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NEW TYPE SINGULAR OPERATORS ON PRODUCT SPACES

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Abstract. In this article we give sufficient conditions on kernels of singular integral operators on product spaces to be bounded on weighted L^p-spaces for $2 < p < \infty$. Applications include the weighted norm inequalities of holomorphic functional calculi of elliptic operators on product spaces.

1. Introduction

The purpose of this article is to give sufficient conditions on kernels of singular integral operators on product spaces to be bounded on weighted L^p-spaces for $2 < p < \infty$. For the basic facts about the classical singular integral operators on product domains, see, for example, [9, 10, 11, 12] and [13].

To begin with, let us recall some results of the one-parameter theory. Let T be a bounded linear operator on $L^2(\mathbb{R})$ with an associated kernel $k(x,y)$ in the sense that

$$(Tf)(x) = \int_{\mathbb{R}} k(x,y)f(y)dy$$

where $k(x,y)$ is a measurable function, and the above formula holds for each continuous function f with compact support, and for almost all x not in the support of f. One important result of Calderón-Zygmund operator theory is that T is bounded on $L^p(\mathbb{R})$ for $2 < p < \infty$ if there exist constants C and $c > 1$ so that the Hörmander integral condition on the kernel $k(x,y)$ holds, i.e.,

$$\int_{|x-y|\geq c|x_1-x|} |k(x,y) - k(x_1,y)|dy \leq C \tag{1.1}$$

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for all $x, x_1 \in \mathbb{R}$ ([18]). In [6], Duong and McIntosh can weaken the Hörmander integral condition and still conclude that T is bounded on $L^p(\mathbb{R})$ for all $2 < p < \infty$. Roughly speaking, assume that there exists a class of operators A_t with kernels $a_t(x, y)$, which play the role of approximation to the identity, so that the kernels $k_t(x, y)$ of the composite operators TA_t satisfy the condition

$$\int_{|x-y|\geq ct^{1/m}} |k(x,y) - k_t(x,y)|dy \leq C \tag{1.2}$$

for some constants m, c, C, uniformly in $x \in \mathbb{R}$ and $t > 0$, then T is bounded on $L^p(\mathbb{R})$ for $2 < p < \infty$. Moreover, it was proved in [7] that under the condition (1.2), T is also bounded from $L^\infty(\mathbb{R})$ to a space $\mathrm{BMO}_A(\mathbb{R})$.

The most important feature of the class of Duong and McIntosh is the uncertainty of the choice of the class $\{A_t\}_{t>0}$. For example, if T is a classical singular operator one can choose $a_t(x, y) = \frac{1}{t}\chi_{|x-y|<t}$. If $T = b(L)$, where $b \in H_\infty(S_{v+})$, and L is the elliptic operator with holomorphic functional calculi, then one may choose $A_t = e^{-tL}$. See [6] for more details.

A natural problem is weather results in [6] can be extended to the product spaces. This paper is devoted to solve this problem. Firstly we recall that the strong maximal operator M_s is defined by:

$$M_s(f)(x) = \sup_{x\in R} \frac{1}{|R|} \int_R |f(x)|dx$$

where the $\sup$ is taken over all rectangles R in $\mathbb{R}^2$ which contain x. In this paper, all rectangles' both sides must be parallel to the coordinate axes.

Given a function $f(x_1, x_2)$ on $\mathbb{R} \times \mathbb{R}$, and a rectangle R, in [11], R.Fefferman introduced the mean oscillation of f over R, $osc_R(f)$, by

$$osc_R(f) = \inf_{f_1,f_2} \Big(\frac{1}{|R|} \iint\limits_R |f(x_1,x_2) - f_1(x_1) - f_2(x_2)|^2 dx_1 dx_2\Big)^{\frac{1}{2}}$$

where the inf is taken over all pairs of functions f_1, f_2 depending only on the x_1 and x_2, respectively. Although in [3] L.Carleson disproved the fact that f is in the dual of $H^1(\mathbb{R} \times \mathbb{R})$ if and only if $osc_R(f) \leq c$, Fefferman proved that the mean oscillation over R can be used to obtain the boundedness of singular operators. To be precise, in [11], Fefferman proved:

(1) Suppose that T is a bounded linear operator on $L^2(\mathbb{R}^2)$. Suppose further that for any rectangle R in $\mathbb{R}^2$,

$$osc_R(Tf) \leq c\gamma^{-\delta}||f||_\infty$$

whenever f is an L^∞ function supported outside of γR for all $\gamma \geq 2$ and some fixed $\delta > 0$, where γR is the rectangle γ-fold dilation of R. Then T maps $L^\infty(\mathbb{R}^2)$ boundedly into $BMO(\mathbb{R} \times \mathbb{R})$.

(2) Suppose $T^{\#}$ is an operator defined on positive locally square integrable functions. $T^{\#}$ is monotone, i.e., $T^{\#}f(x) \leq T^{\#}g(x)$ when $f(x) \leq g(x)$ for all $x \in \mathbb{R}^2$. Suppose further that for any function f supported outside of γR,

$$osc_R(Tf) \leq \gamma^{-\delta} T^{\#} f(x), \text{ for all } x \in R,$$

where $\delta > 0$. Then $T^{\#}$ is called a sharp operator of T. Fefferman proved the following result:

$$\iint_{\mathbb{R}^2} S(Tf)(x)^2 \phi(x) dx \leq c \iint_{\mathbb{R}^2} (I + T^{\#})(|f|)(x)^2 M(\phi)(x) dx.$$

As a corollary, Fefferman also proved: if T is a bounded linear operator on $L^2(\mathbb{R}^2)$ whose sharp operator is $T^{\#} = M_s(f^2)^{\frac{1}{2}}$, then for $p > 2$,

$$\iint_{\mathbb{R}^2} |Tf|^p(x)\omega(x)dx \leq c \int_{\mathbb{R}^2} |f(x)|^p \omega(x) dx$$

whenever $\omega \in A^{p/2}(\mathbb{R} \times \mathbb{R})$. (A positive function $\omega(x_1, x_2)$ is said to be in $A^r(\mathbb{R} \times \mathbb{R})$ if only if $\omega(\cdot, x_2) \in A^r(\mathbb{R})$ with A^r norm bounded independently of x_2, and $\omega(x_1, \cdot) \in A^r(\mathbb{R})$ with A^r norm bounded independently of x_1.)

In this paper, we will extend the above results to more general setting of singular operators on product spaces. In order to simplify, we always denote $t = (t_1, t_2)$, $t^m = (t_1^m, t_2^m)$, $x = (x_1, x_2)$ and $\frac{dxdt}{t}$ means $\frac{dx_1 dx_2 dt_1 dt_2}{t_1 t_2}$.

The following two theorems are our main theorems in this paper:

Theorem 1.1. *Let T be a bounded linear operator on $L^2(\mathbb{R}^2)$. The sharp operator $T^{\#}$ is monotone and for any functions f supported outside of γR, the γ-fold dilation of R,*

$$C_R(Tf) \leq c\gamma^{-\delta} T^{\#} f(x)$$

for all $x \in R$, where $\delta > 0$,

$$C_R(f) = \Big(\frac{1}{|R|} \iint\limits_R \int_0^{|I|} \int_0^{|J|} |\psi_{t^m}(f)(x)|^2 \frac{dxdt}{t}\Big)^{\frac{1}{2}}.$$

(see Section 2 below for the exact definition of ψ_{t^m}) Then

(1) *We have the following a priori duality estimate:*

$$\iint_{\mathbb{R}^2} S(Tf)(x)^2 \phi(x) dx \leq c \iint_{\mathbb{R}^2} (I + T^{\#})(|f|)(x)^2 M(\phi)(x) dx$$

where I denotes the identity operator and $M(\phi) = M_s(M_s(M_s(M_s(\phi))))$. (*see Section 2 below for the exact definition of S function*)

(2) *If $T^{\#}$ is bounded from $L^\infty(\mathbb{R}^2)$ to $L^\infty(\mathbb{R}^2)$, then T is bounded from $L^\infty \cap L^2$ to $BMO_A \cap L^2$(see Section 2 for the definition) with estimate:*

$$||Tf||_{BMO_A} \leq c||f||_\infty.$$

The following theorem can be viewed as an extension of Duong and McIntosh's results to product spaces.

Theorem 1.2. *Let T be a bounded linear operator on $L^2(\mathbb{R}^2)$, which satisfies the following conditions:*

$$\int_{|x_1-y_1|\geq 2^k t_1} ||k^{(1)}_{t_1}(x_1,y_1)||^2_{L^2\to L^2} dy_1 \leq c\frac{t_1^{2\delta}}{(2^k t_1)^{1+2\delta}}, \tag{1.3}$$

$$\int_{|x_2-y_2|\geq 2^k t_2} ||k^{(2)}_{t_2}(x_2,y_2)||^2_{L^2\to L^2} dy_2 \leq c\frac{t_2^{2\delta}}{(2^k t_2)^{1+2\delta}}, \tag{1.4}$$

$$\iint_{\substack{|x_1-y_1|\geq 2^k t_1\\ |x_2-y_2|\geq 2^l t_2}} |k^{(3)}_t(x,y)|^2 dy \leq c\frac{t_1^{2\delta}}{(2^k t_1)^{1+2\delta}}\frac{t_2^{2\delta}}{(2^l t_2)^{1+2\delta}}, \tag{1.5}$$

(here $\delta > 0$ is given)for all $k, l \geq 2$, where $k^{(1)}_{t_1}(x_1,y_1)$ and $k^{(2)}_{t_2}(x_2,y_2)$ are operators satisfying:

$$k^{(1)}_{t_1}(x_1,y_1)g(z) = \iint_{\mathbb{R}^2} \psi_{t_1^m}(x_1,z_1)k(z_1,y_1,z,y_2)g(y_2)dz_1dy_2$$

$$k^{(2)}_{t_2}(x_2,y_2)g(z) = \iint_{\mathbb{R}^2} \psi_{t_2^m}(x_2,z_2)k(z,y_1,z_2,y_2)g(y_1)dz_2dy_1$$

and the function

$$k^{(3)}_t(x,y) = \iint_{\mathbb{R}^2} \psi_{t_1^m}(x_1,z_1)\psi_{t_2^m}(x_2,z_2)k(z_1,y_1,z_2,y_2)dz_1dz_2.$$

Then the operator T satisfies all the assumptions of Theorem 1.1, with $T^{\#}f = M_s(|f|^2)^{\frac{1}{2}}$. As a consequence, for $p > 2$,

$$\iint_{\mathbb{R}^2} |Tf|^p(x)\omega(x)dx \leq c\int_{\mathbb{R}^2} |f(x)|^p\omega(x)dx$$

whenever $\omega \in A^{p/2}(\mathbb{R}\times\mathbb{R})$. Moreover,

$$||Tf||_{BMO_A} \leq c||f||_\infty.$$

The paper is organized as follows. In section 2, we give some assumptions and introduce the Journé's covering lemma. In section 3, we give the proof of the main theorems. In section 4, we show that the C class of singular integral operators defined by Fefferman is contained in our cases. We note that the classical singular operators whose kernels satisfy the pointwise estimate must belong to the C class. In section 5, we prove that the functional calculus of elliptic operators also satisfy the conditions given in the above two theorems. We remark that operators given in section 5 may not be handled by classical singular integral operators theory.

2. Assumptions and Preliminaries

Assumptions suppose that T is a bounded linear operator from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$, and its kernel is $k(x_1, y_1, x_2, y_2)$. There is a class of integral operators A_t, defined on $L^2_{loc}(\mathbb{R})$ initially, which plays the role of approximations to the identity. Also we assume that operators A_t can be represented by kernels $a_t(x, y)$ in the sense that

$$A_t u(x) = \int_{\mathbb{R}} a_t(x, y)u(y)dy$$

for every function $u \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, and the kernel $a_t(x, y)$ satisfies the following conditions:

$$|a_t(x, y)| \leq h_t(x, y) \tag{2.1}$$

for all $x, y \in \mathbb{R}$, where $h_t(x, y)$ is a function satisfying

$$h_t(x, y) = \frac{1}{2t^{\frac{1}{m}}} s\Big(\frac{|x-y|}{t^{\frac{1}{m}}}\Big) \tag{2.2}$$

for some $m > 0$ and s is a positive, bounded, decreasing function satisfying: $\lim\limits_{r\to\infty} r^{1+\epsilon}s(r) = 0$.

In one-parameter case, we denote $\psi_t = A_t(I - A_t)$ and $S(f)(x) = (\iint_{\Gamma(x)} |\psi_{t^m}(f)(y)|^2 \frac{dydt}{t^2})^{\frac{1}{2}}$, where $\Gamma(x) = \{(y, t)| \in \mathbb{R}^2_+, |y - x| < t\}$. we assume that

$$c_1||f||_{L^2} \leq ||S(f)||_{L^2} \leq c_2||f||_{L^2} \tag{2.3}$$

for all $f \in L^2(\mathbb{R})$.

We now consider two-parameter case. For any $t = (t_1, t_2)$ and a function $f(x)$ defined on $\mathbb{R}^2$, we let $\psi_t = \psi_{t_1} \otimes \psi_{t_2}$, where $\psi_{t_i} = A_{t_i}(I - A_{t_i})$, $i = 1, 2$. Also we let $S(f)(x) = (\iint_{\Gamma(x)} |\psi_{t^m}(f)(y)|^2 \frac{dydt}{t_1^2 t_2^2})^{\frac{1}{2}}$, where $\Gamma(x) = \{(y, t)| \in \mathbb{R}^2_+ \times \mathbb{R}^2_+, |y_1 - x_1| < t_1, |y_2 - x_2| < t_2\}$. We assume that:

$$c_1||f||_{L^2} \leq ||S(f)||_{L^2} \leq c_2||f||_{L^2} \tag{2.4}$$

for all $f \in L^2(\mathbb{R}^2)$.

We remark that there are many choices of $\{A_t\}$ that satisfy the above assumptions(see section 4 and section 5 for example), but ψ_t may not satisfy the cancellation condition(i.e. $\int \psi_t(f)(x)dx = 0,\ \forall f \in L^2$).

Definition 2.1. We define the bounded mean oscillation space $BMO_A \cap L^2$ by

$$BMO_A \cap L^2 = \{f| \in L^2(\mathbb{R}^2),\ \tfrac{1}{|\Omega|} \iint\limits_{\widehat{\Omega}} |\psi_{t^m}(f)(x)|^2 \tfrac{dxdt}{t_1 t_2} < \infty,$$
$$\text{for all open set } \ \Omega \subset \mathbb{R}^2\}$$

where $\widehat{\Omega} = \{(x,t)|R(x,t) \subset \Omega\}$, here $R(x,t)$ is a rectangle in $\mathbb{R}^2$ with center at x, and sidelengths t_1 and t_2, respectively.

2.1. Journé's covering lemma

To show the Theorem 1.1, we recall the Journé's covering lemma ([12, 13]) and its extension ([11]).

Lemma 2.2. *For any open set $\Omega \subset \mathbb{R}^2$, let $m(\Omega)$ denote the collection of all maximal dyadic subrectangles in Ω, Similarly, let $m_1(\Omega)$and $m_2(\Omega)$ denote the families of dyadic subrectangles in Ω, which are maximal in the x_1 and x_2 directions, respectively. Given a rectangle $R = I \times J \in m_2(\Omega)$, let $\hat{I}$ be the largest dyadic interval containing I, and such that $\hat{I} \times J \subset \widetilde{\Omega}$, where $\widetilde{\Omega} = \{x| \in \mathbb{R}^2, M_s(\chi_\Omega)(x) > \frac{1}{2}\}$. Define $\gamma_1(R) = \frac{|\hat{I}|}{|I|}$, then we have the following inequality:*

$$\sum_{R \in m_2(\Omega)} |R|\gamma_1(R)^{-\delta} \le c_\delta |\Omega|,$$

moreover,

$$\sum_{R \in m_2(\Omega)} |R|\gamma_1(R)^{-\delta} \inf_{x \in R} |f(x)| \le c_\delta \int\limits_{\Omega} |f(x)|dx.$$

3. Proof of the Main Theorems

3.1. Proof of Theorem 1.1

Basically, the proof of Theorem 1.1 is similar to the proof of Theorem 1 in [11]. We, however, remark that the operators$\{\psi_t\}$ in the Theorem 1.1 may not satisfy the cancellation condition, which is needed in the proof of Theorem 1 in [11]. We let $O_k = \{x| \in \mathbb{R}^2, M(\phi)(x) > 2^k\}$, $\mathcal{B}_k = \{R|$ are a dyadic rectangles, $|R \cap O_k| > \frac{1}{2}|R|$ and $|R \cap O_{k+1}| \le \frac{1}{2}|R|$ $\}$, and $R_+ = \{(y,t)| \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ : y \in R, \frac{1}{2}|I| < t_1 \le |I|, \frac{1}{2}|J| < t_2 \le |J|\}$.

Then

$$
\begin{aligned}
&\iint_{\mathbb{R}^2} |S(Tf)(x)|^2 \phi(x)dx \\
\leq &\iint_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} |\psi_{t^m}(Tf)(y)|^2 \frac{1}{t} \iint_{|y-x|<t} \phi(x)dx \frac{dydt}{t} \\
\leq &\sum_k \sum_{R\in \mathcal{B}_k} 2^{k+1} \iint_{R_+} |\psi_{t^m}(Tf)(y)|^2 \frac{dydt}{t}.
\end{aligned}
$$

The last inequality is based on the fact that $\frac{1}{t}\iint_{|y-x|<t} \phi(x)dx \leq \inf\limits_{y\in R} M_s(\phi)(y)$ and $|R \cap O_{k+1}| \leq \frac{1}{2}|R|$.

We claim that:

$$
\sum_{R\in \mathcal{B}_k} \iint_{R_+} |\psi_{t^m}(Tf)(y)|^2 \frac{dydt}{t} \leq c \iint_{\widetilde{\widetilde{\widetilde{O}}}_k} (I + T^{\#})(|f|)(x)^2 dx,
$$

then it is easy to see that the theorem's part(1) follows readily.

To show the claim above, we decompose f into two parts: $f = f\chi_{\widetilde{\widetilde{\widetilde{O}}}_k} + f\chi_{\widetilde{\widetilde{\widetilde{O}}}_k^c} = f^0 + f^1$, then

$$
\begin{aligned}
\sum_{R\in \mathcal{B}_k} \iint_{R_+} |\psi_{t^m}(Tf^0)(y)|^2 \frac{dydt}{t} &\leq \iint_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} |\psi_{t^m}(Tf^0)(y)|^2 \frac{dydt}{t} \\
&\leq c||f^0||_2^2 \\
&= \iint_{\widetilde{\widetilde{\widetilde{O}}}_k} |f(x)|^2 dx.
\end{aligned}
$$

In order to estimate f^1, we further decompose it into two parts. Firstly, we note that $R \in \mathcal{B}_k$ implies $R \subseteq \widetilde{O}_k$. Let $\hat{I}$ be the maximal dyadic interval, containing I, such that $\hat{I} \times J \subset \widetilde{\widetilde{O}}_k$, and denote $\widetilde{R} = \hat{I} \times J$. Also let $\hat{J}$ be the maximal dyadic interval, containing J, such that $\hat{I} \times \hat{J} \subset \widetilde{\widetilde{\widetilde{O}}}_k$. Write $\gamma_1(R) = \frac{|\hat{I}|}{I}$, $\gamma_2(\widetilde{R}) = \frac{|\hat{J}|}{J}$. Let $f_0^1 = f^1\chi_{\hat{I}^c \times \mathbb{R}}$, and $f_1^1 = f^1 \chi_{\hat{I}\times \hat{J}^c}$, then

$$
\begin{aligned}
\sum_{R\in \mathcal{B}_k} \int_{R_+} |\psi_{t^m}(Tf_0^1)(y)|^2 \frac{dydt}{t} &\leq \sum_{R\in m(\widetilde{O}_k)} \iint_R \int_0^{|I|} \int_0^{|J|} |\psi_{t^m}(Tf_0^1)(y)|^2 \frac{dydt}{t} \\
&\leq c \sum_{R\in m(\widetilde{O}_k)} \gamma_1(R)^{-2\delta} |R| \inf_{x\in R} T^{\#} f(x)^2 \\
&\leq c \iint_{\widetilde{O}_k} T^{\#}(|f|)(x)^2 dx.
\end{aligned}
$$

The last inequality uses the Lemma 2.2. Also we have

$$\begin{aligned}
&\sum_{R\in\mathcal{B}_k}\int_{R_+}|\psi_{t^m}(Tf_1^1)(y)|^2\frac{dydt}{t}\\
&\leq \sum_{S\in m_1(\tilde{\tilde{O}}_k)}\sum_{\substack{R\in m(\tilde{O}_k)\\ \tilde{R}=S}}\iint_{R=I\times J}\int_0^{|I|}\int_0^{|J|}|\psi_{t^m}(Tf_1^1)(y)|^2\frac{dydt}{t}\\
&\leq \sum_{S\in m_1(\tilde{\tilde{O}}_k)}\iint_{S=I'\times J'}\int_0^{|I'|}\int_0^{|J'|}|\psi_{t^m}(Tf_1^1)(y)|^2\frac{dydt}{t}\\
&\leq c\sum_{S\in m_1(\tilde{\tilde{O}}_k)}\gamma_2(S)^{-2\delta}|S|\inf_{x\in S}T^{\#}f(x)^2\\
&\leq c\iint_{\tilde{\tilde{O}}_k}T^{\#}(|f|)(x)^2dx.
\end{aligned}$$

This completes the proof of part (1).

Now we turn to the proof of part(2). For any open set $\Omega\in\mathbb{R}^2$, we decompose f into two parts f^0 an f^1, such that $f^0=f\chi_{\tilde{\tilde{\Omega}}}$, $f=f^0+f^1$. Using the L^2 boundedness of S function and the L^2 boundedness of T, we have:

$$\frac{1}{|\Omega|}\iint_{\widehat{\Omega}}|\psi_{t^m}(Tf^0)(x)|^2dx\leq c\frac{1}{|\Omega|}||f^0||_2^2\leq c||f||_\infty.$$

Now we turn to estimate f^1. For any maximal dyadic rectangle $R=I\times J\subset\widetilde{\Omega}$, let $\hat{I}$ be the maximal dyadic interval, containing I, such that $\hat{I}\times J\subset\tilde{\tilde{\Omega}}$. Also let $\hat{J}$ be the maximal dyadic interval, containing J, such that $\hat{I}\times\hat{J}\subset\tilde{\tilde{\tilde{\Omega}}}$. Write $\gamma_1(R)=\frac{|\hat{I}|}{I}$, $\gamma_2(R)=\frac{|\hat{J}|}{J}$. Let $f_0^1=f^1\chi_{\hat{I}^c\times\mathbb{R}}$, and $f_1^1=f^1\chi_{\hat{I}\times\hat{J}^c}$. Then $f^1=f_0^1+f_1^1$, and

$$\begin{aligned}
&\frac{1}{|\Omega|}\iint_{\widehat{\Omega}}|\psi_{t^m}(Tf^1)(x)|^2\frac{dxdt}{t}\\
&\leq \frac{c}{|\Omega|}\Bigg(\sum_{R\in m(\widetilde{\Omega})}\iint_R\int_0^{|I|}\int_0^{|J|}|\psi_{t^m}(Tf_0^1)(x)|^2\frac{dxdt}{t}+\\
&\sum_{R\in m(\widetilde{\Omega})}\iint_R\int_0^{|I|}\int_0^{|J|}|\psi_{t^m}(Tf_1^1)(x)|^2\frac{dxdt}{t}\Bigg)\\
&\leq \frac{c}{|\Omega|}\Big(\sum_{R\in m(\widetilde{\Omega})}\gamma_1(R)^{-2\delta}|R|+\sum_{R\in m(\widetilde{\Omega})}\gamma_2(R)^{-2\delta}|R|\Big)||f||_\infty^2\\
&\leq c||f||_\infty^2.
\end{aligned}$$

The second inequality above uses the assumption on $T^{\#}$, and the third inequality uses the Journé's covering lemma. This completes the proof of Theorem 1.1.

Similar to [11], as a consequence of Theorem 1.1, we have

Corollary 2.3. *If T is a bounded linear operator $L^2(\mathbb{R}^2)$ whose sharp operator is $T^{\#}f = M_s(f^2)^{\frac{1}{2}}$, then*

(1) *for $p > 2$,*

$$\iint |Tf|^p(x)\omega(x)dx \le c\int_{\mathbb{R}^2} |f(x)|^p\omega(x)dx$$

whenever $\omega \in A^{p/2}(\mathbb{R}\times\mathbb{R})$.

(2) *T is bounded from $L^2 \cap L^\infty$ to $BMO_A \cap L^2$.*

See the proof of the corollary of Theorem 1 in [11].

Proof of Theorem 1.2. Let $R = I \times J$ be a rectangle in $\mathbb{R}^2$, and $x^0 \in R$. Without loss of generality we suppose that the center of R is the origin. We just need to show that when supp$f \subseteq (\gamma I)^c \times \mathbb{R}$, we have

$$\frac{1}{|R|}\iint\limits_R \int_0^{|I|}\int_0^{|J|} |\psi_{t^m}(Tf)(x)|^2\frac{dxdt}{t} \le c\gamma^{-2\delta}M_s(|f|^2)(x^0). \tag{3.1}$$

(since we can decompose f into two parts f_1, f_2, such that $f = f_1 + f_2$ with $f_1 = f\chi_{(\gamma I)^c\times\mathbb{R}}$, $f_2 = f\chi_{\mathbb{R}\times(\gamma J)^c}$)

Let $f = f\chi_{(\gamma I)^c\times(2J)} + f\chi_{(\gamma I)^c\times(2J)^c} = f_1 + f_2$, and apply S function's L^2 estimates, Minkoski inequality, then we have

$$\begin{aligned}
&\frac{1}{|R|}\iint\limits_R \int_0^{|I|}\int_0^{|J|} |\psi_{t^m}(Tf_1)(x)|^2\frac{dxdt}{t}\\
\le\ &\frac{c}{|R|}\int_I\int_0^{|I|}\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}[k_{t_1}^{(1)}(x_1,y_1)f_1(y_1,\cdot)](x_2)dy_1\Big|^2 dx_2\frac{dx_1dt_1}{t_1}\\
\le\ &\frac{c}{|R|}\int_I\int_0^{|I|}\left(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|[k_{t_1}^{(1)}(x_1,y_1)f_1(y_1,\cdot)](x_2)|^2dx_2\Big)^{\frac{1}{2}}dy_1\right)^2\frac{dx_1dt_1}{t_1}\\
\le\ &\frac{c}{|R|}\int_I\int_0^{|I|}\Bigg\{\sum_{k:\ 2^k>\gamma}\Big(\int\limits_{|y_1|\sim 2^k|I|}||k_{t_1}^{(1)}(x_1,y_1)||^2_{L^2\to L^2}dy_1\Big)^{\frac{1}{2}}\\
&\qquad\Big(\int\limits_{|y_1|\sim 2^k|I|}|f_1(y_1,x_2)|^2dx_2dy_1\Big)^{\frac{1}{2}}\Bigg\}^2\frac{dx_1dt_1}{t_1}
\end{aligned}$$

$$\begin{aligned} &\leq \ c\int_I\int_0^{|I|}\Big\{\sum_{k:\ 2^k>\gamma}\frac{t_1^\delta}{(2^k|I|)^{\frac12+\delta}}2^{\frac12 k}M_s(|f|^2)(x^0)^{\frac12}\Big\}^2\frac{dx_1dt_1}{t_1} \\ &\leq \ c\gamma^{-2\delta}M_s(|f|^2)(x^0). \end{aligned}$$

Similarly,

$$\begin{aligned} &\frac{1}{|R|}\iint_R\int_0^{|I|}\int_0^{|J|}|\psi_{t^m}(Tf_2)(x)|^2\frac{dxdt}{t} \\ &\leq \frac{1}{|R|}\iint_R\int_0^{|I|}\int_0^{|J|}\Bigg\{\sum_{\substack{k:\ 2^k>\gamma\\ l:\ l\geq 1}}\Big(\iint_{\substack{|y_1|\sim 2^k|I|\\ |y_2|\sim 2^l|J|}}|k_t^{(3)}(x_1,y_1,x_2,y_2)|^2dy_1dy_2\Big)^{\frac12} \\ &\quad \Big(\int_{\substack{|y_1|\sim 2^k|I|\\ |y_2|\sim 2^l|J|}}|f_1(y_1,y_2)|^2dy_2dy_1\Big)^{\frac12}\Bigg\}^2\frac{dxdt}{t} \\ &\leq c\iint_R\int_0^{|I|}\int_0^{|J|}\Bigg\{\sum_{\substack{k:\ 2^k>\gamma\\ l:\ l\geq 1}}\frac{t_1^\delta}{(2^k|I|)^{\frac12+\delta}}\frac{t_2^\delta}{(2^l|J|)^{\frac12+\delta}}2^{\frac12 k}2^{\frac12 l}M_s(|f|^2)(x^0)^{\frac12}\Bigg\}^2\frac{dxdt}{t} \\ &\leq c\gamma^{-2\delta}M_s(|f|^2)(x^0). \end{aligned}$$

This completes the proof of Theorem 1.2.

4. Example 1: The C Class Singular Integral Operators of Fefferman

In this section, we will verify that the singular integral operators of C class that defined initially by R.Fefferman satisfy all assumptions of Theorem 1.2.

Firstly, let us recall the definition of the singular integral operators of C class. If T is a bounded linear operator on $L^2(\mathbb{R}^2)$, with kernel $k(x_1,y_1,x_2,y_2)$, in the sense that

$$Tf(x_1,y_1)=\iint_{\mathbb{R}^2}k(x_1,y_1,x_2,y_2)f(y_1,y_2)dy_1dy_2$$

whenever $(x_1,x_2)\notin \text{supp}f$.

We define $k^{(1)}(x_1,y_1)$ to be the integral operator on $\mathbb{R}$ whose kernel is $k^{(1)}(x_1,y_1)(x_2,y_2)=k(x_1,y_1,x_2,y_2)$. Define $k^{(2)}$ similarly with the kernel $k^{(2)}(x_2,y_2)(x_1,y_1)=k(x_1,y_1,x_2,y_2)$. Suppose $T_hf(x)=\int_{\mathbb{R}}h(x,y)f(y)dy$ is bounded on $L^2(\mathbb{R})$. We define $|T_h|_{CZ}=||T_h||_{L^2\to L^2}+\inf C$ where the inf is taken over all $C>0$ so that

$$\left(\int_{|x-y|>\gamma|x-x'|}|h(x,y)-h(x',y)|^2dy\right)^{\frac12}\leq C\gamma^{-\frac12-\delta}|x-x'|^{-\frac12}$$

(here $\delta > 0$ is given) for all $\gamma \geq 2$. We say that T is in the C class if

$$(4.1)\quad \left(\int_{|x_1-y_1|>\gamma|x_1-y_1'|} |k^{(1)}(x_1,y_1)-k^{(1)}(x_1',y_1)|_{CZ}^2 dy_1\right)^{\frac{1}{2}} \leq c\gamma^{-\frac{1}{2}-\delta}|x_1-x_1'|^{-\frac{1}{2}}$$

and similarly,

$$(4.2)\quad \left(\int_{|x_2-y_2|>\gamma|x_2-y_2'|} |k^{(2)}(x_2,y_2)-k^{(2)}(x_2',y_2)|_{CZ}^2 dy_2\right)^{\frac{1}{2}} \leq c\gamma^{-\frac{1}{2}-\delta}|x_2-x_2'|^{-\frac{1}{2}}$$

for all $\gamma \geq 2$.

In order to prove all operators in the C class satisfy our assumptions in Theorem 1.2, we should choose a suitable class of operators $\{A_t\}_{t>0}$. To do this, let $\eta(x) \in C_0^\infty(\mathbb{R})$, with supp$\eta \subset [-1,1]$ and $\int \eta(x)dx = 1$. We set $A_t(f)(x) = f * \eta_t(x)$, where $\eta_t(x) = \frac{1}{t}\eta(\frac{x}{t})$. Then for any $t_1 > 0$, $\psi_{t_1}f(x) = \psi_{t_1} * f(x)$, where $\psi(x) = \eta(x)-\eta(x)*\eta(x)$, and hence $\psi(x) \in C_0^\infty(\mathbb{R})$, and $\int \psi(x)dx = 0$. This implies that the S function defined in (2.3) is the classical S function. Therefore the assumptions (2.1), (2.3) and (2.4) hold with $m = 1$.

Theorem 4.1. *The operators in the C class satisfy the hypothesis of Theorem 1.2.*

Proof. We firstly verify the condition(1.3). If $|x_1 - y_1| \geq 2^k t_1$, where $k \geq 2$, then

$$\begin{aligned} k_{t_1}^{(1)}(x_1,y_1)g(z) &= \iint_{\mathbb{R}^2} \psi_{t_1}(x_1,z_1)k(z_1,y_1,z,y_2)g(y_2)dz_1dy_2 \\ &= \iint_{\mathbb{R}^2} \psi_{t_1}(x_1,z_1)[k(z_1,y_1,z,y_2)-k(x_1,y_1,z,y_2)]g(y_2)dz_1dy_2 \end{aligned}$$

by the cancellation condition of ψ. And thus

$$||k_{t_1}^{(1)}(x_1,y_1)||_{L^2\to L^2} \leq c\int_{|x_1-z_1|\leq 2t_1} \frac{1}{t_1}|k^{(1)}(z_1,y_1)-k^{(1)}(x_1,y_1)|_{CZ}dz_1.$$

This gives:

$$\begin{aligned} &\int_{|x_1-y_1|\geq 2^k t_1} ||k_{t_1}^{(1)}(x_1,y_1)||_{L^2\to L^2}^2 dy_1 \\ &\leq c\int_{|x_1-y_1|\geq 2^k t_1} \frac{1}{t_1}\int_{|x_1-z_1|\leq 2t_1} |k^{(1)}(z_1,y_1)-k^{(1)}(x_1,y_1)|_{CZ}^2 dz_1dy_1 \\ &\leq c\int_{|x_1-z_1|\leq 2t_1} \frac{1}{t_1}\Big(\frac{2^k t_1}{|x_1-z_1|}\Big)^{-1-2\delta}|x_1-z_1|^{-1}dz_1 \\ &\leq c\frac{t_1^{2\delta}}{(2^k t_1)^{1+2\delta}}. \end{aligned}$$

The second inequality above uses the assumption(4.1). The proof of the condition(1.4) is similar.

Finally, we turn to verify the condition(1.5). If $|x_1-y_1| \geq 2^k t_1$ and $|x_2-y_2| \geq 2^l t_2$, then

$$\begin{aligned} k_t^{(3)}(x,y) &= \iint_{\mathbb{R}^2} \psi_{t_1}(x_1,z_1)\psi_{t_2}(x_2,z_2)k(z_1,y_1,z_2,y_2)dz_1dz_2 \\ &= \iint_{\mathbb{R}^2} \psi_{t_1}(x_1,z_1)\psi_{t_2}(x_2,z_2)\Big\{[k(z_1,y_1,z_2,y_2) \\ &\quad -k(x_1,y_1,z_2,y_2)] - [k(z_1,y_1,x_2,y_2) - k(x_1,y_1,x_2,y_2)]\Big\}dz_1dz_2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\iint_{\substack{|x_1-y_1|\geq 2^k t_1 \\ |x_2-y_2|\geq 2^l t_2}} |k_t^{(3)}(x,y)|^2 dy \\ &\leq c \iint_{\substack{|x_1-y_1|\geq 2^k t_1 \\ |x_2-y_2|\geq 2^l t_2}} \iint_{\substack{|x_1-z_1|\leq 2t_1 \\ |x_2-z_2|\leq 2t_2}} \frac{1}{t_1t_2}\Big|\big[k(z_1,y_1,z_2,y_2)-k(x_1,y_1,z_2,y_2)\big] \\ &\quad -\big[k(z_1,y_1,x_2,y_2)-k(x_1,y_1,x_2,y_2)\big]\Big|^2 dz_1dz_2dy_1dy_2 \\ &\leq \frac{c}{t_1t_2} \iint_{\substack{|x_1-z_1|\leq 2t_1 \\ |x_2-z_2|\leq 2t_2}} \int_{|x_1-y_1|\geq 2^k t_1} |k^{(1)}(z_1,y_1) \\ &\quad -k^{(1)}(x_1,y_1)|_{CZ}^2 \Big(\frac{2^l t_2}{|x_2-z_2|}\Big)^{-1-2\delta} |x_2-z_2|^{-1} dy_1dz_1dz_2 \\ &\leq \frac{c}{t_1t_2} \iint_{\substack{|x_1-z_1|\leq 2t_1 \\ |x_2-z_2|\leq 2t_2}} \Big(\frac{2^k t_1}{|x_1-z_1|}\Big)^{-1-2\delta} |x_1-z_1|^{-1} \\ &\quad \Big(\frac{2^l t_2}{|x_2-z_2|}\Big)^{-1-2\delta} |x_2-z_2|^{-1} dz_1dz_2 \\ &\leq c\frac{t_1^{2\delta}}{(2^k t_1)^{1+2\delta}} \frac{t_2^{2\delta}}{(2^l t_2)^{1+2\delta}}. \end{aligned}$$

This completes the proof of Theorem 4.1.

5. Example 2: Holomorphic Functional Calculi of Elliptic Operators

In this section, we will give another example of operators that satisfy all the assumptions of Theorem 1.2, which may not be contained in the C class(see section 4 above and [11]). We firstly give some preliminary definitions for introducing holomorphic functional calculi of operators. (For more details see [1, 4] and [14])

For $0 \leq \omega < \mu < \pi$, define the closed and open sectors in the (extended)complex plane:

$$S_\omega = \{z| \in \mathbb{C} : |\arg(z)| \leq \omega\} \cup \{0, \infty\}, \quad S_{\mu+} = \{z| \in \mathbb{C} : |\arg(z)| < \mu\}.$$

Also we let $S^0_{\mu+} = \{z| \in S_{\mu+}, z \neq 0\}$. Denote by $H(S_{\mu+})$ the space of all holomorphic functions on $S_{\mu+}$ and by $H_\infty(S_{\mu+}) = \{b \in H(S_{\mu+}) : ||b||_\infty < \infty\}$ where $||b||_\infty = \sup\{|b(z)| : z \in S_{\mu+}\}$. A closed operator L in $L^2(\mathbb{R})$ is said to be type ω if its spectrum $\sigma(L) \subset S_\omega$, and for each $\mu > \omega$, there exists a constant c_μ such that

$$||(\zeta I - L)^{-1}||_{L^2 \to L^2} \leq c_\mu |\zeta|^{-1}, \quad \zeta \notin S_\mu.$$

In what following, we assume that L is a one-one linear operator of type ω on $L^2(\mathbb{R})$ with $\omega < \pi/2$, and hence L generates a holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \frac{\pi}{2} - \omega$. Also we assume the following two conditions:

(1) The holomorphic semigroup e^{-zL}, $|\arg(z)| < \frac{\pi}{2} - \omega$, is represented by kernels $a_z(x, y)$ which satisfy, for all $\omega < \theta < \frac{\pi}{2}$, an estimate

$$|a_z(x, y)| \leq c_\theta h_{|z|}(x, y)$$

for $x, y \in \mathbb{R}$ and $|\arg(z)| < \frac{\pi}{2} - \theta$, where h_t is defined on $\mathbb{R}^2$ that satisfies(2.2).

(2) The operator L has bounded holomorphic functional calculus in $L^2(\mathbb{R})$. That is, for any $\mu > \omega$ and $b \in H_\infty(S^0_\mu)$, the operator $b(L)$ satisfies

$$||b(L)||_2 \leq C_\mu ||f||_\infty. \tag{5.1}$$

In what following, we choose $A_t = e^{-tL}$, as in section 2, we define the operator $\psi_t = A_t(I - A_t)$, and the S function $S_L(f)(x) = (\iint_{\Gamma(x)} |\psi_{t^m}(f)(y)|^2 \frac{dydt}{t^2})^{\frac{1}{2}}$, where $\Gamma(x) = \{(y, t)| \in \mathbb{R}^2_+, |y - x| < t\}$.
Then inequality(5.1) is equivalent to the square function estimate and its reverse:

$$C_1 ||f||_2^2 \leq \int_{\mathbb{R}} S_L(f)(x)^2 dx \leq C_2 ||f||_2^2.$$

Also we have:

$$C_1 ||f||_2^2 \leq \int_{\mathbb{R}} S_{L^*}(f)(x)^2 dx \leq C_2 ||f||_2^2.$$

See [14] for more details.

Remark. There are many operators satisfying the assumptions above. For example,

(1) the magnetic Schrödinger operator:

$$L = -(\frac{\partial}{\partial x} - ia)^2 + V(x)$$

where a is real value function and $0 \leq V(x) \in L_{\text{loc}}(\mathbb{R})$.

(2) the divergent operator:

$$L = -a_1 \frac{d}{dx}(a_2 \frac{d}{dx})$$

where $a_i \in L^\infty(\mathbb{R}, \mathbb{C})$, and $\text{Re}(a_i) \geq k > 0, i = 1, 2$.

Now we consider the two-parameter case. Suppose that $f(\zeta_1, \zeta_2)$ defined on $\mathbb{C}^2$, and it satisfies the following property: for any fixed ζ_2, $f(\zeta_1, \zeta_2)$ is holomorphic in $S_{\mu+}$ for the ζ_1 variable, and for any fixed ζ_1, $f(\zeta_1, \zeta_2)$ is holomorphic in $S_{\mu+}$ for the ζ_2 variable. The collection of all such functions is denoted by $H(S_{\mu+} \times S_{\mu+})$. Similar to one-parameter case, we write

$\Psi(S_{\mu+} \times S_{\mu+}) = \{f| \in H(S_{\mu+} \times S_{\mu+}), |f(\zeta_1, \zeta_2)| \leq C\frac{|\zeta_1|^s}{1+|\zeta_1|^{2s}} \frac{|\zeta_2|^s}{1+|\zeta_2|^{2s}}, \forall \zeta_1, \zeta_2 \in S_{\mu+}$, for some $s > 0\}$, and $H_\infty(S_{\mu+} \times S_{\mu+}) = \{f| \in H(S_{\mu+} \times S_{\mu+}), |f(\zeta_1, \zeta_2)| \leq C, \forall \zeta_1, \zeta_2 \in S_{\mu+}\}$.

We define the symbol $\otimes$ by $T_1 \otimes T_2 f(x, y) = \iint K_1(x, u)K_2(y, v)f(u, v)dudv$, where T_1 and T_2 are operators on $L^2(\mathbb{R})$ with kernels K_1 and K_2, respectively.

If $\phi \in \Psi(S_{\mu+} \times S_{\mu+})$, then we define

$$\phi(L) = (\frac{1}{2\pi i})^2 \int_\gamma \int_\gamma (L - \zeta_1 I)^{-1} \otimes (L - \zeta_2 I)^{-1} \phi(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2$$

where γ is the contour $\{\zeta = re^{\pm i\theta} : r \geq 0\}$ parameterized clockwise around $S_{\mu+}$, and $\omega < \theta < \mu$. Clearly, this integral is absolutely convergent. Applying Cauchy theorem, we conclude that the definition is independent of the choice of $\theta \in (\omega, \mu)$.

Let $\phi(\zeta_1, \zeta_2) = \frac{\zeta_1}{(1+\zeta_1)^2} \frac{\zeta_2}{(1+\zeta_2)^2}$. Then $\phi \in \Psi(S_{\mu+} \times S_{\mu+})$ and $\phi(L)$ is one-one. For all $b \in H_\infty(S_{\mu+} \times S_{\mu+})$, we define

$$b(L) = (\phi(L))^{-1}(b\phi)(L).$$

We can verify many properties in one-parameter case are also satisfied in our case(see [1]). For example, the following convergence Lemma holds.

Lemma 5.1. *Let $\{f_\alpha\}$ be a uniformly bounded net in $H_\infty(S^0_{\mu+} \times S^0_{\mu+})$, which converges to $f \in H_\infty(S^0_{\mu+} \times S^0_{\mu+})$ uniformly on compact subsets of $S^0_{\mu+} \times S^0_{\mu+}$, such that $\{f_\alpha(L)\}$ is a uniformly bounded net. Then $f(L) \in B(L^2(\mathbb{R}^2))$, $f_\alpha(L)u \to f(L)u$ for all $u \in L^2(\mathbb{R}^2)$, and $||f(L)|| \leq \sup_\alpha ||f_\alpha(L)||$.*

Theorem 5.1. *If $b \in H_\infty(S_{\mu+} \times S_{\mu+})$, then the operator $T = b(L)$ satisfies all the assumptions of Theorem 1.2. As a consequence, we have:*

(1) *The operator* $T = b(L)$ *is bounded from* $L^p(\mathbb{R}^2, \omega)$ *to* $L^p(\mathbb{R}^2, \omega)$*, for all* $\omega \in A_{\frac{p}{2}}(\mathbb{R} \times \mathbb{R})$*, where* $p > 2$*.*

(2) *The operator* $T = b(L)$ *is bounded from* $L^2 \cap L^\infty$ *to* $BMO_L \cap L^2$*.*

Firstly we give a lemma.

Lemma 5.2. *If* $t, s, \beta > 0$*, then*

$$\int_0^\infty \frac{tu}{(1+tu)^3} e^{-\beta su} du \leq c \frac{t}{(s+t)^2}.$$

Proof. If $s \leq t$, then

$$\begin{aligned}\int_0^\infty \frac{tu}{(1+tu)^3} e^{-\beta su} du &\leq c\Big(\int_0^{\frac{1}{t}} tudu + \int_{\frac{1}{t}}^{\frac{1}{s}} \frac{1}{(tu)^2} du + \int_{\frac{1}{s}}^\infty \frac{1}{(tu)^2} \frac{1}{(su)^2} du \Big) \\ &\leq \frac{c}{t}.\end{aligned}$$

If $t \leq s$, then

$$\begin{aligned}\int_0^\infty \frac{tu}{(1+tu)^3} e^{-\beta su} du &\leq c\Big(\int_0^{\frac{1}{s}} tudu + \int_{\frac{1}{s}}^{\frac{1}{t}} tu \frac{1}{(su)^3} du + \int_{\frac{1}{t}}^\infty \frac{1}{(tu)^2} \frac{1}{(su)^2} du \Big) \\ &\leq c \frac{t}{s^2}.\end{aligned}$$

This completes the proof of Lemma 5.2.

Proof of Theorem 5.1. We choose $A_t = e^{-tL}$. Applying the convergent lemma above, we just need to show that when $b \in \Psi(S^0_{\mu+} \times S^0_{\mu+})$, the conditions (1.3),(1.4)and(1.5) hold. We choose $\omega < \theta < \upsilon < \mu$, and denote

$$\gamma_+ = \{se^{i\upsilon} | s \geq 0\},\ \gamma_- = \{se^{-i\upsilon} | s \geq 0\},\ \gamma = \gamma_+ \cup \gamma_-,$$

$$\Gamma_+ = \{se^{i(\frac{\pi}{2}-\theta)} | s \geq 0\},\ \Gamma_- = \{se^{-i(\frac{\pi}{2}-\theta)} | s \geq 0\},\ \Gamma = \Gamma_+ \cup \Gamma_-.$$

It is easy to see that

$$\begin{gathered}k^{(1)}(x_1, y_1)g(z) = (\frac{1}{2\pi i})^2 \int_\gamma \int_\gamma e^{-t_1^m \zeta_1}(1 - e^{-t_1^m \zeta_1}) \\ b(\zeta_1, \zeta_2)(L - \zeta_1 I)^{-1}(x_1, y_1)(L - \zeta_2 I)^{-1} g(z) d\zeta_1 d\zeta_2.\end{gathered}$$

Applying the fact that $(L-\zeta_1 I)^{-1} = c\int_\Gamma e^{\lambda\zeta_1}e^{-\lambda L}d\lambda$, we have

$$
\begin{aligned}
&||k_{t_1}^{(1)}(x_1,y_1)g(z)||_2 \\
\leq\ & \frac{1}{(2\pi)^2}\int_\gamma |e^{-t_1^m\zeta_1}(1-e^{-t_1^m\zeta_1})|\ \sup_{\zeta_2}|b(\zeta_1,\zeta_2)|\ |(L-\zeta_1 I)^{-1}(x_1,y_1)|\ ||g||_2 d|\zeta_1| \\
\leq\ & c\int_{\gamma+\cup\gamma-} |e^{-t_1^m\zeta_1}(1-e^{-t_1^m\zeta_1})|\ \sup_{\zeta_2}|b(\zeta_1,\zeta_2)| \\
& \quad \int_{\Gamma+\cup\Gamma-}|e^{\lambda\zeta_1}|\ |a_\lambda(x_1,y_1)|\ d|\lambda|\ ||g||_2\ d|\zeta_1| \\
\leq\ & c||b||_\infty||g||_2\int_0^\infty\int_0^\infty \frac{t_1^m u}{(1+t_1^m u)^3}e^{-\beta su}h_s(x_1,y_1)duds,
\end{aligned}
$$

where $\beta > 0$ is a constant.

If $|x_1-y_1|\sim 2^k t_1$, we apply Lemma 5.2, then we have

$$
\begin{aligned}
||k_{t_1}^{(1)}(x_1,y_1)||_{L^2\to L^2} &\leq c\int_0^\infty\int_0^\infty \frac{t_1^m u}{(1+t_1^m u)^3}e^{-\beta su}\frac{1}{s^{1/m}}(1+\frac{2^k t_1}{s^{1/m}})^{-1-\epsilon}duds \\
&\leq c\int_0^\infty\int_0^\infty \frac{t_1^m u}{(1+t_1^m u)^3}e^{-\beta s^m u}\frac{1}{s}(1+\frac{2^k t_1}{s})^{-1-\epsilon}s^{m-1}duds \\
&\leq c\int_0^\infty \frac{{t_1}^m}{(s+t_1)^{2m}}\frac{1}{s}(1+\frac{2^k t_1}{s})^{-1-\epsilon}s^{m-1}ds \\
&\leq c\frac{t_1^\delta}{(2^k t_1)^{1+\delta}}
\end{aligned}
$$

where $\delta = min(m,\epsilon)$.

As a result,

$$
\int_{|x_1-y_1|>2^k t_1}||k_{t_1}^{(1)}(x_1,y_1)||^2_{L^2\to L^2}dy_1 \leq c\sum_{l:\ l\geq k}\frac{t_1^{2\delta}}{(2^l t_1)^{2+2\delta}}2^l t_1 \leq c\frac{t_1^{2\delta}}{(2^k t_1)^{1+2\delta}}.
$$

The conditions (1.4), (1.5) can be verified by the similar way. Here we omit details.

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