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COMMON INVARIANT SUBSPACES FOR N-TUPLES OF POSITIVE OPERATORS ACTING ON TOPOLOGICAL VECTOR SPACES

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Abstract. Let $(T_1, \ldots, T_N)$ be a $N-$tuple of positive operators with respect a Markushevich basis which are defined on a Hausdorff topological vector space.

In this work we extend the notion of weak local quasinilpotence to $N-$tuples of operators (not-necessarily commuting). Under the hypothesis of existence of positive vectors, joint weak locally quasinilpotent we will obtain the existence of common invariant subspaces.

1. Introduction

One of the most important unsolved problems of operator theory is the invariant subspace problem. Does every operator on an infinite-dimensional Hilbert space have a non-trivial invariant subspace? Positive results are known for some special classes of operators: N. Aronszajn and K. T. Smith [2] proved that compact operators have non-trivial closed invariant subspaces. A. R. Bernstein and A. Robinson [3] and subsequently P. R. Halmos [7] proved this for polynomially compact operators, Lomonosov [8] for every continuous operator which commutes with an non-zero compact operator, and S. W. Brown [4] for subnormal operators on Hilbert spaces. P. Enflo [6] was the first to construct a continuous operator on a separable Banach space without a non-trivial closed invariant subspace, and C. J. Read [12] presented an example of a continuous operator on l_1 without a non-trivial closed invariant subspace. More recently, Y. A. Abramovich, C. D. Aliprantis and O.

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Burkinshaw [1] proved the existence, in Banach spaces, of non-trivial closed invariant subspaces for positive operators that commute with a quasinilpotent positive operator which dominates a non-zero compact operator, for positive kernel operators which commutes with a quasinilpotent positive operator and, for quasinilpotent positive Dunford-Pettis operators.

In this work we will study the existence of common invariant subspaces for the $N-$tuple $T = (T_1, \ldots, T_N)$, where operators T_i are positive operators defined on a Hausdorff topological vector space X. That is, the existence of non-trivial closed subspace $F \subset X$ such that $T_i(F) \subset F$ for every $i = 1, ..., N$. We extend the results of Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw in [1] to a more general context using the natural generalization of the concept of weak local quasinilpotence to $N-$tuples of operators. The surveys of Marek Ptak (see [10] and [11]) are a good resorce on the invariant subspace problem for N-tuples of operators.

Definition 1.1. Let $T = (T_1, \ldots, T_N)$ be a $N-$tuple of operators on a Banach space X. Then we say that T is *joint locally quasinilpotent at* y_0 if

$$\lim_{n\to\infty} \|T_{i_1} \cdots T_{i_n}(y_0)\|^{1/n} = 0,$$

where $i_j \in \{1, \ldots, N\}$ for every $j \in \mathbb{N}$. We denote

$$Q_T = \{x \in X : \ T \text{ is joint locally quasinilpotent at } x\}$$

Definition 1.2. Let $T = (T_1, \ldots, T_N)$ be a $N-$tuple of operators on a Hausdorff topological vector space X. Then we say that T is *joint weak locally quasinilpotent at* y_0 if

$$\lim_{n\to\infty} |f(T_{i_1} \cdots T_{i_n}(y_0))|^{1/n} = 0$$

for each $f \in X^*$ and each $i_j \in \{1, \ldots, N\};\ j \in \mathbb{N}$. We denote

$$wQ_T = \{x \in X : \ T \text{ is joint weak locally quasinilpotent at } x\}$$

During this paper we will denote $T = (T_1, \ldots, T_N)$ a $N-$tuple of not-necessarily commuting operators defined on a non-zero Hausdorff topological vector space X with Markushevich basis $\{(x_n, f_n)\}_n \subseteq X \times X^*$. That is, span$\{x_n : n \in \mathbb{N}\}$ is dense in X, $f_n(x_n) = 1$, $f_n(x_m) = 0$ for every $n \neq m$ and $\{f_n\}_n$ is separating points of X.

We will say that $x \in X$ is positive with respect to $\{(x_n, f_n)\}_n$ if $f_n(x) \geq 0$ for each $n \in \mathbb{N}$, we denote $0 \leq x$. Consequently, we will write $x \leq y$ if $0 \leq y - x$. An operator T on X is called positive if $T(x) \geq 0$ for all $x \geq 0$.

The above definition will allow us to obtain common invariant subspaces for a $N-$tuple $T = (T_1, \ldots, T_N)$ of non-zero positive operators which is joint weak locally quasinilpotent at a positive vector. The main result of this work is the following.

Theorem 1.3. *Let X be a Hausdorff topological vector space with a Markushevich basis (x_n, f_n) and $T = (T_1, \ldots, T_N)$ be a $N-$tuple of non-zero positive operators. If T is joint weak locally quasinilpotent at $y_0 > 0$, then $\{T_1, \ldots, T_N\}$ have a common non-trivial closed invariant subspace.*

Moreover, using this Theorem we deduce new results about non-trivial common invariant subspaces for $N-$tuples of operators positive operators (see Corollary 3.2, Theorem 3.3). We will conclude this article with a section including open problems and further directions.

2. Joint Weak Local Quasinilpotence

Firstly, let us see some results about the set wQ_T. They show that this set is a common invariant subspace for all the operators T_i and, if we consider the operators acting on a Banach space, the sets Q_T and wQ_T are the same.

Proposition 2.1. *Let $T = (T_1, \ldots, T_N)$ be an $N-$tuple of continuous linear operators on a Haussdorff topological vector space X, then the set wQ_T is a common invariant subspace for $\{T_1, \ldots, T_N\}$.*

Proof. It is not difficult to check that wQ_T is a vector subspace of X.

We fix $y_0 \in wQ_T$ and let us see that $T_k(y_0) \in wQ_T$ for each $k \in \{1, \ldots, N\}$. But, this is clear because

$$\lim_{n\to\infty} |f(T_{i_1} \cdots T_{i_n}(T_k(y_0))|^{\frac{1}{n}} \leq \lim_{n\to\infty} (|f(T_{i_1} \cdots T_{i_n} \cdot T_k)(y_0)|^{\frac{1}{n+1}})^{\frac{n+1}{n}} = 0.$$

for every $f \in X^*$, $i_j \in \{1, \ldots, N\}$ and $j \in \mathbb{N}$. Therefore, wQ_T is a common invariant subspace for $\{T_1, \ldots, T_N\}$. ∎

The problem of finding a common invariant subspace for the operators $T_1, ..., T_N$ has not been solved yet because we do not know if the space wQ_T is trivial, that is, it is different from $\{0\}$ and the whole space X.

Proposition 2.2. *Let $T = (T_1, \ldots, T_N)$ a $N-$tuple of operators on a Banach space X. Then $Q_T = wQ_T$.*

Proof. It is clear that $Q_T \subset wQ_T$. Let us suppose that there exists $x \in X$ such that

$$|f(T_{i_1} \cdots T_{i_n} x)|^{1/n} = 0 \text{ for each } f \in X^* \quad \text{and} \quad \lim_{n\to\infty} \|T_{i_1} \cdots T_{i_n} x\|^{1/n} \neq 0$$

We can suppose (taking a subsequence n_k if it was necessary) that there exist $0 < \varepsilon < 1$ and a natural number n_0 such that

$$|f(T_{i_1}\cdots T_{i_n}x)|^{1/n} < \varepsilon^2 < \varepsilon < \inf_n \|T_{i_1}\cdots T_{i_n}x\|^{1/n}$$

for each $n_0 \leq n$. Let us consider the sequence $x_n = \dfrac{T_{i_1}\cdots T_{i_n}x}{\varepsilon^n}$. Then $x_n \to 0$ and $\liminf \|x_n\| > 0$. By Bessaga-Pelczynsky selection principle (see [5]) there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) is a basis of $Y = \overline{\text{span}}\{x_{n_k} : k = 1, 2, \ldots\}$. Let $\{f_k\}_k$ be a uniformly bounded sequence in Y^* with $f_k(x_{n_i}) = \delta_{ki}$, where $\delta_{ii} = 1$ and $\delta_{ik} = 0$ if $i \neq k$. Let $F_k \in X^*$ be a bounded linear extension of f_k with the same norm. Let us define

$$F = \sum_{k=1}^{\infty} \frac{F_{n_k}}{2^{n_k}}.$$

Then $F \in X^*$ and

$$F(T_{i_1}\cdots T_{i_{n_k}}x) = \varepsilon^{n_k} F(x_{n_k}) = \frac{\varepsilon^{n_k}}{2^{n_k}} \Rightarrow |F(T_{i_1}\cdots T_{i_{n_k}}(x))|^{1/n_k} = \frac{\varepsilon}{2} \not\to 0.$$

This contradiction completes the proof. ∎

To finish this section we consider the following definition.

Definition 2.3. Let $T = (T_1, \ldots, T_N)$ be a $N-$tuple of operators on a Banach space X. We denote by T^n the collection of all possible products of n elements in T. Then we say that T is *uniform joint locally quasinilpotent at* y_0 if

$$\lim_{n\to\infty} \max_{S\in T^n} \|S(y_0)\|^{1/n} = 0.$$

We denote

$$UQ_T = \{x \in X : \ T \text{ isuniform joint locally quasinilpotent at } x\}.$$

Analogously, we will say that T, acting on a Hausdorff topological vector space X, is *uniform joint weak locally quasinilpotent at* y_0 if

$$\lim_{n\to\infty} \max_{S\in T^n} |f(S(y_0))|^{1/n} = 0.$$

for every $f \in X^*$.

We denote

$$wUQ_T = \{x \in X : \ T \text{ isuniform joint weak locally quasinilpotent at } x\}.$$

The preceding results are valid if we replace UQ_T by Q_T and wUQ_T by wQ_T.

The notion of uniform joint local quasinilpotence is closely related with the joint spectral radius defined by G. C. Rota and G. Strang [13]. It is possible to find more information about spectral theory for N- tuples of operators in the book [9].

3. Main Result

In this section we present the main result of this work, we obtain a common non-trivial closed invariant subspace for a N-tuple which is joint weak locally quasinilpotent at a positive vector. As a consequence, we obtain the same result for a N-tuple which is joint locally quasinilpotent at a positive vector.

Theorem 3.1. *Let X be a Hausdorff topological vector space with a Markushevich basis $\{(x_n, f_n)\}_n$ and $T = (T_1, \ldots, T_N)$ be a $N-$tuple of non-zero positive operators. If T is joint weak locally quasinilpotent at $y_0 > 0$, then $\{T_1, \ldots, T_N\}$ have a common non-trivial closed invariant subspace.*

Proof. Let us suppose that there exists x_k such that $T_i x_k = 0$ for all $i \in \{1, \ldots, N\}$. Then $\bigcap_{i=1}^{N} \ker(T_i)$ is a common non trivial invariant subspace for each $T_1, \ldots, T_N$. Thus, we can suppose that for every $k \in \mathbb{N}$ there exists $i(k) \in \{1, \ldots, N\}$ such that $T_{i(k)} x_k \neq 0$.

Since $y_0 > 0$ there exists $j \in \mathbb{N}$ such that $f_j(y_0) > 0$. Now, replacing (if it is necessary) y_0 by λy_0, for an appropriate scalar $\lambda > 0$, we can suppose that $f_j(y_0) > 1$. This implies that $y_0 - x_j \geq 0$. Indeed, if $i \neq j$ then $f_i(y_0 - x_j) = f_i(y_0) \geq 0$ and $f_j(y_0 - x_j) = f_j(y_0) - 1 > 0$. That is, $f_i(y_0 - x_j) \geq 0$ for each $i \in \mathbb{N}$.

Let us consider the projection operator P from X onto the vector subspace generated by x_j, defined by $P(x) = f_j(x) x_j$. We claim that

$$PT_{i_1} \cdots T_{i_m} x_j = 0 \tag{1}$$

for every $m > 0$ and $\{i_1, \ldots, i_m\} \subset \{1, \ldots, N\}$. To see this, we fix $m > 0$ and let us suppose $PT_{i_1} \cdots T_{i_m} x_j = \alpha x_j$ for some scalar $\alpha \geq 0$. Then, taking into account that $0 \leq P \leq I$, we have

$$0 \leq \alpha^n x_j \leq (PT_{i_1} \cdots T_{i_m})^n x_j \leq (T_{i_1} \cdots T_{i_m})^n x_j \leq (T_{i_1} \cdots T_{i_m})^n y_0$$

and, since $T = (T_1, \ldots, T_N)$ is joint weak locally quasinilpotent at y_0, we get

$$0 \leq \alpha \leq (f_j(T_{i_1} \ldots T_{i_m})^n y_0)^{1/n} = \left((f_j(T_{i_1} \ldots T_{i_m})^n y_0)^{\frac{1}{nm}}\right)^m \to 0.$$

Therefore, $\alpha = 0$ and condition (1) must be true.

Now let us consider the linear subspace Y of X generated by the set

$$\{T_{i_1} \ldots T_{i_m} x_j : \; m = 1, 2, \ldots; \; i_k \in \{1, \ldots, N\} \text{ for all } k \in \mathbb{N}\}.$$

Clearly, Y is invariant for each T_k; $k \in \{1, \ldots, N\}$ and, since $0 \neq T_{i(j)} x_j \in Y$ for some $i(j) \in \{1, \ldots, N\}$, we have $Y \neq \{0\}$. From (1) and $f_j(x_j) = 1$ we conclude that $x_j \notin \overline{Y}$. Hence $\overline{Y}$ is the required closed invariant subspace. ■

When X is a Banach space we have the following corollary, which proof can be easily deduced from the above theorem and Proposition 2.2.

Corollary 3.2. *Let $T = (T_1, \ldots, T_N)$ be a $N-$tuple of bounded positive operators on a Banach space X with a Markushevich basis (x_n, f_n). If T is joint locally quasinilpotent at $y_0 > 0$, then $\{T_1, \ldots, T_N\}$ have a common non trivial closed invariant subspace.*

Theorem 3.3. *Let X be a Banach space with a Markushevich basis $\{(x_n, f_n)\}_n$. Assume that the matrix $A_k = (a^k_{ij})_{i,j}$ defines a continuous positive operator T_k for all $k \in \{1, \ldots, N\}$, such that the $N-$tuple $T = (T_1, \ldots, T_N)$ is joint weak locally quasinilpotent at a non-zero positive vector y_0. Let $(w^k_{ij})_{i,j}$ be a matrix of complex numbers for every $k \in \{1, \ldots, N\}$. If the weighted matrix $B_k = (w^k_{ij} a^k_{ij})_{i,j}$ defines a continuous operator B_k for every $k \in \{1, \ldots, N\}$, then $B_1, \ldots, B_N$ have a common non-trivial closed invariant subspace.*

Proof. Arguing as in the proof of Theorem 3.1 we know that, for some $t \in \mathbb{N}$, $f_t(x_0) > 0$ and $x_0 - x_t \geq 0$. If we suppose there exists x_k such that $T_i x_k = 0$ for all $i \in \{1, \ldots, N\}$, and easy argument shows that $B_i x_k = 0$ for all $i \in \{1, \ldots, N\}$. Then $\bigcap_{i=1}^{N} \ker(B_i)$ is a common non trivial invariant subspace for each $B_1, \ldots, B_N$. Thus, we can suppose that for every $k \in \mathbb{N}$ there exists $i(k) \in \{1, \ldots, N\}$ such that $B_{i(k)} x_k \neq 0$. We also proved in Theorem 3.1 that $PT_{i_1} \cdots T_{i_m} x_t = 0$ for every $m > 0$ and $\{i_1, \ldots, i_m\} \subset \{1, \ldots, N\}$, where $P(x) = f_t(x) x_t$. Therefore, since

$$0 = PT_{i_1} \cdots T_{i_m} x_t = f_j(T_{i_1} \cdots T_{i_m} x_j) x_t,$$

we have $f_t(T_{i_1} \cdots T_{i_m} x_t) = 0$ for every $m > 0$ and $\{i_1, \ldots, i_m\} \subset \{1, \ldots, N\}$. In consequence, for every positive operator S acting on X such that $0 \leq S \leq T_{i_1} \cdots T_{i_m}$, for some $m > 0$ and $\{i_1, \ldots, i_m\} \subset \{1, \ldots, N\}$, we obtain

$$0 \leq f_t(Sx_t) \leq f_t(T_{i_1} \cdots T_{i_m} x_t) = 0. \tag{2}$$

Now, we consider the vector subspace Y generated by the set

$$\{Sx_t : 0 \leq S \leq T_{i_1} \cdots T_{i_m} \text{ for some } m > 0 \text{ and } \{i_1, \ldots, i_m\} \subset \{1, \ldots, N\}\}.$$

It is clear that Y is a invariant subspace for each operator R satisfying $0 \leq R \leq T_k$ for some $k \in \{1, \ldots, N\}$. From (2) it is followed that $f_t(y) = 0$ for every $y \in \overline{Y}$. As $T_{i(t)}x_t \neq 0$ and $f_t(x_t) \neq 0$, we obtain that $\overline{Y}$ is a non-trivial closed subspace of X.

We consider now, for every $k \in \{1, \ldots, N\}$, the operators A_{ij}^k defined by

$$A_{ij}^k(x_j) = a_{ij}^k x_j \quad \text{and} \quad A_{ij}^k(x_m) = 0 \ \text{ for } \ m \neq j.$$

Since A_{ij}^k satisfies $0 \leq A_{ij}^k \leq T_k$ for every $k \in \{1, \ldots, N\}$, it is followed that $\overline{Y}$ is invariant for all operators A_{ij}^k. Therefore, the vector subspace $\overline{Y}$ is invariant under the operators

$$B_n^k = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}^k A_{ij}^k$$

for every $n \in \mathbb{N}$ and $k \in \{1, \ldots, N\}$. Using now that the sequence $\{B_n^k\}_n$ converges in the strong operator topology to B_k for every $k \in \{1, \ldots, N\}$, we conclude that $\overline{Y}$ is a common non-trivial closed invariant subspace of $B_1, \ldots, B_N$. ■

4. Concluding Remarks and Open Problems

We have introduced several notions of joint local quasinilpotence and joint weak local quasinilpotence. It will be interesting to know the relations among them. Our conjecture is that the sets Q, UQ and wQ, wUQ are equal in the majority of the cases.

The results of our paper are true only for a finite number of operators, nevertheless, the joint local quasinilpotence can be defined for subsets of (not necessarily finite) operators. It would be interesting to extend these results to the case of an infinite number of operators.

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