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ON A NEW CLASS OF SEQUENCES RELATED TO THE ℓ_p SPACE DEFINED BY ORLICZ FUNCTION

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Abstract. In this paper we introduce new sequence space $m(M, A, \phi, p)$ defined by combining an Orlicz function and an infinite matrix. We study its different properties and obtain some inclusion relations involving the space $m(M, A, \phi, p)$.

1. Introduction

Throughout this paper w, ℓ_∞ and ℓ_p, denote the spaces all, bounded and p absolutely summable sequences, respectively.Also $\wp_s$ denotes the set of all subsets of N , those do not contain more than s elements. Further(ϕ_s) will denote a non-decreasing sequence of positive real numbers such that $n\phi_n \leq (n+1)\phi_{n+1}$,for all $n \in N$ .The class of all the sequences (ϕ_s) satisfying this property is denoted by Φ.

The space $m(\phi)$ introduced and studied by Sargent [13] is defined as follows :

$$m(\phi) = \{x = (x_i) \in s : \|x\|_{m(\phi)} = \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} |x_i| < \infty\}.$$

Sargent [13] studied some of its properties and obtained its relationship with the space ℓ_p .

Later on it was investigated from sequence space point of view by Rath [10], Rath and Tripathy [11], Tripathy and Sen [14], Tripathy and Mahanta [15], and others.

Lindentrauss and Tzafirir [7] used the idea of Orlicz function to defined the following sequence spaces.

$$\ell_M = \{x = (x_i) \in s : \sum_{i=1}^{\infty} M(\frac{|x_i|}{\rho}) < \infty, \rho > 0\}$$

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which is called an Orlicz sequence spaces. The space ℓ_M is a Banach space with the norm,

$$\|x\| = \inf \rho > 0 : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\rho}\right) \leq 1\}.$$

The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with, $M(x) = x^p, 1 \leq p < \infty$.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. It is well known that if M is a convex function and $M(0) = 0$;then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda \leq 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $L > 0$ such that $M(2u) \leq LM(u), u \geq 0$ (see, Krasnoselskii and Rutitsky [6]).

In the later stage different Orlicz sequence spaces were introduced and studied by Bhardwaj and Singh [1], Bilgin [2],Güngör et al [5], Tripathy and Mahanta [15], Esi [3], Esi and Et [4], Parashar and Choudhary [9] and many others.

The following inequality will be used troughout the paper;

(1) $$|a_i + b_i|^{p_i} \leq \max(1, 2^{H-1})(|a_i|^{p_i} + |b_i|^{p_i})$$

where a_i and b_i are complex numbers ,and H = sup $p_i < \infty$

Recently Tripathy and Mahanta [15] defined and studied the following sequence space. Let M be an Orlicz function, then

$$m(M, \Delta, \phi) = \{x = (x_i) \in s : \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M(\frac{|\Delta x_i|}{\rho}) < \infty, \text{ for some } \rho > 0\}.$$

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. We write

$$Ax = (A_i(x)) \text{ if } A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$$

converges for each i.

In this paper we introduce the following sequence space: Let A=(a_{ik})be an infinite matrix of complex numbers,M be an Orlicz function and a $p = (p_i)$be bounded sequence of positive real numbers such that $0 < h = inf p_i \leq p_i \leq \text{supp}_i = H < \infty$, then

$$m(M, A, \phi, p) = x = (x_i) \in s : \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i} < \infty, \text{ for some } \rho > 0\}.$$

where, for convenince, we put

$$M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i} \text{ instead of } \left[M\left(\frac{|\Delta x_i|}{\rho}\right)\right]^{p_i}.$$

If $p_i = 1$, for all $i \in N$ and $a_{ik} = \begin{cases} 1, & k = i \\ -1, & k = i+1 \\ 0, & \text{others} \end{cases}$ the space $m(M, A, \phi, p)$ reduce to the above sequence space $m(M, \Delta, \phi)$. When $A = I$ unit matrix, then write the space $m(M, \phi, p)$ in place of the space $m(M, A, \phi, p)$.

2. Main Results

In this section we prove some results involving the sequence space $m(M, A, \phi, p)$.

Theorem 2.1. *Let M be an Orlicz function, then the space* $m(M, A, \phi, p)$ *is a linear space over the complex field* C.

Proof. Suppose that $x, y \in m(M, A, \phi, p)$ and $\alpha, \beta \in C$. Then there exist some positive numbersρ_1 and ρ_2 such that

$$\sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho_1}\right)^{p_i} < \infty \text{ and } \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho_2}\right)^{p_i} < \infty.$$

Let $\rho_3 = max(2\,|\alpha|\,\rho_1, 2\,|\beta|\,\rho_2)$. Since M is non-decreasing and convex, from inequality (1), we therefore have

$$\begin{aligned}
&\sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(\alpha x + \beta y)|}{\rho_3}\right)^{p_i} \\
&= \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{\left|\sum\limits_{k=1}^{\infty}(\alpha a_{ik}x_k + \beta a_{ik}y_k)\right|}{\rho_3}\right)^{p_i} \\
&= \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|\alpha A_i(x) + \beta A_i(y)|}{\rho_3}\right)^{p_i} \\
&\leq \max(1, 2^{H-1})[\sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho_1}\right)^{p_i} \\
&\quad + \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(y)|}{\rho_2}\right)^{p_i}] < \infty.
\end{aligned}$$

This implies that $\alpha x + \beta y \in m(M, A, \phi, p)$. This proves that $m(M, A, \phi, p)$ is linear.

Theorem 2.2. *Let $Ax \to \infty$, as $x \to \infty$, then the space $m(M, A, \phi, p)$ is a linear topological space paranormed by*

$$g(x) = \left\{ \rho^{p_r/H} : [\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i}]^{1/H} \leq 1, r = 1, 2, 3, ... \right\}$$

Proof. Clearly $g(x) = g(-x)$. Since $M(0) = 0$, we get $Ax = 0$ for $x = 0$, therefore $g(x) = 0$. Let $\rho = \rho_1 + \rho_2$, $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho_1}\right)^{p_i} \leq 1 \text{ and} \, v \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(y)|}{\rho_2}\right)^{p_i} \leq 1$$

Then we have

$$\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(x+y)|}{\rho}\right)^{p_i} \leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)^h \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho_1}\right)^{p_i} + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)^h \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(y)|}{\rho_2}\right)^{p_i}$$

Hence we get $g(x + y) \leq g(x) + g(y)$. Next, for $\lambda \in C$, without loss of generality, let $\lambda \neq 0$, then

$$g(\lambda x) = \left\{ \rho^{p_r/H} : [\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(\lambda x)|}{\rho}\right)^{p_i}]^{1/H} \leq 1, r = 1, 2, 3, ... \right\}$$
$$= \left\{ |\lambda| \, \rho_1^{p_r/H} : [\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho_1}\right)^{p_i}]^{1/H} \leq 1, r = 1, 2, 3, ... \right\}$$

where$\rho = |\lambda| \, \rho_1$. Hence we get $g(\lambda x) \leq max(1, |\lambda|) g(x)$

So, the continuity of the scalar multiplication follows from the above inequality.This completes the proof.

Now we give relation between $m(M, A, \phi^1, p)$ and $m(M, A, \phi^2, p)$ with respect to an Orlicz function.

Theorem 2.3. *$m(M, A, \phi^1, p) \subseteq m(M, A, \phi^2, p)$ if and only if*

$$\sup_{s\geq 1} \frac{\phi_s^1}{\phi_s^2} < \infty.$$

Proof. Let $x \in m(M, A, \phi^1, p)$ and $T = \sup_{s\geq 1} \frac{\phi_s^1}{\phi_s^2} < \infty$.Then we can write

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s^2} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i} \leq \sup_{s\geq 1} \frac{\phi_s^1}{\phi_s^2} \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s^1} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i}$$
$$= T \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s^1} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i}$$

Therefore $x \in m(M, A, \phi^2, p)$.Conversely, let $m(M, A, \phi^1, p) \subseteq m(M, A, \phi^2, p)$ and $x \in m(M, A, \phi^1, p)$. Then there exists $\rho > 0$ such that

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s^1} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i} < \infty$$

Suppose that $\sup_{s\geq 1} \frac{\phi_s^1}{\phi_s^2} = \infty$. Then there exists a sequence of natural numbers (s_j) such that $\lim_{j\to\infty} \frac{\phi_{s_j}^1}{\phi_{s_j}^2} = \infty$. Hence we can write

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s^2} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i} \geq \sup_{j\geq 1} \frac{\phi_{s_j}^1}{\phi_s^2} \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_{s_j}^1} \sum_{i\in\sigma} M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i} = \infty$$

Therefore $x \notin m(M, A, \phi^2, p)$ which is a contradiction to the fact that $m(M, A, \phi^1, p) \subseteq m(M, A, \phi^2, p)$. Hence $\sup_{s\geq 1} \frac{\phi_s^1}{\phi_s^2} < \infty$.

The following result is a consequence of Theorem 2.3.

Proposition 2.4. *Let M be an Orlicz function $m(M, A, \phi^1, p) = m(M, A, \phi^2, p)$ if and only if $\sup_{s\geq 1} \frac{\phi_s^1}{\phi_s^2} < \infty$ and $\sup_{s\geq 1} \frac{\phi_s^2}{\phi_s^1} < \infty$.*

Theorem 2.5. *Let M and M_1 be an Orlicz functions which satisfies the Δ_2-condition. Then $m(M, A, \phi, p) \subseteq m(MoM_1, A, \phi, p)$*

Proof. Let $x \in m(M_1, A, \phi, p)$ and $\varepsilon > 0$ be given and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(M_1[\frac{|A_i(x)|}{\rho}]\right)^{p_i} = \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s} \sum_{1} M\left(M_1[\frac{|A_i(x)|}{\rho}]\right)^{p_i}$$
$$+ \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s} \sum_{2} M\left(M_1[\frac{|A_i(x)|}{\rho}]\right)^{p_i},$$

where the summation $\sum_1$ is over $M_1[\frac{|A_i(x)|}{\rho}] \leq \delta$ and the summation $\sum_2$ is over $M_1[\frac{|A_i(x)|}{\rho}] > \delta$. Since M is continuous, we have

$$\begin{aligned}
&\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_1 M\left(M_1[\frac{|A_i(x)|}{\rho}]\right)^{p_i} \\
&\leq \max\{1, M(1)^H\} \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_1 M_1\left(\frac{|A_i(x)|}{\rho}\right)^{p_i} \\
&\leq \max\{1, M(1)^H\} \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{i\in\sigma} M_1\left(\frac{|A_i(x)|}{\rho}\right)^{p_i}
\end{aligned}$$

For $M_1[\frac{|A_i(x)|}{\rho}] > \delta$, we use the fact that

$$M_1[\frac{|A_i(x)|}{\rho}] < M_1[\frac{|A_i(x)|}{\rho}]\delta^{-1} \leq 1 + M_1[\frac{|A_i(x)|}{\rho}]\delta^{-1}.$$

Since M satisfies the Δ_2-condition , then there exists $L > 1$ such that

$$\begin{aligned}
M(M_1[\tfrac{|A_i(x)|}{\rho}]) &< M(1 + M_1[\frac{|A_i(x)|}{\rho}]\delta^{-1}) \\
&\leq \frac{1}{2}M(2) + \frac{1}{2}M(2M_1[\frac{|A_i(x)|}{\rho}]\delta^{-1}) \\
&\leq \frac{1}{2}LM(2)M_1[\frac{|A_i(x)|}{\rho}]\delta^{-1} + \frac{1}{2}LM(2)M_1[\frac{|A_i(x)|}{\rho}]\delta^{-1} \\
&= LM(2)\delta^{-1}M_1[\frac{|A_i(x)|}{\rho}].
\end{aligned}$$

We have

$$\begin{aligned}
&\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_2 M\left(M_1[\frac{|A_i(x)|}{\rho}]\right)^{p_i} \\
&\leq \max\{1, (LM(2)\delta^{-1})^H\} \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_2 M_1\left(\frac{|A_i(x)|}{\rho}\right)^{p_i}. \\
&\leq \max\{1, (LM(2)\delta^{-1})^H\} \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{i\in\sigma} M_1\left(\frac{|A_i(x)|}{\rho}\right)^{p_i}
\end{aligned}$$

Hence

$$\begin{aligned}
&\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{i\in\sigma} M(M_1[\frac{|A_i(x)|}{\rho}])^{p_i} \\
&\leq \max\{1, M(1)^H\} \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{i\in\sigma} M_1\left(\frac{|A_i(x)|}{\rho}\right)^{p_i} \\
&\quad + \max\{1, (LM(2)\delta^{-1})^H\} \sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{i\in\sigma} M_1\left(\frac{|A_i(x)|}{\rho}\right)^{p_i}
\end{aligned}$$

It follows that $x \in m(MoM_1, A, \phi, p)$.

The proof of the following result is a routine verification in view of the Theorem 2.5

Proposition 2.6. *For an Orlicz function M which satisfies the Δ_2- condition, we have $m(A, \phi, p) \subseteq m(M, A, \phi, p)$.*

The proof of the following result is a consequence of Theorem 2.3 and Proposition 2.6.

Proposition 2.7. *For an Orlicz function M which satisfies the Δ_2 - condition, we have $m(A, \phi^1, p) \subseteq m(M, A, \phi^2, p)$ if and only if $\sup_{s\geq 1} \frac{\phi_s^1}{\phi_s^2} < \infty$.*

(p_i) and (q_i) are any bounded sequences of strictly positive real numbers.We are able to prove $m(M, A, \phi, q) \subseteq m(M, A, \phi, p)$ only under additional conditions.

Theorem 2.8. *Let $0 < p_i \leq q_i$ for all $i \in N$ and let (q_i/p_i) be bounded.Then $m(M, A, \phi, q) \subseteq m(M, A, \phi, p)$.*

Proof. If we take $t_i = M\left(\frac{|A_i(x)|}{\rho}\right)^{p_i}$ for all $i \in N$, then using the same technique of Theorem 2 of Öztürk and Bilgin [8], it is easy to prove of the theorem.

Proposition 2.9.

(i) *If $0 < inf p_i \leq 1$ for all $i \in N$, then $m(M, A, \phi) \subseteq m(M, A, \phi, p)$*

(ii) *$1 \leq p_i \leq sup p_i = H < \infty$, then $m(M, A, \phi, p) \subseteq m(M, A, \phi)$*

Proof.

(i) Follows from Theorem 2.8 for $q_i = 1$ for all $i \in N$.

(ii) Follows from Theorem 2.8 for $p_i = 1$for all $i \in N$ and $q_i = p_i$ for all $i \in N$.

Let X be a sequence space. Then X is called Solid (or normal) if $(a_i x_i) \in X$ whenever $(x_i) \in X$ for all sequences (a_i) of scalars with $|a_i| \leq 1$ for all $i \in N$.

Theorem 2.10. *The sequence space $m(M, \phi, p)$ is solid.*

Proof. Let $\alpha = (\alpha_i)$ be a sequence of scalars such that $|\alpha_i| \leq 1$, for all $i \in N$. We get

$$\sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|\alpha_i x_i|}{\rho}\right)^{p_i} \leq \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{\sup|\alpha_i|\,|x_i|}{\rho}\right)^{p_i}$$
$$\leq \sup_{s\geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{i\in\sigma} M\left(\frac{|x_i|}{\rho}\right)^{p_i}$$

Then the result follows from the above inequality.

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