leq n} \|f_i - \omega_0\| \end{aligned}$$

Therefore

$$\max_{1\leq i\leq n} \|f_i - \omega_0\| = \max_{1\leq i\leq n} \|f_i(q_0) - \omega_0(q_0)\| = d(S_{q_0}, G),$$

and we have $\omega_0(q_0) \in \mathcal{S}_G(S_{q_0})$.

$(ii) \Rightarrow (i)$. Assume that (ii) holds. Then there exists $q_0 \in Q$ such that $\omega_0(q_0) \in \mathcal{S}_G(S_{q_0})$, and (5.11) holds. Therefore, in view of Theorem 5.2, we obtain

$$\begin{aligned} d(S, C(Q, G)) &\leq \max_{1\leq i\leq n} \|f_i - \omega_0\| \\ &= \max_{1\leq i\leq n} \|f_i(q_0) - \omega_0(q_0)\| = d(S_{q_0}, G) \\ &\leq d(S, C(Q, G)). \end{aligned}$$

This implies that $\omega_0 \in \mathcal{S}_W(S)$, and the proof is complete. ∎

Corollary 5.1. *Let* Q *be a compact Hausdorff space. Assume* $W = C_{\mathbb{R}}(Q)$ *is considered as a subspace of* $X = C_{\mathbb{C}}(Q)$. *Let* $\epsilon > 0$ *be given and let* $S = \{f_1, f_2, ..., f_n\}$ *be a finite set in* $C(Q, Y)$ *such that* $S \cap W = \emptyset$ *and* $\max_{1\leq j\leq n} d(f_j(Q), \mathbb{R}) < \frac{\epsilon}{2}$. *If* $\omega_0 \in W$, *then the following assertions are equivalent:*

(i) $\omega_0 \in \mathbf{S}_{C_{\mathbb{R}}(\mathbb{Q})}(S)$,

(ii) *There exists* $q_0 \in Q$ *such that* $\omega_0(q_0) \in \mathcal{S}_{\mathbb{R}}(S_{q_0})$ *and* $\max_{1\leq i\leq n} \|f_i - \omega_0\| = \max_{1\leq i\leq n} \|f_i(q_0) - \omega_0(q_0)\| = d(S_{q_0}, \mathbb{R})$.

Proof. This is an immediate consequence of Theorem 5.3. ∎

Theorem 5.4. *Under the hypotheses of Theorem 5.1, for each* $\epsilon > 0$ *and each finite set* $S = \{f_1, f_2, ..., f_n\}$ *in* X *such that* $S \cap W = \emptyset$, $\max_{1\leq j\leq n} d(f_j(Q), G) < \frac{\epsilon}{2}$ *and* $\omega_0 \in W$, *then the following assertions are equivalent:*

(i) $\omega_0 \in \mathbf{S}_W(S)$,

(ii) *There exist* $q_0 \in Q$ *and bounded linear functionals* $\varphi_i \in Y^*$ $(i = 1, ..., n)$ *such that*

$$\sum_{i=1}^{n} \|\varphi_i\| = 1,$$

$$\sum_{i=1}^{n} \varphi_i(g - \omega_0(q_0)) \leq 0 \qquad (g \in G),$$

and

$$\sum_{i=1}^{n} \varphi_i(f_i(q_0) - \omega_0(q_0)) = \max_{1\leq i\leq n} \|f_i - \omega_0\|.$$

Proof. $(i) \Rightarrow (ii)$. Suppose (i) holds. Since $\omega_0 \in \mathbf{S}_W(S)$, it follows from Theorem 5.3 (the implication $(i) \Rightarrow (ii)$) that $\omega_0(q_0) \in \mathbf{S}_G(S_{q_0})$. Therefore, by Theorem 3.1, there exist linear functionals $\varphi_i \in Y^*$ $(i = 1, 2, ..., n)$ such that

$$\sum_{i=1}^{n} \|\varphi_i\| = 1,$$

$$\sum_{i=1}^{n} \varphi_i(g - \omega_0(q_0)) \leq 0 \qquad (g \in G),$$

and

$$\max_{1\leq i\leq n} \|f_i - \omega_0\| = \sum_{i=1}^{n} \varphi_i(f_i(q_0) - \omega_0(q_0)).$$

$(ii) \Rightarrow (i)$. Assume (ii) holds. Then there exist $q_0 \in Q$ and bounded linear functionals $\varphi_i \in Y^*$ $(i = 1, ..., n)$ such that

$$\begin{aligned}\max_{1\leq i\leq n} \|f_i - \omega_0\| &= \sum_{i=1}^{n} \varphi_i(f_i(q_0) - \omega_0(q_0)) \\ &= \sum_{i=1}^{n} \varphi_i(g - \omega_0(q_0)) + \sum_{i=1}^{n} \varphi_i(f_i(q_0) - g)\end{aligned}$$

$$\leq \sum_{i=1}^{n} \varphi_i(f_i(q_0) - g)$$
$$\leq \max_{1\leq i\leq n} \|f_i(q_0) - g\|,$$

for all $g \in G$. Therefore, we have

$$\begin{aligned}\max_{1\leq i\leq n} \|f_i - \omega_0\| &\leq \inf_{g\in G} \max_{1\leq i\leq n} \|f_i(q_0) - g\| = d(S_{q_0}, G) \\ &\leq \max_{1\leq i\leq n} \|f_i(q_0) - \omega_0(q_0)\| \\ &\leq \max_{1\leq i\leq n} \|f_i - \omega_0\|.\end{aligned}$$

This implies that

$$\max_{1\leq i\leq n} \|f_i - \omega_0\| = \max_{1\leq i\leq n} \|f_i(q_0) - \omega_0(q_0)\| = d(S_{q_0}, G),$$

and $\omega_0(q_0) \in \mathbf{S}_G(S_{q_0})$. Thus, by Theorem 5.3 (the implication $(ii) \Rightarrow (i)$), we obtain $\omega_0 \in \mathbf{S}_W(S)$, and the proof is complete. ■

Acknowledgment

We thank the referee for his comments which improved the presentation of the paper significantly. This research was supported by Linear Algebra and Optimization Central of Excellence of Shahid Bahonar University of Kerman.

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