TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 4, pp. 1167-1179, August 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

ON DISTANCE TWO LABELLING OF UNIT INTERVAL GRAPHS

Peter Che Bor Lam[1]*, Tao-Ming Wang[2], Wai Chee Shiu and Guohua Gu

Abstract. An $L(2,1)$-labelling of a graph G is an assignment of non-negative integers to the vertices of G such that vertices at distance at most two get different numbers and adjacent vertices get numbers which are at least two apart. The $L(2,1)$-labelling number of G, denoted by $\lambda(G)$, is the minimum range of labels over all such labellings. In this paper, we first discuss some necessary and sufficient conditions for unit interval graph G to have $\lambda(G) = 2\chi(G) - 2$ and then characterize all unit interval graphs G of order no more than $3\chi(G) - 1$, where $\chi(G)$ is the chromatic number of G. Finally, we discuss some subgraphs of unit interval graphs G on more than $2\chi(G) + 1$ vertices with $\lambda(G) = 2\chi(G)$.

1. Introduction

The study of distance two labellings of graphs is motivated from the channel/frequency assignment problem (FAP) introduced by Hale [5]. The FAP is the assignment of frequencies to television and radio transmitters subject to restrictions imposed by the distance between transmitters. This problem was first formulated as a graph coloring problem by Hale, who introduced the notion of the T-coloring of a graph. There has been a considerable effort to study the T-coloring problem over the past decade. In 1988, Roberts (in a private communication with Griggs) proposed a variation of the FAP in which "close" transmitters must receive different channels and "very close" transmitters must receive channels at least two apart. Motivated by this variation, Griggs and Yeh [4] first proposed and studied the $L(2,1)$-labelling of a simple graph with a condition at distance two. This is followed by many other works. For examples, see [9, 2, 7, 1, 11, 6, 10, 3].

Received October 21, 2006, accepted November 23, 2007.
Communicated by Xuding Zhu.
2000 *Mathematics Subject Classification*: 05C78.
Key words and phrases: $L(2,1)$-labelling, Unit interval graph.
[1] Partially supported by National Science Council grant NSC 096-2811-M-029-001.
[2] Partially supported by National Science Council grant NSC 95-2115-M-029-003.
*Correspondence author.

Let G be a simple graph. A complete subgraph of G is called a *clique*. The *size* of a clique is the number of its vertices. A *t-clique* of G, denoted by K_t, is a clique of size t. The *clique number* of G, denoted by $\omega(G)$, is the size of the maximum clique of G. We also use the usual notations $V(G)$, $E(G)$, $n(G)$, $\Delta(G)$, $\chi(G)$ and $\lambda(G)$ to denote the vertex set, the edge set, the number of vertices, the maximum degree, the chromatic number and the λ-number of G, respectively. The reference to G will sometimes be omitted if no confusion is possible.

Suppose $0 \leq a \leq b$ are integers. We shall use $[a, b]$ to denote the set $\{a, a+1, \cdots, b-1, b\}$. An $L(2,1)$-labelling f of G is a function $f : V(G) \to [0, k]$, such that $|f(u)-f(v)| \geq 2$ if $uv \in E(G)$; and $|f(u)-f(v)| \geq 1$ if $d_G(u, v) = 2$, where $d_G(u, v)$ is the length (number of edges) of a shortest path between u and v in G. Elements of the image under f are called *labels*, and the *span* of f, denoted by span(f), is the difference between the maximum and minimum labels of f. Without loss of generality, we assume that the minimum label of $L(2,1)$-labellings of G is 0, and so span(f) is the maximum label. The *$L(2,1)$-labelling number*, or *λ-number* of G, $\lambda(G)$, is the minimum span over all such labellings. If span$(f) = \lambda(G)$, then f is a called a *span labelling*.

A graph is a *unit m-sphere graph* if each vertex represents a closed sphere in R^m of unit diameter and edges correspond to pairs of spheres that overlap. In the FAP, the interference graph is usually represented by a unit m-sphere graph for $m = 1$, 2 , or 3. The unit 1-sphere graphs are called *unit interval* graphs, or simply *UI*-graphs. The class of UI-graphs and its generalization are of particular interest in the FAP. In [8], Roberts showed that a graph G on n vertices is UI if and only if there exists an ordering $v_1, v_2, \ldots, v_n$ of V such that if v_i and v_j are adjacent for some $i \leq j$, then $\{v_i, v_{i+1}, \ldots, v_{j-1}, v_j\}$ induces a $(j-i+1)$-clique in G denoted by $[v_i, v_j]$. Such an ordering is called a *compatible* ordering of G. Henceforth, all UI-graphs will come with a listing $v_1, v_2, \ldots, v_n$ according to some compatible ordering. So when we say that a vertex v precedes or follows an other vertex w, we mean that v appears before or after w respectively, according to that ordering. Two distinct cliques in a UI-graph are *adjacent* if there exists a vertex in one clique which is adjacent to at least one vertex in the other, otherwise *separated*. Any set of vertices lying between (according to the compatible ordering) two separated χ-cliques is called a set of *separators* of the two χ-cliques. Two consecutive χ-cliques K' and K'' are *properly separated* if (i) they are separated and (ii) for any set of separators S, the number of vertices in $K' \cup K''$ adjacent to all vertices of S is at most $\chi - |S| - 1$. Throughout this paper, we always assume that a UI-graph is connected.

In [9], Sakai proved that each UI-graph G has only three possible λ-numbers: $2\chi - 2$, $2\chi - 1$ and 2χ. She also characterized UI-graphs G on $2\chi + 1$ vertices with $\lambda(G) = 2\chi(G)$ by the following theorem.

Theorem 1.1. [9]. *Let G be a UI-graph on $n = 2\chi + 1$ vertices and $\chi > 2$. There is a compatible vertex ordering $v_1,\ v_2,\ \cdots,\ v_n$ of vertices such that either*

(1) $v_1v_\chi, v_{\chi+2}v_n, v_qv_{q+\chi-1} \in E$ *for some* $3 \leq q \leq \chi$, *or*

(2) $v_1v_\chi, v_2v_{\chi+1}, v_{\chi+1}v_{n-1}, v_{\chi+2}v_n \in E$

if and only if $\lambda = 2\chi$.

In [9], Sakai also stated three unsolved problems. Two of them are as follows.

(1) Generalize the characterization of UI-graphs on more than $2\chi + 1$ vertices with $\lambda = 2\chi$.

(2) Characterize UI-graphs with $\lambda = 2\chi - 2$ and with $\lambda = 2\chi - 1$.

In this paper, we study the above problems. In Section 2, we first discuss some necessary conditions and some sufficient conditions for UI-graphs to have $\lambda = 2\chi - 2$. We then give a characterization of UI-graphs on at most $3\chi - 1$ vertices with $\lambda = 2\chi - 2$. In Section 3, we obtain some sufficient conditions for UI-graphs to have $\lambda = 2\chi$.

2. Unit Interval Graphs with $\lambda = 2\chi - 2$

In this section, we present some necessary conditions and some sufficient conditions for a UI-graph to have $\lambda = 2\chi - 2$.

Theorem 2.1. *Let G be a UI-graph on $n \geq 2\chi + 1$ vertices with $\chi > 2$. If $\lambda = 2\chi - 2$, then any two consecutive χ-cliques are properly separated.*

Proof. Let G be a UI-graph on $n \geq 2\chi + 1$ vertices with $\chi > 2$ and $\lambda = 2\chi - 2$. If f is a λ-labelling of G, then for any χ-clique K_χ of G, $f(V(K_\chi)) = \{0, 2, 4, \ldots, 2\chi - 2\}$, the set of all available even labels. Let K' and K'' be two consecutive χ-cliques. If they are not separated, then there exists $v \in K' \setminus K''$ such that $\{v\} \cup K''$ is of diameter two. So each vertex of $\{v\} \cup K''$ has to be assigned a distinct label from $\{0, 2, 4, \ldots, 2\chi - 2\}$, which is impossible. Therefore K' and K'' are separated. It remains to show that they are properly separated.

Suppose S is a set of separators of K' and K'' such that there exists $S^* \subset K' \cup K''$ with $|S^*| \geq \chi - |S|$ and each vertex in S^* is adjacent to all vertices in S. Clearly $S^* \not\subset K'$ and $S^* \not\subset K''$, otherwise $S^* \cup S$ is a χ-clique which is not separated from K' or K''. Therefore each of $K' \cup S$, $K'' \cup S$ and $S \cup S^*$ is of diameter two. Consequently, each vertex of S and S^* has to be assigned a distinct label from $\{1, 3, \cdots, 2\chi - 3\}$ and $\{0, 2, 4, \ldots, 2\chi - 2\}$ respectively. But $S^* \cup S$ is of diameter two, so after labelling S, at least $|S| + 1$ labels in $\{0, 2, 4, \ldots, 2\chi - 2\}$

cannot be used to label S^*. That means only $\chi-|S|-1 \leq |S^*|-1$ labels are available for S^*. The contradiction shows that K' and K'' are properly separated. ∎

It is straight-forward to show that the necessary condition of Theorem 2.1 is satisfied when a UI-graph G has only one χ-clique, or any two consecutive χ-cliques of G are separated by χ or more vertices. However, this condition is not sufficient for a UI-graph G to have λ-number $2\chi-2$. To see this fact, we give two UI-graphs H_I and H_{II}, each of which satisfies the necessary condition of Theorem 2.1, but $\lambda \geq 2\chi-1$.

H_I is the UI-graph on $3\chi+s$ ($\chi > 2, \chi-3 \geq s \geq 0$) vertices $v_1, v_2, \cdots, v_{3\chi+s}$ such that $[v_1, v_\chi]$ and $[v_{2\chi+s+1}, v_{3\chi+s}]$ are two χ-cliques and $[v_\chi, v_{2\chi-2}]$, $[v_{\chi+1}, v_{2\chi-1}]$ are two $(\chi-1)$-cliques, and $[v_{2\chi}, v_{2\chi+s+1}]$ is an $(s+2)$-clique. H_{II} is the UI-graph on $4\chi-5$ ($\chi > 3$) vertices $v_1, v_2, \cdots, v_{4\chi-5}$ such that $[v_1, v_\chi]$ and $[v_{3\chi-4}, v_{4\chi-5}]$ are two χ-cliques; and $[v_\chi, v_{2\chi-2}]$ and $[v_{2\chi-2}, v_{3\chi-4}]$ are two $(\chi-1)$-cliques (see Figure 1).

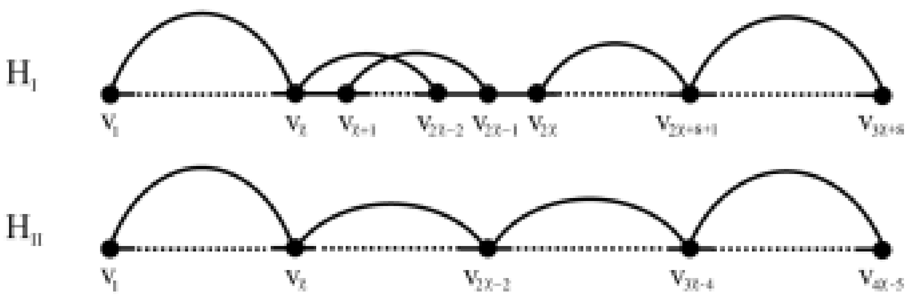

Fig. 1. UI-Graphs H_I and H_{II}.

Lemma 2.2. $\lambda(H_I) \geq 2\chi-1$ *and* $\lambda(H_{II}) \geq 2\chi-1$.

Proof. We first prove $\lambda(H_I) \geq 2\chi-1$. Suppose to the contrary that $\lambda(H_I) = 2\chi-2$ and f is a span labelling of H_I. Then $f([v_1, v_\chi]) = f([v_{2\chi+s+1}, v_{3\chi+s}]) = \{0, 2, \cdots, 2\chi-2\}$. Since $[v_1, v_{2\chi-2}]$ is of diameter two, $f([v_{\chi+1}, v_{2\chi-2}]) \subset \{1, 3, \cdots, 2\chi-3\}$. Similarly, $f([v_{2\chi}, v_{2\chi+s}]) \subset \{1, 3, \cdots, 2\chi-3\}$. The label of v_χ must be either $2\chi-2$ or 0, otherwise there are at most $\chi-3$ odd integers in $\{1, 3, \cdots, 2\chi-3\}$ available to label $v_{\chi+1}, \cdots, v_{2\chi-2}$, which is impossible. Assume that $f(v_\chi) = 2\chi-2$ (the case of $f(v_\chi) = 0$ is similar). Then $f([v_{\chi+1}, v_{2\chi-2}]) = \{1, 3, \cdots, 2\chi-5\}$ and $f(v_{2\chi-1}) = 2\chi-3$. Hence any odd integer in $\{1, 3, \cdots, 2\chi-3\}$ cannot be used to label $v_{2\chi}$ because $v_{2\chi}$ is at a distance of at most two from any vertex in $[v_{\chi+1}, v_{2\chi-1}]$, a contradiction.

To prove that $\lambda(H_{II}) \geq 2\chi-1$, we assume to the contrary that $\lambda(H_{II}) = 2\chi-2$. Let f be a span labelling of H_{II}. With the same argument as above, we may assume that $f(v_\chi) = 2\chi - 2$ and hence $f([v_{\chi+1}, v_{2\chi-2}]) = \{1, 3, \cdots, 2\chi - 5\}$. Similar argument leads to $f(v_{3\chi-4}) = 0$ or $2\chi-2$. But since v_χ is at distance 2 from $v_{3\chi-4}$, we have $f(v_{3\chi-4}) = 0$ and $f([v_{2\chi-2}, v_{3\chi-5}]) = \{3, \cdots, 2\chi - 3\}$. This means that $\chi - 1$ labels are available to label $[\chi + 1, 3\chi - 5]$, a graph on $2\chi - 5$ vertices with diameter two. Because $\chi \geq 5$, we have $\chi - 1 < 2\chi - 5$ and a contradiction. ■

Therefore, we have the following necessary condition for a UI-graph to have $\lambda = 2\chi - 2$.

Theorem 2.3. *Let G be a UI-graph on $n \geq \chi$ vertices. If $\lambda = 2\chi - 2$, then no subgraph of G is isomorphic to H_I or H_{II}.*

We also have the following sufficient condition for a UI-graph to have $\lambda = 2\chi - 2$.

Theorem 2.4. *Let G be a UI-graph. If*

- (a) *G contains exactly one χ-clique, or*
- (b) *G has at least two χ-cliques and the number of vertices between any pair of consecutive χ-cliques is $(2m-1)\chi - m$ for some integer $m \geq 1$,*

then $\lambda = 2\chi - 2$.

Proof. We first observe that the infinite periodic sequence

$$\cdots, \underbrace{1, 3, \cdots, 2\chi - 3,}\overbrace{0, 2, \cdots, 2\chi - 2,}\underbrace{1, 3, \cdots, 2\chi - 3,}\overbrace{0, 2, \cdots, 2\chi - 2,}\cdots \qquad (A)$$

has the following properties:

(1) Period of the sequence is $2\chi - 1$, and elements in the same period are distinct from each other,

(2) All elements in any string of length $\chi - 1$ differs from each other by at least two, and

(3) Between any two strings of even integers $0, 2, \cdots, 2\chi - 2$, there are $(2m - 1)\chi - m$ elements for some $m \geq 1$.

Now we assign the string $\{0, 2, \cdots, 2\chi - 2\}$ to one χ-clique. If there are more χ-cliques, then because of (b) and (3), all other χ-cliques will be fitted with the string $\{0, 2, \cdots, 2\chi - 2\}$. After trimming off excess elements from the sequence, we shall see that we have in fact obtained an $L(2, 1)$-labelling of G with span $2\chi - 2$.

Suppose u and v are two adjacent vertices. If their positions in the compatible ordering differs by $\chi - 1$, then they must belong to the same χ-clique and so each are assigned distinct even labels. If their positions in the compatible ordering differs by $\chi - 2$, then by (2), their labels differ by at least 2.

Suppose a shortest path connecting two vertices u and v has length two with intermediate vertex w. Since $[u, w]$ and $[w, v]$ cannot be both χ-cliques, there are at most $2\chi-4$ vertices lying between u and v. So by (1), they get distinct labels. ∎

Although the condition of Theorem 2.1 is not sufficient for a general UI-graph G to have $\lambda = 2\chi - 2$, it is sufficient for UI-graph in which any two consecutive χ-cliques are separated by at most $\chi - 1$ vertices.

Theorem 2.5. *Let G be a UI-graph such that any two consecutive χ-cliques K' and K'' are properly separated by at most $\chi - 1$ vertices. Then $\lambda(G) = 2\chi - 2$.*

Proof. We only need to give an $L(2, 1)$-labelling of G with span $2\chi - 2$. If G contains exactly one χ-clique, then the labelling can be obtained as in Theorem 2.4.

Suppose G contains at least two χ-cliques. We shall denote the strings of labels $\{1, 3, \cdots, 2\chi - 3\}$ and $\{0, 2, \cdots, 2\chi - 2\}$ by I_o and I_e respectively. We first take a sequence (A) truncated just after the end of one I_e, fit that string to the first χ-clique and trim the unused elements in the front. Similarly, we take a sequence (A) truncated just before the beginning of one I_e, fit that string to the last χ-clique and trim the unused elements at the rear. For other χ-cliques, if any, we assign one I_e. For vertices between two consecutive χ-cliques, we label them with one I_o, preserving the order, but with segments appropriately trimmed off if necessary. We shall describe this process in the following paragraphs.

Let $K' = [v_1, v_\chi]$ and $K'' = [v_{l+1}, v_{l+\chi}]$, be two consecutive χ-cliques properly separated by the set $S = [v_{\chi+1}, v_l]$ with $l - \chi = s \le \chi - 1$. For each $q \in [1, s]$, we put $q' = |N_{K'}(v_{\chi+q})|$ and $q* = |N_{K''}(v_{\chi+q})|$. Note that either q' or q^* can possibly be zero. We set $f'(v_{\chi+q}) = 2q - 1$ and $f''(v_{\chi+q}) = 2q^* + 1$ for each $q \in [1, s]$.

If $f'(v_{\chi+j}) \ge f''(v_{\chi+j})$ for all $j \in [1, s]$, then label $v_{\chi+j}$ with $f'(v_{\chi+j})$ for all $j \in [1, s]$. Since $f'(v_{\chi+j})$ is odd and $2s - 1 \le 2(\chi - 1) - 1 = 2\chi - 3$, all vertices of S have been labelled with elements from I_o. If there exists $j \in [1, s]$ such that $f'(v_{\chi+j}) < f''(v_{\chi+j})$, then let q be the smallest of such integers. We shall call $v_{\chi+q}$ a *critical* vertex. We label $v_{\chi+j}$ with $f'(v_{\chi+j})$ for all $j \in [1, q-1]$, and label $v_{\chi+q}$ with $f''(v_{\chi+q})$. Consider the fact that

$$(1) \qquad f''(v_{\chi+q}) - f'(v_{\chi+q}) = (2q^* + 1) - (2q - 1) = 2k > 0.$$

Since both $2q^* + 1$ and $2q - 1$ are odd integers, it follows that k is a natural number. For $q < j \le s$, we adjust the value of $f'(v_{\chi+j})$ to $2(j+k)-1$. Since $v_{\chi+q}$ is adjacent to v_{l+q^*}, the clique $[v_{\chi+q}, v_{l+q^*}]$ is of order $(s - q + 1) + q^* \le \chi - 1$. Using this inequality and (1), we can deduce that $2(s+k)-1 = 2(s-q+1+q^*)-1 \le 2\chi-3$.

Therefore all adjusted values of $f'(v_{\chi+j})$, $q < j \le s$, belong to I_o. By comparing values of $f'(v_{\chi+j})$ and $f''(v_{\chi+j})$ for all $j \in [q+1, s]$, we repeat the above process, adjusting f' at critical vertices, if any, until all vertices of S have been labelled.

We shall denote the above labelling by f and show that f is in fact an $L(2,1)$-labelling of G. Suppose that v and w are two adjacent vertices. If they both belong to the first χ-clique together with its preceding vertices, or to the last χ-clique together with its succeeding vertices, then by Theorem 2.4 their labels differ by at least two. If they both belong to the separators of two consecutive χ-cliques, then they were assigned distinct odd labels, and so their labels differ by at least two. Since it is impossible to have a χ-clique between two adjacent vertices, the only remaining case is that $v = v_{\chi+q}$ belongs to the separators of two consecutive χ-cliques $K' = [v_1, v_\chi]$ and $K'' = [v_{l+1}, v_{l+\chi}]$, and $w = v_r$ belongs to one of the two χ-cliques. If $v_r \in K''$, then $r - l - \chi + 1 \le q^*$ and $f(v_{\chi+q}) \ge 2q^* + 1$. Therefore $f(v_r) = 2(r - l - \chi) \le 2q^* - 2 \le f(v_{\chi+q}) - 3$. If $v_r \in K'$, then $\chi - q' + 1 \le r \le \chi$ and $f(v_r) = 2(r-1) \ge 2\chi - 2q'$. Suppose $f(v_{\chi+q}) = 2q - 1$, then $f(v_r) - f(v_{\chi+q}) \ge 2\chi - 2q' - 2q + 1 = 2(\chi - 1 - q - q') + 3$. Since $[v_{\chi-q'+1}, v_{\chi+q}]$ is a clique of size $q + q'$ and is not separated from K', therefore $\chi - 1 \ge q + q'$. It follows that $f(v_r) \ge f(v_{\chi+q}) + 3$. Suppose $f(v_{\chi+q}) > 2q - 1$, then we can determine $v_{\chi+q_o}$, the first vertex in S preceding $v_{\chi+q}$ (possibly $q_o = q$), for which $f(v_{\chi+q_o}) = 2q_o^* + 1$. Consider the set $S^* = [v_{\chi+q_o}, v_{\chi+q}]$, a set of separators for K' and K''. Because of the property of compatible ordering, all vertices in S are adjacent to q' and q_o^* vertices in K' and K'' respectively. Because K' and K'' are properly separated, we have

$$(q - q_o + 1) + q' + q_o^* \le \chi - 1. \tag{2}$$

Because $f(v_{\chi+q_o}) = f''(v_{\chi+q_o}) = 2q_o^* + 1$, we have $f(v_{\chi+q}) = 2q_o^* + 1 + 2(q - q_o)$. Using (2), we can deduce that $f(v_{\chi+q}) \le 2\chi - 2q' - 3 \le f(v_r) - 3$.

Let v_r and v_s, $r \le s$, be two vertices at distance two. We may assume that both of them belong to χ-cliques, or none of them belongs to a χ-clique. Suppose both of them belong χ-cliques, then because they are at distance two from each other, they belong to consecutive χ-cliques, say $K' = [v_1, v_\chi]$ and $K'' = [v_{l+1}, v_{l+\chi}]$, where $\chi + 1 \le l \le 2\chi - 1$. Since no vertex in K' is adjacent to any vertex of K'', there exists a vertex $v \in [v_{\chi+1}, v_l]$ adjacent to both v_r and v_s. From the previous paragraph, we have shown that $f(v_r) - 3 \ge f(v) \ge f(v_s) + 3$, and hence $f(v_r) > f(v_s)$. Finally, suppose v_r and v_s comes before and after a χ-clique K, respectively. If $v \in K$ is adjacent to both v_r and v_s, then again by the above argument, we have $f(v_r) - 3 \ge f(v) \ge f(v_s) + 3$ (same conclusion holds if either v_r or v_s precedes the first χ-clique or follows the last χ-clique respectively). A gain, we have $f(v_r) > f(v_s)$. ■

With Theorems 2.1, 2.4 and 2.5, we can characterize all UI-graphs on at most

$3\chi-1$ vertices with $\lambda=2\chi-2$. In particular, we have the following Theorem and Corollary.

Theorem 2.6. *Let G be a UI-graph on $n<3\chi$ vertices and $\chi>3$. Then $\lambda=2\chi-2$ if and only if (1) G contains exactly one χ-clique or (2) G contains precisely two properly separated χ-cliques.*

Corollary 2.7. *Let G be a UI-graph on $2\chi+1$ vertices and $\chi>3$. Then $\lambda=2\chi-2$ if and only if (1) G contains exactly one χ-clique or (2) G contains precisely two nonadjacent χ-cliques and the remaining vertex of G has degree at most $\chi-2$.*

The following theorem follows from Corollary 2.5 in [[9]] and Theorem 2.1.

Theorem 2.8. *Let G be a UI-graph on n vertices, $\chi+1\le n\le 2\chi$. Then*

1. $\lambda=2\chi-2$ *if and only if G contains exactly one χ-clique,*
2. $\lambda=2\chi-1$ *if and only if G contains at least two χ-cliques.*

Theorem 1.1 and Corollary 2.7 together characterizes all UI-graphs on $2\chi+1$ vertices with any possible λ, i.e. $2\chi-2\le\lambda\le 2\chi$. Theorem 2.8 characterize all UI-graphs on $\le 2\chi$ vertices with any possible λ, i.e. $2\chi-2\le\lambda\le 2\chi-1$.

3. Unit Interval Graphs with $\lambda=2\chi$

If the span of an $L(2,1)$-labeling of a graph G is $2k-2$, then any k-clique of G can only receive one set of labels, namely $\{0,2,\dots,2k-2\}$. However, if the span of L is $2k-1$, then a k-clique can receive any one of the following sets of integers [9].

$$
\begin{array}{rcl}
L_1(k) &=& \{0,\ 2,\ 4,\ 6,\ \cdots\ 2k-8,\ 2k-6,\ 2k-4,\ 2k-2\},\\
L_2(k) &=& \{0,\ 2,\ 4,\ 6,\ \cdots\ 2k-8,\ 2k-6,\ 2k-4,\ 2k-1\},\\
L_3(k) &=& \{0,\ 2,\ 4,\ 6,\ \cdots\ 2k-8,\ 2k-6,\ 2k-3,\ 2k-1\},\\
L_4(k) &=& \{0,\ 2,\ 4,\ 6,\ \cdots\ 2k-8,\ 2k-5,\ 2k-3,\ 2k-1\},\\
\cdots & & \qquad\qquad\cdots\\
\cdots & & \qquad\qquad\cdots\\
L_{k-2}(k) &=& \{0,\ 2,\ 4,\ 7,\ \cdots\ 2k-7,\ 2k-5,\ 2k-3,\ 2k-1\},\\
L_{k-1}(k) &=& \{0,\ 2,\ 5,\ 7,\ \cdots\ 2k-7,\ 2k-5,\ 2k-3,\ 2k-1\},\\
L_k(k) &=& \{0,\ 3,\ 5,\ 7,\ \cdots\ 2k-7,\ 2k-5,\ 2k-3,\ 2k-1\},\\
L_{k+1}(k) &=& \{1,\ 3,\ 5,\ 7,\ \cdots\ 2k-7,\ 2k-5,\ 2k-3,\ 2k-1\},
\end{array}
\qquad (L)
$$

Some nice properties about the above sets of integers are given by the following useful lemmas:

Lemma 3.1. [9] *If $r<s<t$ are integers and $x\in L_r(k)\cap L_t(k)$, then $x\in L_s(k)$.*

Lemma 3.2. [9] *Let G be a graph and let $k > 2$ be an integer. Suppose that there is an $L(2,1)$-labelling with span $2k-1$. If K is a k-clique of G, then the set of labels of K is $L_i(k)$ for some integer i, $1 \le i \le k+1$.*

Suppose f is an $L(2,1)$-labelling of G with span $2k-1$ and K is a k-clique in G. If 1 or $2k-2 \in f(K)$, then $f(K) = L_1(k)$ or $L_{k+1}(K)$ respectively. Lemma 3.1 and Lemma 3.2 lead to Lemma 3.3 that appears below. Recall that an r-path on n-vertices, denoted by P_n^r, is the graph G with $V(G) = \{v_i : i = 1, 2, \ldots, n\}$ and $E(G) = \{v_i v_j : 1 \le |i-j| \le r\}$.

Lemma 3.3. *Let P_{2k}^{k-1} be a $(k-1)$-path on $2k$ vertices $v_1, \ldots, v_{2k}$ and f is an $L(2,1)$-labelling of P_{2k}^{k-1} with span $2k-1$. Then either $f(v_i) = 2(k-i)$ and $f(v_{k+i}) = 2(k-i)+1$ for $i = 1, 2, \ldots, k$; or $f(v_i) = 2(i-1)+1$ and $f(v_{k+i}) = 2(i-1)$ for $i = 1, 2, \ldots, k$.*

The λ-number of a graph cannot be less than that of any of its subgraphs. If a UI-graph G contains a subgraph G' with $\lambda(G') = 2\chi(G)$, we can conclude that $\lambda(G) = 2\chi(G)$. Therefore one method of determining whether a UI-graph has λ-number equal to 2χ is to look for subgraphs having λ-number 2χ. Now we shall discuss three types of UI-graphs with $\lambda = 2\chi$. The first type of UI-graphs is one that satisfies condition 2 of Theorem 1.1.

A graph is called U_I if it is a UI-graph on $2\chi+1$ vertices containing four χ-cliques $[v_1, v_\chi]$, $[v_2, v_{\chi+1}]$, $[v_{\chi+1}, v_{2\chi}]$ and $[v_{\chi+2}, v_{2\chi+1}]$, see Figure 2.

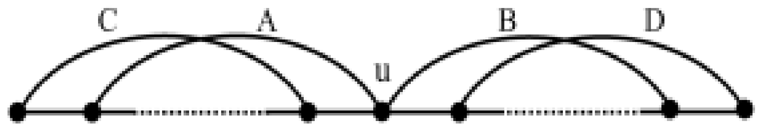

Fig. 2. Sketch of graph U_I.

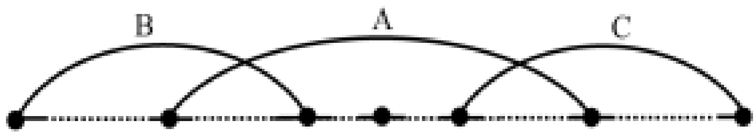

Fig. 3. Sketch of graph U_{II}.

A graph is called U_{II} if it is a UI-graph on at least $2\chi+1$ vertices containing three χ-cliques $[v_r, v_{\chi+r-1}]$, $[v_s, v_{\chi+s-1}]$ and $[v_t, v_{\chi+t-1}]$, where $r < s \le \chi + r - 1 < t - 1$, see Figure 3.

Theorem 3.4. *If a UI-graph contains a subgraph isomorphic to either U_I or U_{II}, then $\lambda = 2\chi$.*

Proof. Since the graph U_I defined above satisfies condition (2) of Theorem 1.1, $\lambda(U_I) = 2\chi$. Although the graph U_{II} is more general than the graph specified in condition (1) of Theorem 1.1, an argument similar to that of Sakai shows that $\lambda(U_{II}) = 2\chi$ (see [9]). ■

A sequence of χ-cliques of a UI-graph G is called a χ-*chain* if the sequence consists of consecutive non-separated χ-cliques one following another. The number of χ-cliques in the chain is called the *length* of the chain. The number of vertices covered by the χ-chain is called the *order* of the chain. Hence the sequence of χ-cliques $K_i = [v_{n_i}, v_{n_i+\chi-1}]$, where $1 \leq i \leq s$ and $n_j < n_{j+1} \leq n_j + \chi - 1$ for $j = 1, 2, \ldots, s-1$, will be called a χ-chain of length s and of order $n_s + \chi - n_1$. A χ chain of length s is *maximal* if its order is $\chi + s - 1$, i.e. $n_j + 1 = n_{j+1}$ for $j = 1, 2, \ldots, s-1$ in the above example. Since any χ-chain of order $2\chi + 1$ contains either U_I or U_{II} as subgraph, the following lemma follows from Theorem 3.4.

Lemma 3.5. *For a UI-graph with $\lambda \leq 2\chi - 1$, the order of any χ-chain is at most 2χ.*

Lemma 3.5 implies the following Theorem.

Theorem 3.6. *If there is a χ-chain of order $2\chi + 1$ or more in a UI-graph , then $\lambda = 2\chi$.*

The third graph U_{III} is a UI-graph consisting of two consecutive maximal χ-chains $\mathcal{C}_1$ and $\mathcal{C}_2$, each of which is of length χ. Moreover if v_i and v_j is the last vertex of $\mathcal{C}_1$ and the first vertex of $\mathcal{C}_2$ respectively, then $[v_i, v_j]$ consists of at least three vertices and is of diameter at most 2.

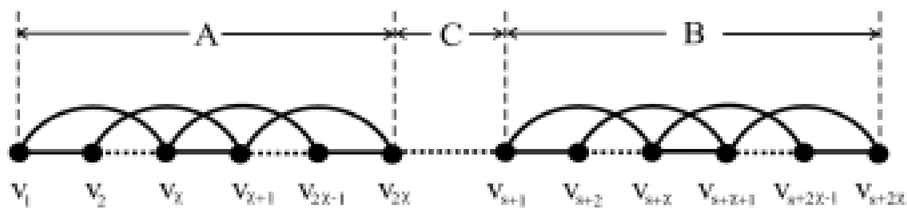

Fig. 4. Sketch of U_{III}.

Theorem 3.7. *If a UI-graph contains a subgraph isomorphic to U_{III}, then $\lambda = 2\chi$.*

Proof. It is sufficient to show that $\lambda(U_{III}) = 2\chi$. Suppose the two maximal χ-chains in U_{III} are $A = [v_1, v_{2\chi}]$ and $B = [v_{s+1}, v_{s+2\chi}]$, where $s + 1 \geq 2\chi + 1$.

Also suppose $C = [v_{2\chi}, v_{s+1}]$ contains at least three vertices and is of diameter at most two (Figure 4).

Suppose for contradiction purposes that $\lambda \le 2\chi - 1$ and f is an $L(2,1)$-labelling of G with span$(f) \le 2\chi - 1$. Since A is a maximal χ-chain of length χ and of order 2χ, then by Lemma 3.3, we must have $f(v_{2\chi}) = 2(\chi - 1)$ or 1 and similarly $f(v_{s+1}) = 2(\chi - 1)$ or 1 also. Since C contains at least three vertices and its diameter is at most 2, there is at least one vertex x between $v_{2\chi}$ and v_{s+1} such that $v_{2\chi}xv_{s+1}$ is a path of length 2. Therefore $v_{2\chi}$ and v_{s+1} cannot get the same label.

If $f(v_{2\chi}) = 2(\chi - 1)$, then $f(v_{s+1}) = 1$. So the cliques $[v_{\chi+1}, v_{2\chi}]$ and $[v_{s+1}, v_{s+2\chi}]$ are labelled with all the integers from 0 to $2\chi - 1$. But x is adjacent to $v_{2\chi}$ and v_{s+1}. So its distance from each vertex of these two cliques is one. That means x cannot be labelled. The contradiction shows that $\lambda(U_{III}) = 2\chi$. ■

4. Concluding Remarks

Although we did not answer all three questions mentioned at the end of [9] completely in this paper, we have completely characterized all UI-graphs G on at most $2\chi + 1$ vertices and all UI-graphs G on at most $3\chi - 1$ vertices for which $\lambda = 2\chi - 2$. In addition, we have given some necessary conditions and some sufficient conditions for a UI-graph with $\lambda = 2\chi - 2$. We have also given some sufficient conditions for a UI-graph to have λ-number 2χ.

However, it seems to be difficult to characterize unit interval graphs G with particular λ-numbers completely. Consiser the following two examples.

Let H_1 be a UI-graph of 19 vertices with $\chi = 4$ with two maximal 4-chains on $[v_1, v_8]$ and $[v_{12}, v_{19}]$ respectively. Further, let $v_8v_9v_{10}$ be a path and $[v_{10}, v_{12}]$ be a 3-clique. We can follow the argument of Theorem 3.7 to get $\lambda(H_1) = 8 = 2\chi$.

Now let H_2 be a UI-graph of 29 vertices with $\chi = 6$ with two maximal 6-chains on $[v_1, v_{12}]$ and $[v_{18}, v_{29}]$ respectively. Further, let $v_{12}v_{13}v_{14}$ be a path and $[v_{14}, v_{18}]$ be a 5-clique. We obtain an $L(2,1)$-labelling f of H_2 with span $11 = 2\chi - 1$ as follows.

$$f(v_i) = f(v_{17+i}) = 2(\chi - i), \qquad \text{for} \quad i = 1,2,3,4,5,6,$$
$$f(v_{6+j}) = f(v_{23+j}) = 2(\chi - j) + 1, \qquad \text{for} \quad j = 1,2,3,4,5,6,$$
$$f(v_{11+k}) = 2(\chi - k) + 1, \qquad \text{for} \quad k = 3,4,5,$$

and $f(v_{13}) = 4$. Thus $\lambda(H_2) = 11 = 2\chi - 1$.

We can see that the structure of H_1 and H_2 are very similar, but $\lambda(H_1) = 2\chi$, whereas $\lambda(H_2) = 2\chi - 1$. It seems that there is something to do with their chromatic numbers.

Acknowledgment

The authors gratefully accept the valuable comments of the referees.

References

1. G. J. Chang, W.-T. Ke, D. Kuo, D. D.-F. Liu and R. K. Yeh, On $L(d,1)$-labellings of graphs, *Discrete Math.*, **220** (2000), 57-66.
2. G. J. Chang and D. Kuo, The $L(2,1)$-labelling problem on graphs, *SIAM J. Discrete Math.*, **9** (1996), 309-316
3. J. P. Georges, D. W. Mauro and M. I. Stein, Labelling products of complete graphs with a condition at distance two, *SIAM J. Discrete Math.*, **14** (2000), 28-35.
4. J. R. Griggs and R. K. Yeh, Labelling graphs with a condition at distance two, *SIAM J. Discrete Math.*, **5** (1992), 586-595.
5. W. K. Hale, Frequency Assignment: Theory and Applications, *Proc. IEEE*, **68** (1980), 1497-1514.
6. J. van den Heuvel, R. A. Leese and M. A. Shepherd, Graph labelling and radio channel assignment, *J. Graph Theory*, **29** (1988), 263-283.
7. D. D.-F. Liu and R. K. Yeh, On distance two labellings of graphs, *Ars Combinatoria*, **47** (1997), 13-22.
8. F. S. Roberts (1971), On the compatibility between a graph and a simple order, *J. Combin. Theory*, **11** (1971), 28-38.
9. D. Sakai, Labeling chordal graphs with a condition at distance two, *SIAM J. Discrete Math.*, **7** (1994), 133-140.
10. M. A. Whittlesey, J. P. Georges and D. W. Mauro, On the λ-number of Q_n and related graphs, *SIAM J. Discrete Math.*, **8** (1995), 499-506.
11. K.-F. Wu and R. K. Yeh, Labelling graphs with the circular difference, *Taiwanese J. Math.*, **4** (2000), 397-405.

Peter Che Bor Lam
Department of Mathematics,
Tunghai University,
Taichung, Taiwan
E-mail: cblam2002@yahoo.com

Tao-Ming Wang
Department of Mathematics,
Tunghai University,
Taichung, Taiwan

Wai Chee Shiu
Department of Mathematics,
Hong Kong Baptist University,
Hong Kong

Guohua Gu
Department of Mathematics, Southeast University,
Nanjing 210018,
P. R. China