

ISOMORPHISM PROBLEM FOR ENDOMORPHISM NEAR-RINGS

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Abstract. Most papers written on the subject of the relationship between groups and near-rings are concerned with the structure of those near-rings which are associated in some way with particular groups. The theme of this paper is in the other direction: recapturing information about the group from the near-rings associated with it. So far most of the work has been concerned with one particular case: obtaining information about $\text{Inn } G$ from $I(G)$. We also will concern ourselves mainly with this situation, although we will provide a very general framework. In the next section we set up the background. The second section gives a view of the results we used as our starting point. Finally we present our contribution to date.

1. INTRODUCTION

We write the group operation both additively and multiplicatively, according to context. So care should be taken. As we write maps on the right, our near-rings are left near-rings. For a given group $(G, +)$ we write $M(G)$, $M_0(G)$, $M_c(G)$, $E(G)$, $A(G)$, $I(G)$ for, respectively, the near-ring of all maps from G to G , the near-ring of all maps from G to G respecting the neutral element 0 of G , the constant maps on G , the d. g. (distributively generated) near-rings generated by all the endomorphisms of G , $\text{End } G$, all the automorphisms of G , $\text{Aut } G$, and all the inner automorphisms of G , $\text{Inn } G$. For general background on the theory and structure of near-rings we would refer the reader to, in chronological order, Pilz [6, 7], Meldrum [4] and Clay [1]. As might be expected, the second of those books ([4]) is the one to which we are closest in spirit. We assume knowledge of the main results about the structure of near-rings and d. g. near-rings.

As mentioned before we write the group operation both additively and multiplicatively. This happens in the context of d. g. near-rings. Groups are normally

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written additively and in particular this is so for near-ring modules, that is the groups on which near-rings act. But when we consider groups of automorphisms generating a d. g. near-ring, or indeed acting on a group, we write them multiplicatively. If we are using multiplicative notation not in the context of a generating set for a d. g. near-ring, we will mention the fact explicitly. This does lead to situations of the following type: a group $(G, +)$, its group of inner automorphisms $(\text{Inn } G, \cdot)$ generating the d. g. near-ring $I(G)$, and questions arising as to the relationship of G with $I(G)$, the group G occurring with both notations. Normally the group operation will not be mentioned and we may write $\text{Inn } G \cong G/Z(G)$, although the group operations in the two groups are different. Note that $Z(G)$ will denote the centre of G and G' or $\delta(G)$ or $\delta_1(G)$ will denote the derived group of G . A perfect group is one in which $G = \delta(G)$. A strictly non-abelian group is one all of whose normal subgroups are perfect, which is equivalent to saying that all the chief factors of the group are non-abelian. A chief series is a maximal series of subgroups each of which is normal in the whole group. Chief factors are the factors from a chief series.

We now set the scene for the problems that we are going to consider. We consider two families of maps. The first is a set of maps from the class of groups to the class of near-rings. For those who have worries about foundational matters, we can restrict our attention to the groups and near-rings contained in some suitable universal set. We give some obvious examples: associate the group G with the near-rings mentioned earlier, $M(G)$, $M_0(G)$, $M_c(G)$, $E(G)$, $A(G)$, $I(G)$, any of these with an appropriate radical factored out, and many more.

Similarly we consider a set of maps from groups to groups, with the same proviso as above concerning foundational matters. Again we give some obvious examples, associating with a group G the groups $Z(G)$, $\delta(G)$, $\text{Inn } G \cong G/Z(G)$, $\text{Aut } G$, $\text{Out } G \cong \text{Aut } G/\text{Inn } G$, $\text{Fitt } G$, the Fitting subgroup of G , $\Phi(G)$, the Frattini subgroup of G , and many more. Any reasonable book on group theory, e. g. Robinson [9], Rotman [10] or Scott [12], will give details of these groups and many more. A slight generalization would associate with each group a family of groups, for instance with G associate the chief factors of G . One particular map of interest here is the identity map $1 : G \rightarrow G$, always denoted by 1.

This leads to the definition which provides the setting in which we work in this paper.

Definition 1.1. Let $\mathcal{X} = \{X_\lambda; \lambda \in \Lambda\}$ be a set of maps from groups to near-rings and let $\mathcal{Y} = \{Y_\mu; \mu \in M\}$ be a set of maps from groups to groups. A class \mathcal{C} of groups is said to be $(\mathcal{X}, \mathcal{Y})$ **constrained** if whenever $G, H \in \mathcal{C}$ and $X_\lambda(G) \cong X_\lambda(H)$ for all $\lambda \in \Lambda$, then $Y_\mu(G) \cong Y_\mu(H)$ for all $\mu \in M$. A class \mathcal{S} of groups is said to be **strongly** $(\mathcal{X}, \mathcal{Y})$ **constrained** if whenever $G \in \mathcal{S}$ and $X_\lambda(G) \cong X_\lambda(H)$ for all $\lambda \in \Lambda$, then $Y_\mu(G) \cong Y_\mu(H)$ for all $\mu \in M$.

Lemma 1.2. *If a class \mathcal{C} of groups is strongly $(\mathcal{X}, \mathcal{Y})$ constrained then it is $(\mathcal{X}, \mathcal{Y})$ constrained.*

Equally obvious is the following result.

Lemma 1.3. *If $\mathcal{X} = \{M_c\}$ and $\mathcal{Y} = \{1\}$, then the class of all groups is strongly $(\mathcal{X}, \mathcal{Y})$ constrained.*

This is because $(M_c(G), +) \cong (G, +)$ for any group G . When \mathcal{X} and/or \mathcal{Y} consist of a single map, we write it as that single map. So we would reword lemma 1.3 as "The class of all groups is strongly $(M_c, 1)$ constrained". While we are on trivial results, here is another one.

Lemma 1.4. *The class of all abelian groups is (I, Inn) constrained.*

Of course if A is abelian, then $\text{Inn}A = e$.

Corollary 1.5. *The class of all cyclic groups is $(I, 1)$ constrained.*

This follows since $I(A)$ for A an abelian group, is just Z_d , the ring of integers modulo d , the exponent of A . The exponent of a cyclic group identifies it up to isomorphism.

Theorem 1.6. *If G is strongly (A, \mathcal{Y}) or strongly (E, \mathcal{Y}) constrained for any set of maps \mathcal{Y} , then it is strongly (M_0, \mathcal{Y}) constrained.*

Proof. If for some group H there is an isomorphism $\phi : M_0(G) \rightarrow M_0(H)$ then the restriction of ϕ to the semigroup $\text{End } G$ (respectively the automorphism group $\text{Aut } G$) is an isomorphism to $\text{End } H$ (respectively $\text{Aut } H$) because $\text{End } G$ is precisely the set of distributive elements of $M_0(G)$ and $\text{Aut } G$ is precisely the set of invertible distributive elements and similarly for H . These maps then extend canonically to isomorphisms between $E(G)$ and $E(H)$ (respectively $A(G)$ and $A(H)$).

Theorem 1.7. *The class of torsion abelian groups is $(E, 1)$ constrained.*

This result is available as exercise 87c, p. 72 of Kaplansky [3].

2. SOME BASIC RESULTS

Here we present results which are either already known, or can be deduced very easily from known results. The first result, although known to most workers in this

field, is not stated and proved explicitly anywhere that we are aware of, and hence we state and prove it here.

Theorem 2.1. *The class of all finite simple non-abelian groups is (I, Inn) constrained and hence $(I, 1)$ constrained.*

Proof. The second statement follows from the first since, for a simple non-abelian group G , we have $\text{Inn } G \cong G$. The result follows easily from two previous results. The first, due to Fröhlich [2] states that for a finite simple non-abelian group G , we have $I(G) \cong M_0(G)$. The second states that if the group $(K, +)$ is a direct sum of non-abelian simple groups, then any normal subgroup of K is a direct sum of a subset of the factors (Scott [12]). In particular any chief factor of K will be isomorphic to one of the non-abelian simple factors. But $M_0(G)$ is a direct sum of copies of G . Hence every chief factor of $(M_0(G), +)$ is a copy of G . This identifies G from $M_0(G)$ and so gives the result.

We next present a result due to Syskin [13] which shows that the situation is far from being simple.

Theorem 2.2. *There are three groups D , G and H , no two of which are isomorphic, all of which have trivial centre, such that $I(D) \cong I(G) \cong I(H)$. These groups are metabelian, i. e. their derived groups are abelian.*

The group D is the direct product of two groups of order 55, each being a non-abelian extension of a group of order 11 by a group of order 5, and G and H are two subgroups of D of index 5 in D .

Corollary 2.3. *The class of metabelian groups is not (I, Inn) constrained.*

The positive result in theorem 2.1 can be extended to a larger class of groups. This is done in Syskin [13] and in Peterson [5]. We present the more general result due to Peterson.

Theorem 2.4. *Let \mathcal{C} be the class of finite perfect groups. Then \mathcal{C} is (I, Inn) constrained.*

In fact Peterson's result is stronger. For G, H in \mathcal{C} he only needs endomorphism near-rings R, S with $R \cong S$, where $I(G) \subseteq R \subseteq A(G)$ and $I(H) \subseteq S \subseteq A(H)$. Syskin's result required all normal subgroups of G and H to be perfect. But his result is about strong constraint.

Theorem 2.5. *Let \mathcal{C} be the class of finite strictly non-abelian groups. Then \mathcal{C} is strongly (I, Inn) constrained.*

This result follows also from a more general result due to Scott [11].

3. MORE RECENT RESULTS

We now come to some recent results which we have obtained and gathered together. The situation that we are considering is that of a group G , a tame endomorphism near-ring R on G with R being distributively generated by a semigroup of endomorphisms A with $A \supseteq \text{Inn } G$. Extra conditions on R and G will be added later.

Gary Peterson has shown that if R has d. c. c. r. (the descending chain condition on right ideals), then G is monogenic if and only if G/G' is monogenic, where G' is the derived group of G . We extend this in two ways. The first consists in removing the d. c. c. r. condition on R , while in theorem 3.4 we weaken the condition slightly and put it on the group instead of R .

Lemma 3.1. *Let G be monogenic. Then G/G' is also monogenic.*

This is immediate, as is the fact that G' is an R -submodule of G . Hence, when considering generalizations of Gary Peterson's result, we only need to show that if G/G' is monogenic then G is monogenic.

Theorem 3.2. *Let G/G' be monogenic and assume that for some $g \in G$ such that G/G' is generated by $g + G'$, we have $\delta_n(G) \subseteq gR$, for some $n \geq 1$, where $\delta_n(G)$ is the n th term of the derived series of G . Then G is monogenic.*

Proof. As G/G' is monogenic we choose $g \in G$ satisfying the hypotheses of the theorem, and we have

$$G = G' + gR.$$

If $G' = \delta_1(G) \subseteq gR$ it follows that $G = gR$ and the result holds. Otherwise we will show by induction on r that

$$(3.3) \quad \delta_r(G)/(\delta_r(G) \cap gR) \cong \delta_{r+1}(G)/(\delta_{r+1}(G) \cap gR).$$

Since $G = G' + gR$, we have by the homomorphism theorems that

$$G/G' = (G' + gR)/G' \cong gR/(G' \cap gR).$$

We also have

$$G/(G' \cap gR) = (G' + gR)/(G' \cap gR) \cong G'/(G' \cap gR) \oplus gR/(G' \cap gR).$$

Now $G'/(G' \cap gR) \cong (G' + gR)/gR = G/gR$, which is just $\delta_1(G)/(\delta_1(G) \cap gR) \cong \delta_0(G)/(\delta_0(G) \cap gR)$, which is the statement of (3.3) for $r = 0$.

So assume that (3.3) is true for r . Then

$$\delta_r(G)/(\delta_r(G) \cap gR) \cong \delta_{r+1}(G)/(\delta_{r+1}(G) \cap gR).$$

Using the homomorphism theorems, this is equivalent to

$$(\delta_r(G) + gR)/gR \cong (\delta_{r+1}(G) + gR)/gR.$$

Take the derived group on both sides of the equation to obtain

$$(\delta_{r+1}(G) + gR)/gR \cong (\delta_{r+2}(G) + gR)/gR$$

thus establishing the induction.

Now by hypothesis we can find n such that $\delta_n(G) \subseteq gR$. Then $(\delta_n(G) + gR)/gR$ is the trivial group. Using (3.3) we deduce that $\delta_0(G)/gR$ is the trivial group and hence that $gR = G$, the conclusion that we wanted.

The next extension consists in weakening slightly the finiteness condition and putting it on the group rather than the near-ring. We also avoid appealing to general structural results.

Theorem 3.4. *Let G be a group, R a tame endomorphism near-ring on G . Let G/G' be a monogenic R -module. If G satisfies the minimum condition on intersections of maximal submodules, then G/N is a monogenic R module, where N is the intersection of all the maximal submodules of G . Furthermore G is a monogenic R module if either N is finitely generated or if for some g which satisfies $(g + N)R = G/N$ there exists a maximal submodule of G containing gR .*

Proof. Consider the set $\{M_\lambda; \lambda \in \Lambda\}$ of all maximal submodules of G . We divide Λ into two mutually disjoint sets $\Lambda = M \cup L$, where $\lambda \in M$ if and only if G/M_λ is abelian, whereas $\lambda \in L$ if and only if G/M_λ is non-abelian, and hence perfect as it is a minimal R -module which is not abelian. The minimum condition in the hypothesis has as an immediate consequence that $\bigcap\{M_\lambda; \lambda \in \Lambda\}$ is the intersection of a finite number of the M_λ and without loss of generality we will assume that Λ is a minimal such finite set. This fact is really what we need to make the proof work.

By a standard procedure we can say that G/N can be embedded as a subdirect sum of

$$(3.5) \quad \bigoplus_{\lambda \in \Lambda} G/M_\lambda = \left(\bigoplus_{\lambda \in M} G/M_\lambda\right) \bigoplus \left(\bigoplus_{\lambda \in L} G/M_\lambda\right)$$

where $N = \bigcap\{M_\lambda; \lambda \in \Lambda\}$. The first term $\bigoplus_{\lambda \in M} G/M_\lambda$ is abelian by definition of M . Let $\bigcap_{\lambda \in M} M_\lambda = M_A$. Then we can reorganise (3.5) as G/N being embedded as a subdirect sum of

$$G/M_A \bigoplus \bigoplus_{\lambda \in L} G/M_\lambda.$$

As remarked above each G/M_λ for $\lambda \in L$ is perfect and hence so is $\bigoplus_{\lambda \in L} G/M_\lambda$. We know that G/M_A is a monogenic R -module: as $M_A \supseteq G'$, since G/M_A is

abelian, and as G/G' is monogenic, it follows that G/M_A is the homomorphic image of a monogenic module, hence monogenic.

Similarly define $M_P = \bigcap_{\lambda \in L} M_\lambda$. Then G/M_P is perfect as a direct sum of a finite number of perfect groups is perfect, and G/N is embedded as a subdirect sum of G/M_A and G/M_P . As one of these groups is abelian and the other is perfect, neither has a homomorphic image isomorphic to a homomorphic image of the other, except for the trivial group. So it follows from lemma 3.6 that $G/N \cong G/M_A \oplus G/M_P$.

We turn our consideration to G/M_P now. We first show that if $M_\lambda \neq M_\mu$, $\lambda, \mu \in L$, then G/M_λ is not isomorphic to G/M_μ . Choose $g \in M_\lambda \setminus M_\mu$, and let $\tau_g \in R$ be the inner automorphism induced by g on G . Then τ_g acts trivially on G/M_λ . We need to show that g can be chosen so that τ_g does not act trivially on G/M_μ . But τ_g acts trivially on G/M_μ only if $g + M_\mu \in Z(G/M_\mu)$. As G/M_μ is perfect, it follows that $Z(G/M_\mu) \subset G/M_\mu$ and so there exists $h + M_\mu \notin Z(G/M_\mu)$. Then $h + M_\mu = k + M_\mu$ for some $k \in M_\lambda$, as $G = M_\lambda + M_\mu$ by the maximality of M_μ . So without loss of generality we can assume that τ_g does not act trivially on G/M_μ and hence $G/M_\lambda \not\cong G/M_\mu$.

The next step is to show by induction on r that

$$G/(M_1 \cap \dots \cap M_r) \cong G/M_1 \oplus \dots \oplus G/M_r$$

where M_1, \dots, M_r are distinct members of $\{M_\lambda; \lambda \in L\}$. Since $G/M_1 \not\cong G/M_2$ and both are minimal, hence simple, R modules, neither can have a non-trivial homomorphic image isomorphic to a non-trivial homomorphic image of the other. Hence

$$G/(M_1 \cap M_2) \cong G/M_1 \oplus G/M_2$$

using lemma 3.6. This starts the induction.

Assume that the result is true for r . So $G/(M_1 \cap \dots \cap M_r) \cong G/M_1 \oplus \dots \oplus G/M_r$. Then $G/(M_1 \cap \dots \cap M_{r+1})$ is a subdirect sum of $G/(M_1 \cap \dots \cap M_r)$ and G/M_{r+1} . Using the induction hypothesis and lemma 3.6, we see that we only need to show that there is no non-trivial homomorphism between G/M_{r+1} and $G/(M_1 \cap \dots \cap M_r) = G/M_1 \oplus \dots \oplus G/M_r$. Since $(M_1 \cap \dots \cap M_r) + M_{r+1} = G$ and $M_1 \cap \dots \cap M_r \cap M_{r+1} \subset M_1 \cap \dots \cap M_r$ and $M_1 \cap \dots \cap M_r \cap M_{r+1} \subset M_{r+1}$ by the minimality of L we can use the method above to obtain $g \in (M_1 \cap \dots \cap M_r) \setminus M_{r+1}$ such that τ_g acts non-trivially on G/M_{r+1} and trivially on $G/(M_1 \cap \dots \cap M_r)$. Thus no homomorphic image of $G/(M_1 \cap \dots \cap M_r)$, on which τ_g would have to act trivially can map onto G/M_{r+1} which is minimal. Hence

$$\begin{aligned} G/(M_1 \cap \dots \cap M_{r+1}) &\cong G/(M_1 \cap \dots \cap M_r) \oplus G/M_{r+1} \\ &\cong G/M_1 \oplus \dots \oplus G/M_r \oplus G/M_{r+1} \end{aligned}$$

by the induction hypothesis. This completes the induction argument.

We are now in the position that

$$G/N \cong G/M_A \bigoplus \bigoplus_{i=1}^n G/M_i$$

where $\bigcap_{i=1}^n M_i = \bigcap_{\lambda \in L} M_\lambda$. Since, for $1 \leq i \leq n$, G/M_i is a minimal R -module and R is tame on G , hence on G/M_i , we can deduce that G/M_i is monogenic for each i , $1 \leq i \leq n$.

Let a generator of G/M_A be $g_0 + M_A$, and of G/M_i be $g_i + M_i$ for $1 \leq i \leq n$. Consider $g = \sum_{i=0}^n g_i$. We claim that $(g + N)R = G/N$. We do this by showing that $g_i + N \in (g + N)R$ for $0 \leq i \leq n$. Since $G/N \cong G/M_A \bigoplus \bigoplus_{i=1}^n G/M_i$, using the standard sequence notation, we have

$$g + N = (g_0 + M_A, g_1 + M_1, \dots, g_n + M_n).$$

Choose j , $1 \leq j \leq n$. Since $\{M_1, \dots, M_n\}$ is minimal, there exists $h \in M_i$ for all i , $0 \leq i \leq n$ except for $i = j$. Then $1 - \tau_h \in R$ and $(g + N)(1 - \tau_h) = g(1 - \tau_h) + N$. Since $h \in M_i$ for $i \neq j$, it follows that $g_i(1 - \tau_h) + M_i = 0 + M_i$. Hence

$$\begin{aligned} (g + N)(1 - \tau_h) &= (g_0(1 - \tau_h) + M_A, g_1(1 - \tau_h) + M_1, \dots, g_n(1 - \tau_h) + M_n) \\ &= (0, \dots, g_j(1 - \tau_h) + M_j, 0, \dots, 0), \end{aligned}$$

and $g_j(1 - \tau_h) + M_j$ is not the zero element of G/M_j . Thus $(g_j(1 - \tau_h) + M_j)R$ contains $g_j + M_j$ since G/M_j is minimal and R is tame on G/M_j . This proves that $g_i + N \in (g + N)R$ for $1 \leq i \leq n$ and hence $g_0 + N \in (g + N)R$. This shows finally that G/N is monogenic.

Suppose that $gR \subset G$ and that for some g satisfying $(g + N)R = G/N$ there exists a maximal submodule M of G containing gR . But $M \supseteq N$ and $(g + N)R = (gR + N)/N \subseteq M/N \subset G/N$, a contradiction to $G/N = (g + N)R$. Hence $gR = G$. Finally suppose that N is finitely generated. Now N , being the intersection of all the maximal submodules of G , is the Frattini submodule. By a straightforward extension of result 7.3.2 of Scott [12], it follows that N consists of those elements of G which can be omitted from a generating set, the non-generators. If N is finitely generated, this means that all of N can be omitted from a generating set for G , and so, as G is generated as R module by $\{g, N\}$, it must be generated by $\{g\}$, i. e. we have $G = gR$.

Lemma 3.6. *Let G be an R subdirect sum of the R groups H and K . Then there exists L an R ideal of H and M an R ideal of K such that*

- (i) $\{0\} + M$ and $L + \{0\}$ are contained in G ;
- (ii) H/L is R isomorphic to K/M .

This lemma is a straightforward extension of result 4.3.1 in Scott [12].

Theorem 3.7. *Let G and H be two groups, with R and S the near-rings generated by A_G and A_H respectively, where $A_G \supseteq \text{Inn } G$, $A_H \supseteq \text{Inn } H$. Let G be a monogenic R module and let H be a monogenic S module and let the minimal condition on annihilators hold for both R and S . If R is isomorphic to S and both G and H are perfect, then $G/Z(G)$ is isomorphic to $H/Z(H)$.*

Proof. This theorem generalizes the result of Peterson [5]. His result assumed that G and H were finite and perfect. The proof follows Peterson's. Since G and H are monogenic, the invertible distributive elements of R (S) are automorphisms of G (H). Let A be the set of invertible distributive elements of R , B those of S . Then the isomorphism ψ from R to S maps A one to one onto B , and $A \supseteq A_G$, $B \supseteq A_H$. Let $K = (\text{Inn } H)\psi^{-1}$. Now $\text{Inn } H \triangleleft \text{Aut } H$. So $\text{Inn } H \triangleleft B$ and $K \triangleleft A$, taking inverse images under ψ . Let $N = \text{Inn } G \cap K$. Let L be the inverse image in G of N under the canonical homomorphism $G \rightarrow G/Z(G) \cong \text{Inn } G$. So $L \supseteq Z(G)$.

We claim that L is an R ideal of G . Let $\ell \in L$, $\sigma \in A$. Then $\tau_\ell \in N$ and $(\tau_\ell)^\sigma = \tau_{\ell\sigma} \in N$ as $N \triangleleft A$, noting that $\text{Inn } G$ is normalized by $\text{Aut } G$. Hence $\tau_{\ell\sigma} \in N \cap \text{Inn } G$ and thus $\ell\sigma \in L$. Also $K \triangleleft A$, $A \supseteq \text{Inn } G$, so $N \triangleleft \text{Inn } G$ and so $L \triangleleft G$. So the claim is proved.

Consider the action of K on G/L . Let $g \in G$, $\sigma \in K$. Then $[\tau_g, \sigma] \in \text{Inn } G \cap K = N$, since $\text{Inn } G \triangleleft A$ and $K \triangleleft A$. Thus $[\tau_g, \sigma] = \tau_\ell$ for some $\ell \in L$ and

$$\tau_{g\sigma} = (\tau_g)^\sigma = \tau_g[\tau_g, \sigma] = \tau_g\tau_\ell = \tau_{g+\ell}.$$

Hence $g\sigma = g + \ell + z$ for some $z \in Z(G)$. But $Z(G) \subseteq L$, so $g\sigma \equiv g \pmod L$ and K acts trivially on G/L . Because G is monogenic, we can find an R module M of G such that $M \supseteq L$ and M is maximal. Thus G/M is a minimal R module on which K acts trivially.

We break off to prove a lemma.

Lemma 3.8. *With the hypotheses of the theorem and the notation used so far in the proof of the theorem, let N be a maximal right ideal of S . If $N \supseteq \text{Ann}(h_1, \dots, h_n)$ but N does not contain the annihilator of any proper subset of $\{h_1, \dots, h_n\}$, then*

$$S/N \cong_S h_n \text{Ann}(h_1, \dots, h_{n-1}) / h_n(\text{Ann}(h_1, \dots, h_{n-1}) \cap N).$$

Proof. Since $N \not\supseteq \text{Ann}(h_1, \dots, h_{n-1})$ and N is maximal, we have $S = N + \text{Ann}(h_1, \dots, h_{n-1})$. So

$$\begin{aligned} S/N &= (N + \text{Ann}(h_1, \dots, h_{n-1})) / N \\ &\cong_S \text{Ann}(h_1, \dots, h_{n-1}) / (N \cap \text{Ann}(h_1, \dots, h_{n-1})). \end{aligned}$$

Now consider the map $S \rightarrow H$ given by $s \mapsto h_n s$. This is an S homomorphism with kernel $\text{Ann}(h_n)$. Let the restriction of this map to $\text{Ann}(h_1, \dots, h_{n-1})$ be denoted by θ_n . Then θ_n maps $\text{Ann}(h_1, \dots, h_{n-1})$ to $h_n \text{Ann}(h_1, \dots, h_{n-1})$ and has kernel $\text{Ann}(h_n) \cap \text{Ann}(h_1, \dots, h_{n-1}) = \text{Ann}(h_1, \dots, h_n)$. So $h_n \text{Ann}(h_1, \dots, h_{n-1}) \cong_S \text{Ann}(h_1, \dots, h_{n-1}) / \text{Ann}(h_1, \dots, h_n)$. As $(N \cap \text{Ann}(h_1, \dots, h_{n-1})) \supseteq \text{Ann}(h_1, \dots, h_n)$, $(N \cap \text{Ann}(h_1, \dots, h_{n-1})) / \text{Ann}(h_1, \dots, h_n)$ is a submodule of $\text{Ann}(h_1, \dots, h_{n-1}) / \text{Ann}(h_1, \dots, h_n)$ whose image under θ_n is $h_n(N \cap \text{Ann}(h_1, \dots, h_{n-1}))$. Taking factors we have

$$\begin{aligned} & h_n \text{Ann}(h_1, \dots, h_{n-1}) / h_n(N \cap \text{Ann}(h_1, \dots, h_{n-1})) \\ & \cong_S \text{Ann}(h_1, \dots, h_{n-1}) / (N \cap \text{Ann}(h_1, \dots, h_{n-1})) \\ & \cong_S S/N \end{aligned}$$

by the above. This finishes the proof of the lemma.

We return to the proof of the theorem. Let $g \in G$ be an R generator of G . Then $G \cong R / \text{Ann}(g)$ and under this isomorphism M corresponds to $\overline{M} / \text{Ann}(g)$, where \overline{M} is a maximal right ideal of R . The isomorphism $\psi : R \rightarrow S$ maps \overline{M} to a maximal right ideal N of S . By lemma 3.8, we can find $h_n \in H$ such that $S/N \cong_S h_n A / h_n(N \cap A)$ where $A = \text{Ann}(h_1, \dots, h_{n-1})$. The map $G/M \rightarrow R/\overline{M}$ is an R isomorphism. So K acts trivially on R/\overline{M} and, using ψ we see that $K\psi = \text{Inn } H$ acts trivially on S/N and hence on $h_n A / h_n(N \cap A)$. So in particular $h_n A / h_n(N \cap A)$ is abelian, and hence so is S/N , hence R/\overline{M} and finally G/M . Thus $M \supseteq G'$ and, as G is perfect, this forces $L = G$, and thus $N = \text{Inn } G$, and $K \supseteq \text{Inn } G$. Similarly $\text{Inn } H \subseteq (\text{Inn } G)\psi$. Hence ψ is an isomorphism from $\text{Inn } G$ onto $\text{Inn } H$. This finishes the proof.

Note that the only place at which we use the fact that G and H are perfect is in the very last part of the proof.

The next stage involves generalizing Syskin's examples. As he was the first to provide an example where the near-rings are isomorphic, but the inner automorphism groups are not, we make the following definition.

Definition 3.9. A set of groups $\{G_\lambda; \lambda \in \Lambda\}$, no two of which have isomorphic inner automorphism groups, but with the property that all near-rings $\{I(G_\lambda); \lambda \in \Lambda\}$ are isomorphic, will be called a **Syskin set**. If there are only two groups in the set, we will use the term **Syskin pair**.

Our main result is to generalise the construction used by Syskin in his paper giving the first example of a Syskin pair. We consider two isomorphic metacyclic groups which are defined as follows.

$$D_1 = \langle a, b; na = mb = 0, b + a = qa + b \rangle$$

$$D_2 = \langle c, d; nc = md = 0, d + c = qc + d \rangle$$

where $(q - 1, n) = 1$, $q^m \equiv 1 \pmod n$ and if we define r by $rq \equiv 1 \pmod n$, then $(r - 1, n) = 1$, and $a + b = b + ra$, $c + d = d + rc$. Note that $(q - 1, n) = 1$ or, equivalently, $(r - 1, n) = 1$, gives $D'_1 = \langle a \rangle$ and $D'_2 = \langle c \rangle$ since $b + a - b - a = (q - 1)a$ or $-a - b + a + b = (r - 1)a$ and $d + c - d - c = (q - 1)c$ or $-c - d + c + d = (r - 1)c$. So in particular D_1/D'_1 and D_2/D'_2 are cyclic. Note also that we do not require either m or n to be prime nor do we require (m, n) to be 1.

The groups that are going to provide what we wish are subgroups of $D = D_1 \oplus D_2$. So we define the group $H(n, m, r, w)$ by

$$H(n, m, r, w) = \langle a, c, b + wd \rangle$$

where w has to satisfy $(r^w - 1, n) = 1$.

Lemma 3.10. *The following formulae hold in D_1 :*

- (i) $ua + vb = vb + ur^v a$ for all positive integers u and v ;
- (ii) $yb + xa = xq^w a + yb$ for all positive integers x and y .

Similar results hold for D_2 with c replacing a and d replacing b .

The proof consists in a straightforward double induction.

Lemma 3.11. *The centres of D_1 , D_2 and $H(n, m, r, w)$ are given by:*

- (i) $Z(D_1) = \{jb; q^j \equiv 1 \pmod n\}$.
- (ii) $Z(D_2) = \{jd; q^j \equiv 1 \pmod n\}$.
- (iii) $Z(H(n, m, r, w)) = \{k(b + wd); q^k \equiv 1 \pmod n\}$.

The proof is again straightforward, and the conditions $(r - 1, n) = 1$ and $(r^w - 1, n) = 1$ are both needed to force the centres to be cyclic. We can also see from the same calculations that $H' = \langle a \rangle \oplus \langle c \rangle = D'$ and that H/H' is cyclic.

Lemma 3.12. *$H/Z(H)$ cannot be isomorphic to $D/Z(D)$.*

Proof. We have $|H/Z(H)| = n^2m/(m/o(q)) = n^2o(q)$, whereas $|D/Z(D)| = n^2m^2/(m^2/o(q)^2) = n^2o(q)^2$, where $o(q)$ is the order of $q \pmod n$. Since q is prime to n , it follows that $o(q)$ is not 1 and hence the two groups have different orders and cannot be isomorphic. In fact it is easily seen that $Z(H)$ is cyclic of order $m/o(q)$, whereas $Z(D)$ is the direct sum of two cyclic groups of order $m/o(q)$.

Theorem 3.13. *$I(D) \cong I(H)$ if, in addition, $(r - r^w, n) = 1$.*

Proof. Note that $H + D_1 = D$ and $H + D_2 = D$. If $\alpha \in I(D)$ annihilates H then α annihilates $H/(D_i \cap H) \cong (H + D_i)/D_i = D/D_i$ and so $D\alpha \subseteq D_i$ for $i = 1, 2$ forcing $D\alpha \subseteq D_1 \cap D_2 = \{0\}$. Hence $I(D)$ acts faithfully on H .

The next stage consists in showing that the generators of $I(D)$, namely the maps τ_a, τ_b, τ_c and τ_d all lie in $I(H)$. In fact we only need to consider τ_b and τ_d as τ_a and τ_c are already in $I(H)$. From earlier calculations or directly from the definition of D , we can see that, if $h = b + wd$ and $g = ia + jb + kc + \ell d$, then

$$g\tau_h = ria + r^w kc + jb + \ell d.$$

First we show that the projection $\varepsilon : D \rightarrow \langle b, d \rangle$ lies in $I(H)$. We have

$$g(\tau_h - 1) = (r - 1)ia + (r^w - 1)kc.$$

As $(r - 1, n) = 1$, there exists x such that $x(r - 1) \equiv -r \pmod{n}$. Then

$$gx(\tau_h - 1) = -ria + x(r^w - 1)kc,$$

so we have

$$\begin{aligned} g(x(\tau_h - 1) + \tau_h) &= (x(r^w - 1)k + r^w k)c + jb + \ell d \\ g(x(\tau_h - 1) + \tau_h)\tau_h &= r^w(x(r^w - 1) + r^w)kc + jb + \ell d \\ g(x(\tau_h - 1) + \tau_h)(\tau_h - 1) &= (r^w - 1)(x(r^w - 1) + r^w)kc. \end{aligned}$$

Since $(r^w - 1, n) = 1$, there exists y such that $y(r^w - 1) \equiv -r^w \pmod{n}$. Then

$$\begin{aligned} g(x(\tau_h - 1) + \tau_h)(y(\tau_h - 1) + \tau_h) &= (-r^w(x(r^w - 1) + r^w) + r^w(x(r^w - 1) + r^w))kc + jb + \ell d \\ &= jb + \ell d \\ &= g\varepsilon. \end{aligned}$$

Next we show that the action of b is also in $I(H)$. We have $g(1 - \varepsilon) = ia + kc$, so $g(1 - \varepsilon)\tau_h = ria + r^w kc$ and $g(1 - \varepsilon)(\tau_h - r^w) = (r - r^w)ia$. Since $(r - r^w, n) = 1$ there exists t with $t(r - r^w) \equiv (r - 1) \pmod{n}$, so

$$g(1 - \varepsilon)(\tau_h - r^w)t = (r - 1)ia,$$

$$g((1 - \varepsilon)t(\tau_h - r^w) + 1) = ria + kc + jb + \ell d = g\tau_b.$$

Finally $g(1 - \varepsilon)(\tau_h - r) = (r^w - r)kc$, so

$$g(1 - \varepsilon)(-t)(\tau_h - r) = (r - 1)kc,$$

$$g((1 - \varepsilon)(-t)(\tau_h - r) + 1) = ia + rkc + jb + \ell d = g\tau_d.$$

Thus the actions of a, b, c and d are all reproduced in $I(H)$ and $I(H) \cong I(D)$.

Finally we show that the various groups $H(n, m, r, w)$ provide us with a large number of groups as w varies, always subject to the conditions laid down in the definition.

Theorem 3.14. *Let $H(n, m, r, w)$ and $H(n, m, r, u)$ be two groups satisfying the conditions in the definition. If $u \neq w$ then the two groups are isomorphic precisely when $uw \equiv 1 \pmod m$.*

Proof. First we point out that we can assume that the two groups have trivial centre. For if $Z(H) \supset \{0\}$, then $H/Z(H) \cong H(n, o(r), r, w)$ and this latter does have trivial centre.

For the purposes of this proof we will use the notation $H(w)$ and $H(u)$, as n , m and r remain the same in both groups. Let θ be the isomorphism from $H(w)$ to $H(u)$. Then θ maps $\langle a \rangle \oplus \langle c \rangle$ to itself as this is just the derived group for both $H(w)$ and $H(u)$. Let $(b + wd)\theta = f(b + ud) + x$ where $x \in \langle a \rangle \oplus \langle c \rangle$. Let $a\theta = ia + jc$, $c\theta = ka + \ell c$. Now using our earlier calculations, we have

$$a + (b + wd) = (b + wd) + ra.$$

$$c + (b + wd) = (b + wd) + r^w c.$$

Taking images under θ and again using earlier calculations we obtain

$$ia + jc + f(b + ud) + x = f(b + ud) + x + ria + rjc.$$

$$ka + \ell c + f(b + ud) + x = f(b + ud) + x + r^w ka + r^w \ell c.$$

But working in D , we obtain

$$ia + jc + f(b + ud) + x = f(b + ud) + x + r^f ia + r^{uf} jc.$$

$$ka + \ell c + f(b + ud) + x = f(b + ud) + x + r^f ka + r^{uf} \ell c.$$

For this to hold we require

- (1) $ri \equiv r^f i \pmod n$,
- (2) $rj \equiv r^{uf} j \pmod n$,
- (3) $r^w k \equiv r^f k \pmod n$ and
- (4) $r^w \ell \equiv r^{uf} \ell \pmod n$.

Since $a\theta$ has to have order n , i or j or both must be prime to n , and k or ℓ or both must be prime to n for $c\theta$ to have order n . Since $(b + wd)\theta$ must have order m we must have f prime to m , and because we are assuming that the two groups have trivial centre, we also have that $m = o(r)$.

First assume that i is prime to n . Then we must have $r - r^f \equiv 0 \pmod n$ by (1) and so $f - 1$ is the order of $r \pmod n$, i. e. m . This forces $f = 1$. By (3) $(r^w - r)k \equiv 0 \pmod n$. As $(r^w - r, n) = 1$ we must have $k \equiv 0 \pmod n$. Hence $(\ell, n) = 1$ and by (4) $r^w - r^u \equiv 0 \pmod n$. But $0 < w, u < m$ and the order of $r \pmod n$ is m . So $r^w \equiv r^u \pmod n$ forces $w = u$.

We now turn to the case when i is not prime to n . Then we must have j prime to n . By (2) $r \equiv r^{fu} \pmod n$. As before that means that $fu \equiv 1 \pmod m$. By (4) $(r^w - r)\ell \equiv 0 \pmod n$. Since $r^w - r$ is prime to n , it follows that $\ell \equiv 0 \pmod n$. Hence k is prime to n and by (3) $r^w - r^f \equiv 0 \pmod n$. As before this forces $f = w$. So $f(b + ud) = wb + d$. It is now clear that we can get an isomorphism between $H(w)$ and $H(u)$ where $wu \equiv 1 \pmod m$, by effectively transposing D_1 and D_2 . This gives us the result that we want.

We now give some examples of Syskin sets.

Example 3.15. In the first place we are not free to choose m and n arbitrarily for if $mn = \prod_1^t p_i^{a_i}$ with $1 \leq a_i \leq 2$ and mn is relatively prime to $\prod_1^t p_i^{a_i} - 1$ then there are no non-abelian groups of order mn (L. Redei [8]). If we take $n = 9$, $m = 6$, $q = 5$, $r = 2$ or $q = 8$, $r = 8$, there are no w satisfying the required conditions. In case n is prime, any $w < o(r)$ will do. The Syskin examples use $n = 11$, $m = 5$, $q = 4$, $r = 3$ and $w = 2$, $w = 4$ to get two non-isomorphic subgroups H_1, H_2 of D so that $I(H_1) \cong I(H_2) \cong I(D)$. Using $w = 3$ gives nothing new by theorem 3.14. To get smaller examples use $n = 7$, $m = 3$, $q = 2$, $r = 4$ which gives D a group of order 9×49 and only one possible value of $w (= 2)$ for a subgroup H of order 3×39 .

Examples 3.16. Let $n = p^t$, $m = \phi(n)$ and let q be a primitive root mod m . Then r is also a primitive root mod m , so for each k ; $1 \leq k \leq p - 1$ there exists a w with $r^w = kp + 1$, i. e. $(r^w - 1, n) = p \neq 1$. It follows that $(r^{w+1} - r, n) = p \neq 1$. So by excluding these $2p - 1$ values of w we have the ones which create a Syskin set. For example if $n = 25$, $m = 20$, $q = 3$, $r = 17$, then $r^{4t} \equiv 1 \pmod 5$ and $r^{4t+1} \equiv r \pmod 5$. So we can choose $w \in \{2, 3, 6, 7, 10, 11, 14, 15, 18, 19\}$. Only one pair satisfies the condition in theorem 3.14, $7 \cdot 3 \equiv 1 \pmod 20$ which leaves 9 subgroups and D itself for a Syskin set of 10 groups.

Finally we show that arbitrarily large finite Syskin sets exist.

Example 3.17. Choose n to be a large prime, and choose $m = n - 1$. Then the multiplicative group of non-zero integers mod n is cyclic and a suitable r can be chosen. Because n is prime all the necessary conditions on r and q are satisfied. Finally let w run through all even numbers up to m . These will give groups $\{H(n, m, r, w); 2 \leq w \leq m, 2|w\}$ no two of which are isomorphic since we cannot have $wu \equiv 1 \pmod m$ in this case. So we have a Syskin set consisting of $(m + 2)/2$ groups.

The final topic that we will consider is the direct sum of two groups. Let $G = H \oplus K$ and let S be a semigroup of endomorphisms of G that contains $\text{Inn } G$. Let $N_G, N_H,$ and N_K be the near-rings generated respectively by S, S_H and S_K respectively, where we are assuming that H and K are S invariant, and S_H, S_K are the restrictions of S to H and K respectively. We will use a similar notation for the restrictions of elements s of S to H and to K . Of course we are particularly interested in $I(G), I(H)$ and $I(K)$. Note that $(\text{Inn } G)_H$ contains $\text{Inn } H$ and $(\text{Inn } G)_K$ contains $\text{Inn } K$.

Define the homomorphisms π_H and π_K as the natural homomorphisms from S_H and S_K respectively into $\text{End } H$ and $\text{End } K$ respectively. Write $\overline{S_H}$ for $S_H\pi_H$ and $\overline{S_K}$ for $S_K\pi_K$. Let $T = \overline{S_H} \otimes \overline{S_K}$, the multiplicative semigroup direct product of $\overline{S_H}$ and $\overline{S_K}$. We can map $S \rightarrow T$ by a map θ defined by

$$s\theta = (s_H\pi_H, s_K\pi_K).$$

It is easy to see that θ is a monomorphism, and that

$$S\theta \subseteq T \subseteq \text{End } G.$$

Indeed $S\theta$ is a subdirect product of $\overline{S_H}$ and $\overline{S_K}$. Note that if $g = h + k, h \in H, k \in K$, then $\tau_g = (\tau_h, \tau_k)$. Write $M = N_H \oplus N_K$ and write N for the near-ring generated by $S\theta$. Any element n of N_G is a word in the elements $\{s_i; s_i \in S, 1 \leq i \leq n\}$, say $w(s_i)$. Let $s_i^H = s_i\pi_H, s_i^K = s_i\pi_K$. Then $s_i\theta = (s_i^H, s_i^K)$. Similarly let $s\theta = (s^H, s^K)$ for any $s \in S$. Then $g = h + k, h \in H, k \in K$, gives us $gs = hs + ks = hs_H + ks_K = hs^H + ks^K$. So

$$\begin{aligned} gw(s_i) &= g(\sum \varepsilon_i s_i) = \sum \varepsilon_i gs_i \\ &= \sum \varepsilon_i (hs_i^H + ks_i^K) = \sum \varepsilon_i hs_i^H + \sum \varepsilon_i ks_i^K \end{aligned}$$

since elements of H commute additively with elements of K ,

$$= hw(s_i^H) + kw(s_i^K).$$

Hence the map $w(s_i) \rightarrow (w(s_i^H), w(s_i^K))$ gives an embedding of N_G in $N_H \oplus N_K = M$. It is easy to see that N is isomorphic to N_G

It is not difficult to see that if $S = \text{End } G$ then $S = T$ and $N = M$. But it is possible to get proper containment, particularly if $S \subseteq \text{Aut } G$. For instance if $H = K$ are both abelian of exponent d and $S = \text{Inn } G$, then $N = Z_d$ and $M = Z_d \oplus Z_d$. In order to have $N = M$ it is enough to show that $\overline{S_H} \otimes \{0\} \subseteq N$. Let $\alpha \in \overline{S_H}$, and assume that H and K are groups with finite exponents which are coprime. Then there exists $\beta \in \overline{S_K}$ such that (α, β) is in $S\theta$. Since the exponents of H and K are finite and coprime, the additive orders of α and β are coprime, say a and

b. Then there exists x and y such that $xa + yb = 1$. So $yb(\alpha, \beta) = (yb\alpha, yb\beta) = (\alpha, 0) \in N\theta$. A similar process will show that for any β in $\overline{S_K}$, we have $(0, \beta) \in N\theta$. Hence $N = M$. We have proved the following theorem.

Theorem 3.18. *Let $G = H \oplus K$ where H and K are groups of finite coprime exponent. Let S be a semigroup of endomorphisms of G containing $\text{Inn } G$. In the notation developed above we have $N_G \cong N_H \oplus N_K$.*

We apply this to obtain a large number (infinite in fact) of Syskin sets.

Theorem 3.19. *Let $\{H_\lambda; \lambda \in \Lambda\}$ be a Syskin set in which all groups have finite exponent. Let K be a group of finite exponent prime to the exponents of all members of the Syskin set. Then $\{H_\lambda \oplus K; \lambda \in \Lambda\}$ is a Syskin set.*

Proof. Let $G_i = H_i \oplus K$ be two members of the family of groups $\{H_\lambda \oplus K; \lambda \in \Lambda\}$ for $i = 1, 2$. By theorem 3.18 we have $I(G_i) \cong I(H_i) \oplus I(K)$ for $i = 1, 2$. Since $I(H_1) \cong I(H_2)$ by hypothesis, it follows that $I(G_1) \cong I(G_2)$. Now $\text{Inn } G_i \cong \text{Inn } H_i \oplus \text{Inn } K$. Any isomorphism between the two groups $\text{Inn } G_1$ and $\text{Inn } G_2$ must map $\text{Inn } H_1$ to $\text{Inn } H_2$ because of the properties of the exponent. But by hypothesis there cannot be such an isomorphism, and this proves the theorem.

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