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MULTIPLICITY RESULTS FOR DOUBLE EIGENVALUE PROBLEMS INVOLVING THE p-LAPLACIAN

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Abstract. The existence of multiple nontrivial solutions for two types of double eigenvalue problems involving the p-Laplacian is derived. To prove the existence of at least two nontrivial solutions we use a Ricceri-type three critical point result for non-smooth functions of S. Marano and D. Motreanu [12]. The existence of at least three nontrivial solutions is shown by combining a result of B. Ricceri [17] and a Pucci-Serrin mountain pass type theorem of S. Marano and D. Motreanu [12].

1. INTRODUCTION

Let $h_p : \mathbb{R}^N \to \mathbb{R}^N$ be the homeomorphism defined by $h_p(x) = |x|^{p-2}x$ for all $x \in \mathbb{R}^N$, where $p > 1$ is fixed and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$.

For $T > 0$, let $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ be a mapping satisfying:

(F_1) for each $M > 0$ there exists some $\alpha_M \in L^1(0, T)$ such that, for a.e. $t \in [0, T]$ and all $x, y \in B_M = \{\xi \in \mathbb{R}^N : |\xi| \le M\}$, it holds

$$|F(t, x) - F(t, y)| \le \alpha_M(t)|x - y|;$$

(F_2) the mapping $F(\cdot, x) : [0, T] \to \mathbb{R}$ is measurable for each $x \in \mathbb{R}^N$ and $F(\cdot, 0) \in L^1(0, T)$;

(F_3) $\lim\limits_{|x|\to\infty} \dfrac{F(t, x) - F(t, 0)}{|x|^p} \le 0$ uniformly for a.e. $t \in [0, T]$.

Let $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty, +\infty]$ be a function having the following properties:

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(J_1) $D(j) = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : j(x, y) < +\infty\} \neq \emptyset$ is a closed convex cone with $D(j) \neq \{(0, 0)\}$;

(J_2) j is a convex and lower semicontinuous (shortly, l.s.c.) function.

Let $\gamma > 0$ be arbitrary. For $\lambda, \mu > 0$ we consider the following double eigenvalue problem involving the p-Laplacian operator:

$$(P_{\lambda,\mu}) \qquad \begin{cases} -[h_p(u')]' + \gamma h_p(u) \in \lambda \bar{\partial} F(t, u) \text{ a.e. } t \in [0, T], \\ \Big(h_p(u')(0), -h_p(u')(T)\Big) \in \mu \partial j(u(0), u(T)), \end{cases}$$

where $u : [0, T] \to \mathbb{R}^N$ is of class C^1 and $h_p(u')$ is absolutely continuous. Note, that $\bar{\partial} F(t, \eta)$ denotes the generalized gradient (in the sense of Clarke) of $F(t, \cdot)$ at $\eta \in \mathbb{R}^N$, while ∂j denotes the subdifferential of j in the sense of convex analysis.

Our approach to problem $(P_{\lambda,\mu})$ is a variational one and it relies on results concerning Motreanu-Panagiotopoulos type functionals (see for example in [13] and [14]), which are extensions of the critical point theory of Szulkin type functionals [18].

Previous results concerning p-Laplacian systems with various types of boundary conditions have been obtained by R. Manásevich and J. Mawhin [8], [9], J. Mawhin [10], [11], L. Gasinski and N. Papageorgiu [4], P. Jebelean and G. Moroşanu [6], [7]. As far as we know, eigenvalue problems for differential inclusions involving the p-Laplacian and having mixed boundary conditions where not studied yet. Eigenvalue problems with no boundary conditions were investigated in the books [13],[14] (see also the references therein).

In order to obtain the *existence of multiple solutions* for problem $(P_{\lambda,\mu})$ we impose some further assumptions on F:

(F_4) $\displaystyle\lim_{|x|\to 0} \frac{F(t, x) - F(t, 0)}{|x|^p} \leq 0$ uniformly for a.e. $t \in [0, T]$;

(F_5) there exists $s_0 \in \mathbb{R}^N$ such that $\displaystyle\int_0^T \Big(F(t, s_0) - F(t, 0)\Big) dt > 0$.

P. Jebelean and G. Moroşanu [6] proved the existence of a nontrivial solution for a differential inclusion problem of the type $(P_{\lambda,\mu})$ by using ”mountain pass theorems”. Our paper completes their results by proving the *existence of at least two nontrivial solutions* for a first type of double eigenvalue problem and the *existence of at least three nontrivial solutions* for a second type of double eigenvalue problem. For this, we need assumptions on the behavior around zero and close to infinity of the function F (see (F_3), (F_4), (F_5)). The two types of problems $(P_{\lambda,\mu})$ rely on different assumptions for the function j, and for this reason we use different tools for their investigation.

The main tool for the first type problem is a Ricceri-type three critical point result for non-smooth functions of S. Marano and D. Motreanu [12, Theorem 3.1]. For the second type problem we use a recent result of B. Ricceri [17, Theorem 4] concerning the existence of multiple solutions and a Pucci-Serrin mountain pass type theorem of S. Marano and D. Motreanu [12, Corollary 2.1].

This paper is organized as follows: in Section 2, there are introduced some notations and important preliminary results for problem $(P_{\lambda,\mu})$. Then, in Section 3 it is proved the existence of at least two nontrivial solutions for the first type double eigenvalue problem $(P_{\lambda,\mu})$ and in Section 4 we complete the results of Section 3 by showing the existence of at least three nontrivial solutions for the second type double eigenvalue problem $(P_{\lambda,\mu})$. Finally, Section 5 contains important results from variational calculus concerning the critical point theory, which are used in our investigations.

2. Notations and Preliminary Results

Let $W^{1,p} = W^{1,p}(0,T;\mathbb{R}^N)$ be the usual Sobolev space equipped with the norm

$$\|u\|_\eta = \Big(\|u'\|_{L^p}^p + \eta\|u\|_{L^p}^p\Big)^{1/p},$$

where $\eta > 0$, and $\|\cdot\|_{L^p}$ is the norm of $L^p = L^p(0,T;\mathbb{R}^N)$

$$\|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt\right)^{1/p}.$$

We consider $C = C([0,T];\mathbb{R}^N)$ endowed with the norm

$$\|u\|_C = \max\{|u(t)| : t \in [0,T]\}.$$

For $\gamma > 0$, we consider $\varphi_\gamma : W^{1,p} \to \mathbb{R}$ defined by

$$\varphi_\gamma(u) := \frac{1}{p}\Big(\|u'\|_{L^p}^p + \gamma\|u\|_{L^p}^p\Big) \text{ for all } u \in W^{1,p}.$$

Note, that φ_γ is convex and $\varphi_\gamma \in C^1(W^{1,p};\mathbb{R})$ with

$$\langle \varphi_\gamma'(u), v\rangle = \int_0^T (h_p(u'), v')dt + \gamma\int_0^T (h_p(u), v)dt \text{ for all } u, v \in W^{1,p}.$$

We define the function $J : W^{1,p} \to]-\infty, +\infty]$ by

$$J(u) = j(u(0), u(T)) \text{ for all } u \in W^{1,p}.$$

J is a proper, convex and l.s.c. function. Note, that

$$D(J) = \{u \in W^{1,p} : (u(0), u(T)) \in D(j)\}.$$

We introduce the constant $\gamma_1 = \gamma_1(p,\gamma) > 0$ by setting

$$\gamma_1 = \inf\left\{ \frac{\|u'\|_{L^p}^p + \gamma\|u\|_{L^p}^p}{\|u\|_{L^p}^p} : u \in W^{1,p} \setminus \{0\}, u \in D(J) \right\}.$$

By computation one has

$$2^{-1/p}\|u\|_{\gamma_1} \le (\|u'\|_{L^p}^p + \gamma\|u\|_{L^p}^p)^{1/p} \le \|u\|_{\gamma_1} \text{ for all } u \in D(J). \tag{2.1}$$

We consider the functional $\hat{\mathcal{F}} : C \to \mathbb{R}$ defined by

$$\hat{\mathcal{F}}(v) = -\int_0^T F(t,v)dt + \int_0^T F(t,0)dt \text{ for all } v \in C$$

and $\mathcal{F} : W^{1,p} \to \mathbb{R}$ defined by $\mathcal{F} = \hat{\mathcal{F}}\Big|_{W^{1,p}}$. The functional $\mathcal{F}$ is sequentially weakly continuous, since the embedding $W^{1,p} \hookrightarrow C$ is compact.

Note that for $1 \le r < p$ and $p < q < p^*$ the embeddings $L^p \hookrightarrow L^r$, $W^{1,p} \hookrightarrow L^q$, $W^{1,p} \hookrightarrow C$ are continuous, hence there exist constants $C_{r,p}, \hat{C}_{q,p}, \hat{c} > 0$ such that

$$\|u\|_{L^r} \le C_{r,p}\|u\|_{L^p}, \quad \|u\|_{L^q} \le \hat{C}_{q,p}\|u\|_{W^{1,p}}, \ \|u\|_C \le \hat{c}\|u\|_{W^{1,p}} \text{ for all } u \in W^{1,p}.$$

Let $\mathcal{E} : [0,\infty) \times [0,\infty) \times W^{1,p} \to]-\infty,\infty]$ be defined by

$$\mathcal{E}(\lambda,\mu,u) = \varphi_\gamma(u) + \lambda\mathcal{F}(u) + \mu J(u).$$

The functional $\mathcal{E}$ is of Motreanu-Panagiotopoulos type.

Proposition 2.1. [6, Proposition 3.2]. *Assume that* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies* (F_1) *and* (F_2) *and* $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty,+\infty]$ *satisfies* (J_1) *and* (J_2). *If* $u \in W^{1,p}$ *is a critical point of* $\mathcal{E}(\lambda,\mu,\cdot)$ *(in the sense of Definition 5.1), then* u *is a solution of* $(P_{\lambda,\mu})$.

Remark 2.1. Let $\varepsilon > 0$ be arbitrary. From (F_1), (F_2) and (F_3) it follows that there exists $\delta_1 > 0$ (depending on ε) such that

$$F(t,x) - F(t,0) \le \varepsilon|x|^p + \alpha_{\delta_1}(t)\delta_1 \quad \text{for all } x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T].$$

Then

$$\mathcal{F}(u) \ge -\varepsilon\|u\|_{L^p}^p - \delta_1\|\alpha_{\delta_1}\|_{L^1(0,T)} \quad \text{for all } u \in W^{1,p}. \tag{2.2}$$

Proposition 2.2. *Assume that* $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *satisfies* (F_1), (F_2) *and* (F_3) *and that* $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty, +\infty]$ *satisfies* (J_1) *and* (J_2). *Then the following properties hold:*

(1) $\mathcal{E}(\lambda, \mu, \cdot)$ *is weakly sequentially lower semicontinuous on* $W^{1,p}$ *for each* $\lambda > 0$, $\mu \geq 0$;

(2) $\lim\limits_{\|u\|_{\gamma_1} \to +\infty} \mathcal{E}(\lambda, \mu, u) = +\infty$ *for each* $\lambda > 0, \mu \geq 0$;

(3) $\mathcal{E}(\lambda, \mu, \cdot)$ *satisfies the* (PS) *condition for each* $\lambda, \mu > 0$.

Proof. (1) The function $\mathcal{E}(\lambda, \mu, \cdot)$ is weakly sequentially l.s.c on $W^{1,p}$, because $\mathcal{F}$ is weakly sequentially l.s.c., while φ_γ and J are convex and l.s.c., hence they are also weakly sequentially l.s.c.

(2) First observe that

$$\|u\|_{L^p}^p \leq \frac{1}{\gamma_1} \|u\|_{\gamma_1}^p \text{ for all } u \in W^{1,p}.$$

In (2.2) we choose $\varepsilon < \frac{\gamma_1}{2\lambda p}$. Using that the embedding $L^p \hookrightarrow L^1$ is continuous and that (2.1) holds, we have for all $u \in D(J)$

$$\begin{aligned} \mathcal{E}(\lambda, \mu, u) &\geq \frac{1}{p}\Big(\|u'\|_{L^p}^p + \gamma \|u\|_{L^p}^p\Big) - \lambda\varepsilon \|u\|_{L^p}^p - \lambda \delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} + \mu J(u) \\ &\geq \frac{\gamma_1 - 2\varepsilon\lambda p}{2\gamma_1 p} \|u\|_{\gamma_1}^p - \lambda\delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} + \mu J(u). \end{aligned}$$

Since J is convex and l.s.c. it is bounded from below by an affine functional and then there exist constants $c_1, c_2, c_3 > 0$ such that for all $u \in D(J)$

$$\mathcal{E}(\lambda, \mu, u) \geq \frac{\gamma_1 - 2\varepsilon\lambda p}{2\gamma_1 p} \|u\|_{\gamma_1}^p - \lambda\delta_1 \|\alpha_{\delta_1}\|_{L^1(0,T)} - c_1|u(0)| - c_2|u(T)| - c_3.$$

By the continuity of the embedding $W^{1,p} \hookrightarrow C$ we have for all $u \in W^{1,p}$

$$\mathcal{E}(\lambda, \mu, u) \geq c_4 \|u\|_{\gamma_1}^p - c_5 \|u\|_{\gamma_1} - c_6,$$

where $c_4, c_5, c_6 > 0$ are constants. Since, $1 < p$ it follows that $\mathcal{E}(\lambda, \mu, \cdot) \to +\infty$ when $\|u\|_{\gamma_1} \to +\infty$.

(3) Let $\{u_n\}$ in $W^{1,p}$ be a sequence satisfying $\mathcal{E}(\lambda, \mu, u_n) \to c$ and

$$\lambda \mathcal{F}^0(u_n; v - u_n) + \varphi_\gamma(v) - \varphi_\gamma(u_n) + \mu J(v) - \mu J(u_n) \geq -\varepsilon_n \|v - u_n\|_{\gamma_1}, \forall v \in W^{1,p},$$

where $\{\varepsilon_n\} \subset [0, \infty)$ with $\varepsilon_n \to 0$. We have a subsequence $\{u_n\} \subset D(J)$ (we just eliminate the finite number of elements of the sequence which do not belong to $D(J)$), since $\mu > 0$ and $\mathcal{E}(\lambda, \mu, u_n) \to c$.

But $\mathcal{E}(\lambda, \mu, \cdot)$ is coercive, this implies that $\{u_n\}$ is bounded in $W^{1,p}$. The embedding $W^{1,p} \hookrightarrow C$ is compact, then we can find a subsequence, which we still denote by $\{u_n\}$, which is weakly convergent to a point $u \in W^{1,p}$ and strongly in C.

In the above inequality we take $v = u_n + s(u - u_n)$, with $s > 0$, then divide both sides of the inequality by s and let $s \searrow 0$, to obtain

$$\lambda \mathcal{F}^0(u_n; u - u_n) + \varphi'_\gamma(u_n; u - u_n) + \mu J'(u_n; u - u_n) \geq -\varepsilon_n \|u - u_n\|_{\gamma_1}, \qquad \forall n \in \mathbb{N}.$$

By the upper semicontinuity of $\hat{\mathcal{F}}^0$ (see [14], Chapter 1), it follows that

$$\liminf_{n\to\infty} \Big(\varphi'_\gamma(u_n; u - u_n) + \mu J'(u_n; u - u_n) \Big) \geq 0.$$

By Lemma 4.1 in [6] it follows that $\{u_n\}$ converges strongly to $u \in W^{1,p}$. ■

Remark 2.2. From (F_1), (F_2), (F_3) and (F_4) it follows that for each $\varepsilon > 0$ there exist $\delta_\varepsilon, \bar{\delta}_\varepsilon > 0$ such that

$$F(t,x) - F(t,0) \leq \varepsilon |x|^p + \frac{\alpha_{\delta_\varepsilon}(t)}{\bar{\delta}_\varepsilon^{r-1}} |x|^r \qquad \text{for all } x \in \mathbb{R}^N, \text{ a.e. } t \in [0,T],$$

where $r \geq 1$. Then, by using the continuity of the embedding $W^{1,p} \hookrightarrow C$ we get

$$\mathcal{F}(u) \geq -\varepsilon \|u\|^p_{L^p} - \frac{\hat{c}^r \|\alpha_{\delta_\varepsilon}\|_{L^1(0,T)}}{\bar{\delta}_\varepsilon^{r-1}} \|u\|^r_\gamma \qquad \text{for all } u \in W^{1,p}. \tag{2.3}$$

Remark 2.3. If $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies (F_1) and (F_4), then $0 \in \bar{\partial} F(t,0)$ for a.e. $t \in [0,T]$. In order to prove this property, let $x \in \mathbb{R}^N$ be fixed. From (F_4) it follows that there exists $\delta > 0$ such that

$$F(t,z) - F(t,0) \leq |z|^p \text{ for each } |z| < \delta \text{ and a.e. } t \in [0,T]. \tag{2.4}$$

But

$$(-F)^0(t,0;x) = \lim_{\varepsilon \searrow 0} \sup_{\substack{0<|w|<\varepsilon \\ 0<s<\varepsilon}} \frac{-F(t, w+sx) + F(t,w)}{s}.$$

Let $\varepsilon > 0$ be fixed and let $\{w_n\}$ be a sequence in $\mathbb{R}^N$ such that $|w_n| \searrow 0$ and $|w_n| < \varepsilon$ for all $n \in \mathbb{N}$. Then for $0 < s < \varepsilon$ and $n \in \mathbb{N}$ we have

$$\frac{-F(t, w_n + sx) + F(t, w_n)}{s} \leq \sup_{\substack{0<|w|<\varepsilon \\ 0<s<\varepsilon}} \frac{-F(t, w+sx) + F(t,w)}{s}.$$

Since $F(t,\cdot)$ is continuous (see (F_1)), we get for $n \to \infty$

$$\frac{-F(t,sx)+F(t,0)}{s} \leq \sup_{\substack{0<|w|<\varepsilon \\ 0<s<\varepsilon}} \frac{-F(t,w+sx)+F(t,w)}{s},$$

when $0 < s < \varepsilon$. By (2.4) it follows that

$$-s^{p-1}|x|^p \leq \sup_{\substack{0<|w|<\varepsilon \\ 0<s<\varepsilon}} \frac{-F(t,w+sx)+F(t,w)}{s},$$

when s is small enough such that $|sx| < \delta$. Finally we take $\varepsilon \searrow 0$ and get

$$0 \leq (-F)^0(t,0;x) = F^0(t,0;-x) \text{ for all } x \in \mathbb{R}^N.$$

This implies, $0 \in \bar{\partial} F(t,0)$ for a.e. $t \in [0,T]$.

3. First Type Problem

In order to obtain the existence of at least two nontrivial solutions for $(P_{\lambda,\mu})$ we impose some further assumptions on the convex function $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty,+\infty]$ which satisfies (J_1) and (J_2):

(J_3) $j(0,0)=0$, $j(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Theorem 3.1. *Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying $(F_1)-(F_5)$ and let $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty,+\infty]$ be a function satisfying $(J_1)-(J_3)$. Then for each fixed $\mu > 0$, there exists an open interval $\Lambda_\mu \subset]0,+\infty[$ such that for each $\lambda \in \Lambda_\mu$, the problem $(P_{\lambda,\mu})$ has at least two nontrivial solutions.*

Proof. Let $\mu > 0$ be fixed. We define the function $g :]0,+\infty[\to \mathbb{R}$, by

$$g(t) = \sup\{-\mathcal{F}(u) \ : \ \varphi_\gamma(u) + \mu J(u) \leq t\ \}, \quad \text{for all } \ t > 0.$$

Using (2.3) for $r \in]p,p^*[$ it follows that for all $u \in W^{1,p}$ we have

$$-\mathcal{F}(u) \leq \frac{\varepsilon}{\gamma}\|u\|_\gamma^p + \frac{\hat{c}^r \|\alpha_{\delta_\varepsilon}\|_{L^1(0,T)}}{\bar{\delta}_\varepsilon^{r-1}} \|u\|_\gamma^r.$$

Since $p < r$, this implies

$$\lim_{t\to 0^+} \frac{g(t)}{t} = 0.$$

Using (F_5) we define $u_0(t) = s_0$ for a.e. $t \in [0,T]$. Then, $u_0 \in W^{1,p} \setminus \{0\}$ and $-\mathcal{F}(u_0) > 0$. Due to the convergence relation above, it is possible to choose a real number t_0 such that $0 < t_0 < \varphi_\gamma(u_0) + \mu J(u_0)$ and

$$\frac{g(t_0)}{t_0} < [\varphi_\gamma(u_0) + \mu J(u_0)]^{-1} \cdot (-\mathcal{F}(u_0)).$$

We choose $\rho_0 > 0$ such that

$$g(t_0) < \rho_0 < [\varphi_\gamma(u_0) + \mu J(u_0)]^{-1} \cdot (-\mathcal{F}(u_0))t_0. \tag{3.1}$$

We apply Theorem 5.2 to the space $W^{1,p}$, the interval $\Lambda =]0, +\infty[$ and the functions $\mathcal{G}, \mathcal{H} : W^{1,p} \to \mathbb{R}, h : \Lambda \to \mathbb{R}$ defined by

$$\mathcal{G}(u) = \varphi_\gamma(u), \psi(u) = \mu J(u), \mathcal{H}(u) = \mathcal{F}(u), h(\lambda) = \rho_0\lambda.$$

By Proposition 2.2 the assumption (a) from Theorem 5.2 is fulfilled.

We prove now the minimax inequality

$$\begin{aligned}&\sup_{\lambda\in\Lambda} \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big)\\ &< \inf_{u\in W^{1,p}} \sup_{\lambda\in\Lambda} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big).\end{aligned}$$

The function

$$\lambda \mapsto \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big)$$

is upper semicontinuous on Λ. Since

$$\inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big) \le \varphi_\gamma(u_0) + \mu J(u_0) + \lambda\mathcal{F}(u_0) + \rho_0\lambda$$

and $\rho_0 < -\mathcal{F}(u_0)$, it follows that

$$\lim_{\lambda\to+\infty} \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big) = -\infty.$$

Thus we can find $\overline{\lambda} \in \Lambda$ such that

$$\begin{aligned}\beta_1 :&= \sup_{\lambda\in\Lambda} \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda\Big)\\ &= \inf_{u\in W^{1,p}} \Big(\varphi_\gamma(u) + \mu J(u) + \overline{\lambda}\mathcal{F}(u) + \rho_0\overline{\lambda}\Big).\end{aligned}$$

In order to prove that $\beta_1 < t_0$, we distinguish two cases:

I. If $0 \le \overline{\lambda} < \frac{t_0}{\rho_0}$, we have

$$\beta_1 \le \varphi_\gamma(0) + \mu J(0) + \overline{\lambda}\mathcal{F}(0) + \rho_0\overline{\lambda} = \overline{\lambda}\rho_0 < t_0.$$

II. If $\overline{\lambda} \ge \frac{t_0}{\rho_0}$, then we use $\rho_0 < -\mathcal{F}(u_0)$ and the inequality (3.1) to get

$$\eta_1 \le \varphi_\gamma(u_0) + \mu J(u_0) + \overline{\lambda}\mathcal{F}(u_0) + \rho_0\overline{\lambda} \le \varphi_\gamma(u_0) + \mu J(u_0) + \frac{t_0}{\rho_0}(\rho_0 + \mathcal{F}(u_0)) < t_0.$$

From $g(t_0) < \rho_0$ it follows that for all $u \in W^{1,p}$ with $\varphi_\gamma(u) + \mu J(u) \leq t_0$ we have $-\mathcal{F}(u) < \rho_0$. Hence

$$t_0 \leq \inf \{\varphi_\gamma(u) + \mu J(u) : \ -\mathcal{F}(u) \geq \rho_0\}.$$

On the other hand,

$$\begin{aligned} \beta_2 &= \inf_{u\in W^{1,p}} \sup_{\lambda\in\Lambda}(\varphi_\gamma(u) + \mu J(u) + \lambda\mathcal{F}(u) + \rho_0\lambda) \\ &= \inf \{\varphi_\gamma(u) + \mu J(u) : -\mathcal{F}(u) \geq \rho_0\}. \end{aligned}$$

We conclude that

$$\beta_1 < t_0 \leq \beta_2.$$

Hence, assumption (b) from Theorem 5.2 holds. Then, by Theorem 5.2 it follws that there exists an open interval $\Lambda_\mu \subseteq]0,\infty)$ such that for each $\lambda \in \Lambda_\mu$ the function $\varphi_\gamma + \mu J + \lambda\mathcal{F}$ has at least three critical points in $W^{1,p}$. By Proposition 2.1 it follows that these critical points are solutions of $(P_{\lambda,\mu})$. Since $0 \in \bar{\partial}F(t,0)$ for a.e. $t \in [0,T]$, we get that at least two of the above solutions are nontrivial. ■

Remark 3.1. The two conditions from (J_3) can be replaced by

(J_3') $j(x,y) \geq j(0,0)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

Then, all the proofs above can be adapted by considering

$$J(u) = j(u(0),u(T)) - j(0,0).$$

Corollary 3.1. *Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying $(F_1) - (F_5)$ and let $b : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a positive, convex and Gâteaux differentiable function with $b(0,0) = 0$. Assume that $S \subset \mathbb{R}^N \times \mathbb{R}^N$ is a nonempty closed convex cone with $S \neq \{(0,0)\}$, whose normal cone we denote by N_S. Then for each fixed $\gamma, \mu > 0$, there exists an open interval $\Lambda_0 \subset]0,+\infty[$ such that for each $\lambda \in \Lambda_0$, the following problem*

$$(\hat{P}_{\lambda,\mu}) \quad \begin{cases} -[h_p(u')]' + \gamma h_p(u) \in \lambda\bar{\partial}F(t,u) \text{ a.e. } t \in [0,T], \\ (u(0),u(T)) \in S, \\ \Big(h_p(u')(0), -h_p(u')(T)\Big) \in \mu\nabla b(u(0),u(T)) + \mu N_S(u(0),u(T)), \end{cases}$$

has at least two nontrivial solutions.

Proof. The statement follows by applying Theorem 3.1 to the function F and the convex function $j : \mathbb{R}^N \times \mathbb{R}^N \to]-\infty,+\infty]$ defined by

$$j(x,y) = b(x,y) + I_S(x,y), \text{ for all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where

$$I_S(x,y) = \begin{cases} 0, \text{ if } (x,y) \in S \\ +\infty, \text{ if } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \setminus S, \end{cases}$$

is the indicator function of the cone S.

Note, that in this case $D(j) = S$ and j satisfies the conditions $(J_1) - (J_3)$. Moreover,

$$\partial j(x,y) = \nabla b(x,y) + \partial I_S(x,y) = \nabla b(x,y) + N_S(x,y) \text{ for all } (x,y) \in S. \quad \blacksquare$$

Example 3.1. We give an example of a function F that satisfies the assumptions (F_1) to (F_5): Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$F(t,x) = f(t) - \min\{|x|^{p+\alpha}, |x|^{p-\beta} + 1\} \text{ for all } t \in [0,T], x \in \mathbb{R}^N,$$

where $\alpha > 0, \beta \in]0,p[, f \in L^1(0,T)$.

Various possible choices of b and S from Corollary 3.1 recover some classical boundary conditions. For instance:

(a) $b = 0$ and $S = \{(x,x) : x \in \mathbb{R}^N\}$ we get periodic boundary conditions $u(0) = u(T), u'(0) = u'(T)$;

(b) $b = 0$ and $S = \mathbb{R}^N \times \mathbb{R}^N$ we get Neumann type boundary conditions $u'(0) = u'(T) = 0$;

(c) $b(z) = \frac{1}{2}(Az,z)_{R^{2N}}$, $z \in \mathbb{R}^{2N}$, where A is a symmetric, positive $2N \times 2N$ real valued matrix, and $S = \mathbb{R}^N \times \mathbb{R}^N$; we get the following mixed boundary conditions

$$\begin{pmatrix} h_p(u')(0) \\ -h_p(u')(T) \end{pmatrix} = A \begin{pmatrix} u(0) \\ u(T) \end{pmatrix}.$$

For these choices of F, b and S it follows by Corollary 3.1 that for each fixed $\gamma, \mu > 0$, there exists an open interval $\Lambda_0 \subset]0,+\infty[$ such that for each $\lambda \in \Lambda_0$ the problem $(\hat{P}_{\lambda,\mu})$ has at least two nontrivial solutions.

4. Second Type Problem

Theorem 4.1. *Let $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying $(F_1) - (F_5)$ and let $j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a convex function. Then, there exist a non-degenerate compact interval $[a,b] \subset]0,+\infty[$ and a number $\sigma_0 > 0$ such that for every $\lambda \in [a,b]$ there exists $\mu_0 > 0$ such that for each $\mu \in]0,\mu_0[$, the problem $(P_{\lambda,\mu})$ has at least three solutions with norms less than σ_0. Moreover, if $0 \notin \partial j(0,0)$, then these solutions are nontrivial.*

Proof. We define the function $g :]0, +\infty[\to \mathbb{R}$, by

$$g(t) = \sup\{-\mathcal{F}(u) \ : \ \varphi_\gamma(u) \le t\ \}, \quad \text{for all} \quad t > 0.$$

Using (2.3) for $r \in]p, p^*[$ it follows that for all $u \in W^{1,p}$ we have

$$-\mathcal{F}(u) \le \frac{\varepsilon}{\gamma}\|u\|_\gamma^p + \frac{\hat{c}^r \|\alpha_{\delta_\varepsilon}\|_{L^1(0,T)}}{\bar{\delta}_\varepsilon^{r-1}}\|u\|_\gamma^r.$$

Since $p < r$, this implies

$$\lim_{t\to 0^+} \frac{g(t)}{t} = 0.$$

As in the proof of Theorem 3.1, by (F_5) there exists $u_0 \in W^{1,p} \setminus \{0\}$ such that $-\mathcal{F}(u_0) > 0$. Due to the convergence relation above, it is possible to choose a real number t_0 such that $0 < t_0 < \varphi_\gamma(u_0)$ and

$$\frac{g(t_0)}{t_0} < [\varphi_\gamma(u_0)]^{-1} \cdot (-\mathcal{F}(u_0)).$$

We choose $\rho_0 > 0$ such that

$$g(t_0) < \rho_0 < [\varphi_\gamma(u_0)]^{-1} \cdot (-\mathcal{F}(u_0))t_0.$$

We apply Theorem 5.3 to the space $W^{1,p}$, the interval $I =]0, +\infty[$ and the function $\Psi : W^{1,p} \times I \to \mathbb{R}$ defined by

$$\Psi(u, \lambda) = \varphi_\gamma(u) + \lambda\left(\rho_0 + \mathcal{F}(u)\right), \text{ for all } (u, \lambda) \in W^{1,p} \times I$$

and $\Phi : W^{1,p} \to \mathbb{R}$ by

$$\Phi(u) = J(u) \text{ for all } u \in W^{1,p}.$$

Clearly, by Proposition 2.2 $\Psi(\cdot, \lambda)$ and Φ are sequentially weakly l.s.c. for all $u \in W^{1,p}$. Moreover, $\Psi(\cdot, \lambda)$ is continuous (the norm φ_γ and $\mathcal{F}$ are continuous functions), coercive (by Proposition 2.2), and obviously $\Psi(u, \cdot)$ is concave for all $u \in W^{1,p}$.

By the same technique as in the proof of Theorem 3.1 we prove the minimax inequality

$$\sup_{\lambda\in I} \inf_{u\in W^{1,p}} \Psi(u, \lambda) < \inf_{u\in W^{1,p}} \sup_{\lambda\in I} \Psi(u, \lambda).$$

Note, that the role of the function $\varphi_\gamma + J + \lambda\mathcal{F} + \rho_0\lambda$ from Theorem 3.1 is now replaced by $\Psi(\cdot, \lambda)$.

We can apply Theorem 5.3. Fix $\delta > \eta_1$, and for every $\lambda \in I$ denote

$$S_\lambda = \left\{u \in W^{1,p} \ : \ \Psi(u, \lambda) < \delta\right\}.$$

There exists a non-empty open set $I_0 \subset]0, +\infty[$ with the following property: for every $\lambda \in I_0$ there exists $\lambda_0 > 0$, such that for each $\mu \in]0, \mu_0[$, the functional

$$u \to \Psi(u, \lambda) + \mu\Phi(u)$$

has at least two local minima lying in the set S_λ. Let $[a, b] \subset I_0$ be a non-degenerate compact interval.

We prove now the assertion of our theorem: Let $\lambda \in [a, b]$ be a real number. From what stated above, there exists $\mu_0 > 0$ such that for all $\mu \in]0, \mu_0[$ the functional $\mathcal{E}(\lambda, \mu, \cdot)$ admits at least two local minima $u^1_{\lambda,\mu},\ u^2_{\lambda,\mu} \in S_\lambda$, therefore by Proposition 5.1 (for $\mathcal{G}(u) = \lambda\mathcal{F}(u), \psi(u) = \varphi_\gamma(u) + \mu J(u), u \in W^{1,p}$) these are critical points of $\mathcal{E}(\lambda, \mu, \cdot)$.

Observe that

$$S := \bigcup_{\lambda\in[a,b]} S_\lambda \subseteq S_a \cup S_b.$$

Since $\Psi(\cdot, \lambda)$ is coercive (see Proposition 2.2 applied for $\mathcal{E}(\lambda, 0, \cdot)$), the latter sets are bounded, hence S is bounded as well. By choosing $\sigma_0 > \sup\limits_{u\in S} \|u\|_{\gamma_1}$, we get

$$\|u^1_{\lambda,\mu}\|_{\gamma_1}, \|u^2_{\lambda,\mu}\|_{\gamma_1} < \sigma_0.$$

To prove the existence of a third critical point for $\mathcal{E}(\lambda, \mu, \cdot)$, we apply Proposition 5.2 (for $\mathcal{G}(u) = \lambda\mathcal{F}(u) + \varphi_\gamma(u) + \mu J(u), \psi(u) = 0, u \in W^{1,p}$; note that, since J is convex and continuous, it is then also locally Lipschitz), since the (PS) condition holds by Proposition 2.2. Finally, by Proposition 2.1 it follows that these critical points are solutions of $(P_{\lambda,\mu})$.

Obviously, if $0 \notin \partial j(0, 0)$, then each solution is nontrivial. ∎

Example 4.1. We give an example of functions F and j that satisfy the assumptions of Theorem 4.1: Let $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$F(t, x) = -f(t) \cdot \min\{|x|^{p+\alpha}, |x|^{p-\beta} + 1\} \text{ for all } t \in [0, T], x \in \mathbb{R}^N,$$

where $\alpha > 0, \beta \in]0, p[, f \in L^1(0, T; \mathbb{R}_+) \setminus \{0\}$, and let $j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be given by

$$j(x, y) = \max\{|(x, y)-(1, 1)|^a+1, |(x, y)-(1, 1)|^b+1\} \text{ for all } (x, y) \in \mathbb{R}^N\times\mathbb{R}^N,$$

where $a > b \geq 1$ and $(1, 1) \in \mathbb{R}^N \times \mathbb{R}^N$ denotes the vector with all coordinates 1. By Theorem 4.1 it follows that in this case there exist at least three nontrivial solutions for the eigenvalue problem $(P_{\lambda,\mu})$.

5. Appendix - Basic Notions and Results

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $\mathcal{G} : X \to \mathbb{R}$ is called *locally Lipschitz* if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|\mathcal{G}(u_1) - \mathcal{G}(u_2)| \leq L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$. The *generalized directional derivative* of $\mathcal{G}$ at the point $u \in X$ in the direction $z \in X$ is

$$\mathcal{G}^0(u; z) = \limsup_{w \to u, s \to 0^+} \frac{\mathcal{G}(w + sz) - \mathcal{G}(w)}{s}.$$

The *generalized gradient* (in the sense of Clarke [1]) of $\mathcal{G}$ at $u \in X$ is defined by

$$\bar{\partial}\mathcal{G}(u) = \{x^* \in X^* : \langle x^*, x\rangle \leq \mathcal{G}^0(u; x),\ \forall x \in X\},$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X.

Let $\mathcal{G} : X \to \mathbb{R}$ be a locally Lipschitz function, and let $\psi : X \to]-\infty, +\infty]$ be a convex, proper, l.s.c. function.

Definition 5.1. [14]. An element $u \in X$ is said to be a critical point of $\mathcal{E} = \mathcal{G} + \psi$, if

$$\mathcal{G}^0(u; v - u) + \psi(v) - \psi(u) \geq 0, \forall v \in X.$$

In this case, $\mathcal{E}(u)$ is a critical value of $\mathcal{E}$.

In the case of differentiable functions one gets the notion of critical point introduced by A. Szulkin [18].

Definition 5.2. [14]. The functional $\mathcal{E} = \mathcal{G} + \psi$ is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ $(shortly, (PS)_c)$ if every sequence $\{u_n\}$ in X satisfying $\mathcal{E}(u_n) \to c$ and

$$\mathcal{G}^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v - u_n\|, \forall v \in X,$$

for a sequence $\{\varepsilon_n\} \subset [0, \infty)$ with $\varepsilon_n \to 0$, contains a convergent subsequence. If $(PS)_c$ is verified for all $c \in \mathbb{R}$, $\mathcal{E}$ is said to satisfy the Palais-Smale condition $(shortly, \text{(PS)})$.

Proposition 5.1. [12, Proposition 2.1]. *Each local minimum of $\mathcal{E} = \mathcal{G} + \psi$ is necessarily a critical point of $\mathcal{E}$.*

Theorem 5.2. [12, Theorem 3.1]. *Assume that X is a separable and reflexive Banach space, Λ is a real interval, $\mathcal{G}, \mathcal{H} : X \to \mathbb{R}$ are locally Lipschitz functions and $\psi : X \to]-\infty, +\infty]$ is a convex, proper, l.s.c. function, such that:*

(a) *for every* $\lambda \in \Lambda$ *the function* $\mathcal{G} + \psi + \lambda\mathcal{H}$ *fulfils the* (PS) *condition, together with*

$$\lim_{\|u\| \to +\infty} \Big(\mathcal{G}(u) + \psi(u) + \lambda\mathcal{H}(u) \Big) = +\infty;$$

(b) *there exists a continuous concave function* $h : \Lambda \to \mathbb{R}$ *satisfying*

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} \Big(\mathcal{G}(u) + \psi(u) + \lambda\mathcal{H}(u) + h(\lambda) \Big) < \inf_{u \in X} \sup_{\lambda \in \Lambda} \Big(\mathcal{G}(u) + \psi(u) + \lambda\mathcal{H}(u) + h(\lambda) \Big).$$

Then, there is an open interval $\Lambda_0 \subseteq \Lambda$ *such that for each* $\lambda \in \Lambda_0$ *the function* $\mathcal{G} + \psi + \lambda\mathcal{H}$ *has at least three critical points in* X.

The following result is proved by Marano and Motreanu and it generalizes results of P. Pucci, J. Serrin [16]:

Proposition 5.2. [12, Corollary 2.1]. *Let* $I = \mathcal{G} + \psi$ *satisfying the Palais-Smale condition* (PS). *If* $\mathcal{E}$ *has two local minima* $u_0, u_1 \in X$, *then it admits at least three critical points.*

The main tool in our investigations is the result of B. Ricceri [17, Theorem 4], which we state for the reader's convenience in a slightly modified form (adapted for the weak topology), suitable for our purposes:

Theorem 5.3. *Let* X *be a real, reflexive, separable Banach space, let* $I \subseteq \mathbb{R}$ *be an interval, and let* $\Psi : X \times I \to]-\infty, +\infty]$ *be a function satisfying the following conditions:*

(1) $\Psi(x, \cdot)$ *is concave in* I *for all* $x \in X$;

(2) $\Psi(\cdot, \nu)$ *is upper semicontinous, coercive and sequentially weakly lower semicontinuous in* X *for all* $\nu \in I$;

(3) $\eta_1 := \sup_{\nu \in I} \inf_{x \in X} \Psi(x, \nu) < \inf_{x \in X} \sup_{\nu \in I} \Psi(x, \nu) =: \eta_2$.

Then, for each $\delta > \eta_1$ *there exists a non-empty open set* $I_0 \subset I$ *with the following property: for every* $\nu \in I_0$ *and every sequentially weakly l.s.c. function* $\Phi : X \to \mathbb{R}$, *there exists* $\tau_0 > 0$ *such that, for each* $\tau \in]0, \tau_0[$, *the function* $\Psi(\cdot, \nu) + \tau\Phi(\cdot)$ *has at least two local minima lying in the set* $\{x \in X \ : \ \Psi(x, \nu) < \delta\}$.

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