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PERIODIC SOLUTIONS OF DELAY EQUATIONS IN BESOV SPACES AND TRIEBEL-LIZORKIN SPACES

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Abstract. Under suitable assumptions on the Fourier transform of the delay operator F, we give necessary and sufficient conditions for the inhomogeneous abstract delay equations: $u'(t) = Au(t) + Fu_t + f(t)$, $(t \in \mathbb{T})$ to have maximal regularity in Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$.

1. INTRODUCTION

The aim of this paper is to study maximal regularity of the following inhomogeneous abstract delay equations:

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{T} := [0, 2\pi], \tag{1.1}$$

here A is a closed linear operator in a complex Banach space X, $u_t(\cdot) = u(t + \cdot)$ is defined on $[-2\pi, 0]$, $f \in \mathcal{F}(\mathbb{T}, X)$, and $F : \mathcal{F}([-2\pi, 0], X) \to X$ is a bounded linear operator, where $\mathcal{F}$ is an X-valued function space, it may be L^p-spaces, Besov spaces $B^s_{p,q}$ or Triebel-Lizorkin spaces $F^s_{p,q}$. We say that (1.1) has $\mathcal{F}$-maximal regularity, if for each $f \in \mathcal{F}(\mathbb{T}, X)$, there exists a unique function u, such that u is a.e. differentiable, $u(t) \in D(A)$ and (1.1) is satisfied for a.e. $t \in \mathbb{T}$, $u', Au, Fu. \in \mathcal{F}(\mathbb{T}, X)$.

J. K. Hale [7] and G. Webb [14] firstly studied the equation (1.1) for $t \in \mathbb{R}$. In [3], A. Bátkai, E. Fasanga and R. Shvidkoy obtained results on the hyperbolicity of delay equations using the theory of operator-valued Fourier multipliers. In [10],

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Y. Latushkin and F. Räbiger studied stability of linear control systems in Banach spaces. Recently, in [11] C. Lizama obtained necessary and sufficient condition for (1.1) to have L^p-maximal regularity using Fourier multiplier theorems on $L^p(\mathbb{T}, X)$, and C^α-maximal regularity of the corresponding equation on the real line has been studied by C. Lizama and V. Poblete [12]. We note that in the special case when $F = 0$, maximal regularity of (1.1) has been studied by W. Arendt and S. Bu [1, 2] in L^p-spaces case and Besov spaces case, S. Bu and J. Kim [6] in Triebel-Lizorkin spaces case. The corresponding integro-differential equations were treated by V. Keyantuo and C. Lizama [8, 9], S. Bu and Y. Fang [5].

In this paper, we are interested in maximal regularity of (1.1) in Besov spaces $B^s_{p,q}(\mathbb{T}, X)$ and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{T}, X)$. The main results are necessary or sufficient conditions for this problem to have maximal regularity in $B^s_{p,q}(\mathbb{T}, X)$ and $F^s_{p,q}(\mathbb{T}, X)$. The main tools we will use are operator-valued Fourier multiplier results on $B^s_{p,q}(\mathbb{T}, X)$ and $F^s_{p,q}(\mathbb{T}, X)$ established in [2] and [6]. We remark that the sufficient condition for a sequence $M = (M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ to be a $B^s_{p,q}-$multiplier is a Marcinkiewicz condition of order 2 [2] , and in the $F^s_{p,q}$-multiplier case one requires a Marcinkiewicz condition of order 3 [6], while it is well known that in the L^p-multiplier case, only a Marcinkiewicz condition of order 1 is needed when X is UMD spaces [1]. This is the reason that, in contrast with the sufficient condition of L^p-maximal regularity of (1.1) given in [11], we have to impose an extra condition on Fourier transform of delay operator F in our sufficient condition of the maximal regularity of (1.1) in $B^s_{p,q}(\mathbb{T}, X)$ and $F^s_{p,q}(\mathbb{T}, X)$. We will see that this extra condition is not needed in the sufficient condition of the maximal regularity of (1.1) in $B^s_{p,q}(\mathbb{T}, X)$ when the underlying Banach space X is B-convex, as in this case a Marcinkiewicz condition of order 1 is sufficient for a sequence $M = (M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$ to be a $B^s_{p,q}-$multiplier.

It is known that for $0 < \alpha < 1$, the periodic α-Hölder continuous function space $C^\alpha_{per}(\mathbb{T}, X)$ coincides with $B^\alpha_{\infty,\infty}(\mathbb{T}, X)$. Thus actually our result gives necessary and sufficient conditions for the problem (1.1) to have C^α-maximal regularity.

The paper is organized as follows. In Section 2, we consider $B^s_{p,q}$-maximal regularity for (1.1). Section 3 will be devoted to $F^s_{p,q}$-maximal regularity for (1.1).

2. Maximal Regularity on Besov SPaces

Let X be a Banach space. For $f \in L^1(\mathbb{T}; X)$, we denote by

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e_{-k}(t)f(t)dt$$

the k-th Fourier coefficient of f, where $k \in \mathbb{Z}$, $\mathbb{T} = [0, 2\pi]$ (the point 0 and 2π are identified), and $e_k(t) = e^{ikt}$. For $k \in \mathbb{Z}$ and $x \in X$, we denote by $e_k \otimes x$ the X-valued function defined by $(e_k \otimes x)(t) = e_k(t)x$.

Firstly, we briefly recall the definition of periodic Besov spaces in the vector-valued case introduced in [2]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_\alpha = sup_{x\in\mathbb{T}}|f^{(\alpha)}(x)|$ for $\alpha \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Let $\mathcal{D}^{'}(\mathbb{T}, X) := \mathcal{L}(\mathcal{D}(\mathbb{T}), X)$ be the space of all bounded linear operator from $\mathcal{D}(\mathbb{T})$ to X. In order to define Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : \ |t| \le 2\}, \ I_k = \{t \in \mathbb{R} : \ 2^{k-1} < |t| \le 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $supp(\phi_k) \subset \bar{I}_k$ for each $k \in \mathbb{N}_0$,

$$\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1 \qquad \text{for} \quad x \in \mathbb{R},$$

and for each $\alpha \in \mathbb{N}_0$

$$sup_{\substack{x\in\mathbb{R}\\ k\in\mathbb{N}_0}} 2^{k\alpha}|\phi_k^{(\alpha)}(x)| < \infty.$$

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \phi(\mathbb{R})$ be fixed. For $1 \le p, q \le \infty$, $s \in \mathbb{R}$, the X-valued periodic Besov space is defined by

$$\begin{aligned} B^s_{p,q}(\mathbb{T}, X) &= \Big\{f \in \mathcal{D}^{'}(\mathbb{T}, X) : \\ \big\|f\big\|_{B^s_{p,q}} &:= \Big(\sum_{j\ge 0} 2^{sjq}\Big\|\sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k)\Big\|_p^q\Big)^{1/q} < \infty\Big\} \end{aligned}$$

with the usual modification if $q = \infty$. The space $B^s_{p,q}(\mathbb{T}, X)$ is independent from the choice of ϕ and different choices of ϕ lead to equivalent norms $\|\cdot\|_{B^s_{p,q}}$ on $B^s_{p,q}(\mathbb{T}, X)$. $B^s_{p,q}(\mathbb{T}, X)$ equipped with the norm $\|\cdot\|_{B^s_{p,q}}$ is a Banach space. See [2, Section 2] for more information about the space $B^s_{p,q}(\mathbb{T}, X)$. If $f \in B^s_{p,q}(\mathbb{T}, X)$, then we will identify f with its periodic extension to $\mathbb{R}$. In this way, if $r \in \mathbb{R}$ is fixed, we say that a function $f : [r, r + 2\pi] \to X$ is in $B^s_{p,q}([r, r + 2\pi], X)$ if and only if its periodic extension to $\mathbb{R}$ is in $B^s_{p,q}([0, 2\pi], X)$. It is easy to verify from the definition that if $u \in B^s_{p,q}(\mathbb{T}, X)$ and $t_0 \in [0, 2\pi]$ is fixed, then the function u_{t_0} defined on $[-2\pi, 0]$ by $u_{t_0}(t) = u(t_0 + t)$, is still an element of $B^s_{p,q}(\mathbb{T}, X)$, and $\|u_{t_0}\|_{B^s_{p,q}} = \|u\|_{B^s_{p,q}}$.

We consider the equation

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{T} = [0, 2\pi] \tag{2.1}$$

where A is a closed linear operator in X, $f \in B^s_{p,q}(\mathbb{T}, X)$ is given,

$$F : B^s_{p,q}([-2\pi, 0], X) \to X$$

is a bounded linear operator. Moreover, for fixed $t \in \mathbb{T}$, u_t is an element of $B^s_{p,q}([-2\pi, 0], X)$ defined by $u_t(s) = u(t+s)$ for $-2\pi \leq s \leq 0$.

Definition 2.1. Let $1 \leq p, q \leq \infty$, $s > 0$ and $f \in B^s_{p,q}(\mathbb{T}, X)$ is given. A function $u \in B^{s+1}_{p,q}(\mathbb{T}, X)$ is called a strong $B^s_{p,q}$-solution of (2.1), if $u(t) \in D(A)$ and (2.1) holds for a.e. $t \in \mathbb{T}$, $Au \in B^s_{p,q}(\mathbb{T}, X)$ and the function $t \to Fu_t$ also belongs to $B^s_{p,q}(\mathbb{T}, X)$. We say that (2.1) has $B^s_{p,q}$-maximal regularity, if for each $f \in B^s_{p,q}(\mathbb{T}, X)$, (2.1) has a unique strong $B^s_{p,q}$-solution.

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. If $X = Y$, we will simply denote it by $\mathcal{L}(X)$. The main tool in the study of $B^s_{p,q}$-maximal regularity of (2.1) is the operator-valued Fourier multiplier theory established in [2].

Definition 2.2. Let X, Y be Banach spaces, $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We say that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier, if for each $f \in B^s_{p,q}(\mathbb{T}, X)$, there exists $u \in B^s_{p,q}(\mathbb{T}, Y)$, such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

The following result has been obtained in [2]:

Theorem 2.3. *Let X, Y be Banach spaces, $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We assume that*

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k(M_{k+1} - M_k)\|) < \infty \tag{2.2}$$

$$\sup_{k\in\mathbb{Z}} \|k^2(M_{k+2} - 2M_{k+1} + M_k)\| < \infty. \tag{2.3}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier. Moreover, if X and Y are B-convex, then the first order condition (2.2) is sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier.

Recall that a Banach space X is B-convex if it does not contain l^n_1 uniformly. This is equivalent to say that X has Fourier type $1 < p \leq 2$, i.e., the Fourier transform is a bounded linear operator from $L^p(\mathbb{R}, X)$ to $l^q(\mathbb{Z}, X)$, where $1/p + 1/q = 1$. It is well known that when $1 < p < \infty$, then $L^p(\mu)$ has Fourier type $\min\{p, \frac{p}{p-1}\}$.

Let $F \in \mathcal{L}(B^s_{p,q}([-2\pi, 0], X), X)$ and $k \in \mathbb{Z}$, we define the operator B_k by $B_k x = F(e_k x)$ for all $x \in X$. It is clear that $\|B_k\| \leq \|F\|$ as $\|e_k\|_{B^s_{p,q}} \leq 1$. We define the spectrum of (2.1) by

$$\sigma(\Delta) = \big\{k \in \mathbb{Z} : \quad ikI - B_k - A \text{ is not invertible from } D(A) \text{ to } X\big\}. \tag{2.4}$$

Since A is closed, if $k \in \mathbb{Z} \setminus \sigma(\Delta)$, then $(ikI - B_k - A)^{-1}$ is a bounded linear operator on X. This is an easy consequence of the Closed Graph Theorem. We will use the following notations: for $k \in \mathbb{Z}$

$$C_k := ikI - B_k;\ N_k := (C_k - A)^{-1};\ M_k := ikN_k. \tag{2.5}$$

We will need the following preparation.

Lemma 2.4. *Let A be a closed linear operator in a Banach space X. Assume that $\sigma(\Delta) = \emptyset$, $(M_k)_{k\in\mathbb{Z}}$ and $(k(B_{k+1} - 2B_k + B_{k-1})_{k\in\mathbb{Z}}$ are uniformly bounded. Then*

$$\sup_{k\in\mathbb{Z}} \|k^2(N_{k+1} - N_k)\| < \infty, \tag{2.6}$$

$$\sup_{k\in\mathbb{Z}} \|k^3(N_{k+2} - 2N_{k+1} + N_k)\| < \infty. \tag{2.7}$$

Proof. For $k \in \mathbb{Z}$,

$$N_{k+1} - N_k = N_{k+1}(B_{k+1} - B_k - i)N_k.$$

Thus $\sup_{k\in\mathbb{Z}} \|k^2(N_{k+1} - N_k)\| < \infty$ by assumption. To show (2.7), we remark that for $k \in \mathbb{Z}$

$$\begin{aligned} &N_{k+2} - 2N_{k+1} + N_k \\ &= N_{k+2}(B_{k+2} - B_{k+1} - i)N_{k+1} - N_k(B_{k+1} - B_k - i)N_{k+1} \\ &= (N_{k+2} - N_k)(B_{k+2} - B_{k+1} - i)N_{k+1} + N_k(B_{k+2} - 2B_{k+1} + B_k)N_{k+1}. \end{aligned}$$

Hence $\sup_{k\in\mathbb{Z}} \|k^3(N_{k+2} - 2N_{k+1} + N_k)\| < \infty$ by assumption and (2.6). The proof is completed ■

Proposition 2.5. *Let A be a closed linear operator in a Banach space X. Suppose that $\sigma(\Delta) = \emptyset$ and $(k(B_{k+2} - 2B_{k+1} + B_k))_{k\in\mathbb{Z}}$ is uniformly bounded. Then the following assertions are equivalent:*

- (i) *$(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier for all (or equivalently, for some) $1 \le p,\ q \le \infty$ and $s \in \mathbb{R}$.*
- (ii) *$(M_k)_{k\in\mathbb{Z}}$ is uniformly bounded.*

Proof. The implication (i) ⇒ (ii) is trivially true (see e.g. [2]). To show the the converse implication, we assume that the sequence $(M_k)_{k\in\mathbb{Z}}$ is uniformly bounded, we are going to show that $(M_k)_{k\in\mathbb{Z}}$ satisfies the Marcinkiewicz conditions (2.2) and (2.3). We have for $k \in \mathbb{Z}$

$$k(M_{k+1} - M_k) = ik^2(N_{k+1} - N_k) + ikN_{k+1}.$$

Thus $(k(M_{k+1} - M_k))_{k\in\mathbb{Z}}$ is uniformly bounded by assumption and Lemma 2.4. This shows that $(M_k)_{k\in\mathbb{Z}}$ satisfies (2.2). To show that $(M_k)_{k\in\mathbb{Z}}$ also satisfies (2.3), we remark that for $k \in \mathbb{Z}$

$$k^2(M_{k+2} - 2M_{k+1} + M_k) = k^2[i(k+2)N_{k+2} - 2i(k+1)N_{k+1} + ikN_k]$$

$$= ik^3(N_{k+2} - 2N_{k+1} + N_k) + 2ik^2(N_{k+2} - N_{k+1})$$

which is uniformly bounded by assumption and Lemma 2.4. Then the result follows from Theorem 2.3. ∎

When X is B-convex, the first order condition (2.2) is sufficient for a sequence $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier by Theorem 2.3. Thus we have the following

Corollary 2.6. *Let A be a closed linear operator in a B-convex Banach space X. Suppose that $\sigma(\Delta) = \emptyset$. Then the following assertions are equivalent:*

- (i) *$(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier for all (equivalently, for some) $1 \le p,\ q \le \infty$, $s \in \mathbb{R}$.*
- (ii) *$(M_k)_{k\in\mathbb{Z}}$ is uniformly bounded.*

The following is the main result of this section.

Theorem 2.7. *Let A be a closed linear operator defined in a Banach space X. Assume that $(k(B_{k+2} - 2B_{k+1} + B_k))_{k\in\mathbb{Z}}$ is uniformly bounded. Then the following assertions are equivalent:*

- (i) *The problem (2.1) has $B^s_{p,q}$-maximal regularity for all (equivalently, for some) $1 \le p,\ q \le \infty$ and $s > 0$.*
- (ii) *$\sigma(\Delta) = \emptyset$ and $(M_k)_{k\in\mathbb{Z}}$ is uniformly bounded.*

Proof. We notice that when $s > 0$, we have $B^s_{p,q}(\mathbb{T}, X) \subset L^p(\mathbb{T}, X)$ [2]. The implication (i)⇒(ii) follows the same lines in the proof of [1, Theorem 2.3] or [11, Proposition 3.3]. We omit the details.

To show that the implication (ii)⇒(i) is true, we assume that $\sigma(\Delta) = \emptyset$ and $(M_k)_{k\in\mathbb{Z}}$ is uniformly bounded. Then $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier by Proposition 2.5. We let $f \in B^s_{p,q}(\mathbb{T}, X)$ be fixed. Since the sequence $(P_k)_{k\in\mathbb{Z}}$ given by $P_k = (I/ik)$ when $k \neq 0$, and $P_0 = I$ is a $B^s_{p,q}$-multiplier by [2, Theorem 4.5], $(N_k)_{k\in\mathbb{Z}}$ is also a $B^s_{p,q}$-multiplier as the product of two $B^s_{p,q}$-multipliers is still a $B^s_{p,q}$-multiplier, and if we change the value of a $B^s_{p,q}$-multiplier at 0, then the resulting sequence is still a $B^s_{p,q}$-multiplier. There exists $u \in B^s_{p,q}(\mathbb{T}, X)$ such that $\hat{u}(k) = N_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. This implies that $\hat{u}(k) \in D(A)$ and

$$(ikI - A - B_k)\hat{u}(k) = \hat{f}(k), \quad k \in \mathbb{Z}. \tag{2.8}$$

Since $M_k = ikN_k$ is a $B^s_{p,q}$-multiplier by Proposition 2.5, there exists $v \in B^s_{p,q}(\mathbb{T}, X)$ such that

$$\hat{v}(k) = ikN_k\hat{f}(k) = ik\hat{u}(k), \quad k \in \mathbb{Z}.$$

By [1, Lemma 2.1], u is differentiable a.e. and $v = u'$. Therefore $u \in B^{s+1}_{p,q}(\mathbb{T}, X)$ by [2, Theorem 2.3].

We claim that $(B_kN_k)_{k\in\mathbb{Z}}$ is also a $B^s_{p,q}$-multiplier. In fact, B_kN_k is uniformly bounded and

$$k(B_{k+1}N_{k+1} - B_kN_k) = B_{k+1}(kN_{k+1}) - B_k(kN_k)$$

is also uniformly bounded by assumption. On the other hand,

$$\begin{aligned} &k^2(B_{k+2}N_{k+2} - 2B_{k+1}N_{k+1} + B_kN_k) \\ = &k^2B_{k+2}(N_{k+2} - N_{k+1}) + k^2(B_{k+2} \\ &-2B_{k+1} + B_k)N_{k+1} + k^2B_k(N_k - N_{k+1}). \end{aligned}$$

is still uniformly bounded by assumption and Lemma 2.4. Thus $(B_kN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier by Theorem 2.3.

Since $(Fu.)^\wedge(k) = F(e_k\hat{u}(k)) = B_k\hat{u}(k) = B_kN_k\hat{f}(k)$, we obtain that $Fu. \in B^s_{p,q}(\mathbb{T}, X)$ $(B_kN_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.

We have $\hat{u}(k) \in D(A)$ and $A\hat{u}(k) = ik\hat{u}(k) - B_k\hat{u}(k) - \hat{f}(k)$ by (2.8). By [1, Lemma 3.1], we conclude that $u(t) \in D(A)$ and $u'(t) = Au(t) + Fu_t + f(t)$ for a.e. $t \in [0, 2\pi]$ by the uniqueness theorem of Fourier coefficients [1, Page 134], and $Au \in B^s_{p,q}(\mathbb{T}, X)$. Thus u is a strong $B^s_{p,q}$-solution of (2.1). This proves the existence.

To show the uniqueness, let $u \in B^{s+1}_{p,q}(\mathbb{T}, X)$ be such that $u'(t) = Au(t) + Fu_t$, $Fu.,\ Au \in B^s_{p,q}(\mathbb{T}, X)$. Then taking Fourier transform on both sides we obtain that $\hat{u}(k) \in D(A)$ by [1, Lemma 3.1], and $(ik - A - B_k)\hat{u}(k) = 0$ for $k \in \mathbb{Z}$. Since $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$, this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$. This proof is ■

When the underlying Banach space X is B-convex and $1 \le p, q \le \infty, s \in \mathbb{R}$, the first order condition (2.2) is sufficient for the sequence $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier by Theorem 2.3. From this fact and the proof of Theorem 2.7, we easily deduce the following result on $B^s_{p,q}$-maximal regularity of the problem (2.1) when X is B-convex.

Corollary 2.8. *Let X be a B-convex Banach space. Then the following statements are equivalent:*

(i) *the problem (2.1) has $B^s_{p,q}$-maximal regularity for all (equivalently, for some) $1 \le p,\ q \le \infty$ and $s > 0$.*

(ii) $\sigma(\Delta) = \emptyset$ *and* $(M_k)_{k\in\mathbb{Z}}$ *is uniformly bounded.*

Periodic Hölder continuous function space is a particular case of periodic Besov space $B^s_{p,q}(\mathbb{T}, X)$. From [2, Theorem 3.1], we have $B^\alpha_{\infty,\infty}(\mathbb{T}, X) = C^\alpha_{per}(\mathbb{T}, X)$ whenever $0 < \alpha < 1$, where $C^\alpha_{per}(\mathbb{T}, X)$ is the space of all X-valued functions f defined on $\mathbb{T}$ satisfying $f(0) = f(2\pi)$ and $\sup_{x\neq y} \frac{\|f(x)-f(y)\|}{|x-y|^\alpha} < \infty$. Moreover the norm $\|f\|_{C^\alpha_{per}} := \max_{t\in\mathbb{T}} \|f(t)\| + \sup_{x\neq y} \frac{\|f(x)-f(y)\|}{|x-y|^\alpha}$ on $C^\alpha_{per}(\mathbb{T}, X)$ is an equivalent norm of $B^\alpha_{\infty,\infty}(\mathbb{T}, X)$. If $0 < \alpha < 1$, we say that the problem (2.1) has C^α_{per}-maximal regularity if for every $f \in C^\alpha_{per}(\mathbb{T}, X)$, there exists a unique $u \in C^{\alpha+1}_{per}(\mathbb{T}, X)$ such that $u(t) \in D(A)$ and equation (1.1) holds true for all $t \in \mathbb{T}$, and $Au, Fu. \in C^\alpha_{per}(\mathbb{T}, X)$, where $C^{\alpha+1}_{per}(\mathbb{T}, X)$ is the space of all functions $u \in C^1(\mathbb{T}, X)$ such that $u' \in C^\alpha_{per}(\mathbb{T}, X)$. Theorem 2.7 and Theorem 2.8 have the following corollary.

Corollary 2.9. *Let* X *be a Banach space,* $0 < \alpha < 1$. *Then*

1. *if* $(k(B_{k+2} - 2B_{k+1} + B_k))_{k\in\mathbb{Z}}$ *is uniformly bounded, then the problem* (*2.1*) *has* C^α_{per}*-maximal regularity for all* (*equivalently, for some*) $0 < \alpha < 1$ *if and only if* $\sigma(\Delta) = \emptyset$ *and* $(M_k)_{k\in\mathbb{Z}}$ *is uniformly bounded.*
2. *if* X *is B-convex, then the problem* (*2.1*) *has* C^α_{per}*-maximal regularity for all* (*equivalently, for some*) $0 < \alpha < 1$ *if and only if* $\sigma(\Delta) = \emptyset$ *and* $(M_k)_{k\in\mathbb{Z}}$ *is uniformly bounded.*

3. Maximal Regularity on Triebel-lizorkin Space

In this section, we study $F^s_{p,q}$-maximal regularity for (1.1) in Triebel-Lizorkin spaces. We first recall the definition of these spaces and operator-valued Fourier multipliers on them. Let X be a Banach space and let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \phi(\mathbb{R})$ be fixed ($\phi(\mathbb{R})$ was defined in the second section). For $1 \leq p < \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$, the X-valued periodic Triebel-Lizorkin space is defined by

$$F^s_{p,q}(\mathbb{T}, X) = \big\{ f \in \mathcal{D}'(\mathbb{T}, X) :$$
$$\|f\|_{F^s_{p,q}} := \Big\| \Big(\sum_{j\geq 0} 2^{sjq} \Big| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k) \Big|^q \Big)^{1/q} \Big\|_p < \infty \big\}$$

with the usual modification if $q = \infty$. The space $F^s_{p,q}(\mathbb{T}, X)$ is independent from the choice of ϕ and different choices of ϕ lead to equivalent norms $\|\cdot\|_{F^s_{p,q}}$ on $F^s_{p,q}(\mathbb{T}, X)$. $F^s_{p,q}(\mathbb{T}, X)$ equipped with the norm $\|\cdot\|_{F^s_{p,q}}$ is a Banach space. See [6] for more information about the spaces $F^s_{p,q}(\mathbb{T}, X)$. If $f \in F^s_{p,q}(\mathbb{T}, X)$, then we will identify f with its periodic extension to $\mathbb{R}$. In this way, if $r \in \mathbb{R}$ is fixed, we say that a function $f : [r, r + 2\pi] \to X$ is in $F^s_{p,q}([r, r + 2\pi], X)$ if and only if its periodic extension to $\mathbb{R}$ is in $F^s_{p,q}([0, 2\pi], X)$. It is easy to verify from the definition that if $u \in F^s_{p,q}(\mathbb{T}, X)$ and $t_0 \in [0, 2\pi]$ is fixed, then the function u_{t_0} defined on $[-2\pi, 0]$ by $u_{t_0}(t) = u(t_0 + t)$, is still an element of $F^s_{p,q}(\mathbb{T}, X)$, and $\|u_{t_0}\|_{F^s_{p,q}} = \|u\|_{F^s_{p,q}}$.

As in the Besov spaces case, the main tool to study $F^s_{p,q}$-maximal regularity for (3.3) is operator-valued Fourier multiplier theorems on $F^s_{p,q}(\mathbb{T}, X)$.

Definition 3.1. Let X, Y be Banach spaces, $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We will say that $(M_k)_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier, if for each $f \in F^s_{p,q}(\mathbb{T}, X)$, there exists $u \in F^s_{p,q}(\mathbb{T}, Y)$, such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

The following result has been obtained in [6]:

Theorem 3.2. *Let X, Y be Banach spaces, $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$ and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X, Y)$. We assume that*

$$\sup_{k\in\mathbb{Z}}(\|M_k\| + \|k(M_{k+1} - M_k)\| + \|k^2(M_{k+2} - 2M_{k+1} + M_k)\|) < \infty \tag{3.1}$$

$$\sup_{k\in\mathbb{Z}} \|k^3(M_{k+3} - 3M_{k+2} + 3M_{k+1} - M_k)\| < \infty. \tag{3.2}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier. Moreover if $1 < p < \infty,\ 1 < q \le \infty$, then the first condition (3.1) is sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be an $F^s_{p,q}$-multiplier.

In this section, we study $F^s_{p,q}$-maximal regularity of the problem

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{T}, \tag{3.3}$$

here as before, A is a closed operator in a Banach space X, F is a bounded linear operator from $F^s_{p,q}([-2\pi, 0], X)$ to X and $f \in F^s_{p,q}(\mathbb{T}, X)$ is given.

Definition 3.3. Let $1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$ and let $f \in F^s_{p,q}(\mathbb{T}, X)$ be given. A function $u \in F^{s+1}_{p,q}(\mathbb{T}, X)$ is called a strong $F^s_{p,q}$-solution of (3.3), if $u(t) \in D(A)$ and (3.3) holds for a.e. $t \in \mathbb{T}$, and $Au,\ Fu_\cdot \in F^s_{p,q}(\mathbb{T}, X)$. We say that the problem (3.3) has $F^s_{p,q}$-maximal regularity, if for each $f \in F^s_{p,q}(\mathbb{T}, X)$, there exists a unique strong $F^s_{p,q}$-solution of (3.3).

Let $F \in \mathcal{L}(F^s_{p,q}([-2\pi, 0], X), X)$ and $k \in \mathbb{Z}$, we define the operator B_k by $B_k x = F(e_k x)$ for all $x \in X$. It is clear that $B_k \in \mathcal{L}(X)$ and $\|B_k\| \le \|F\|$ as $\|e_k\|_{F^s_{p,q}} \le 1$. We define the spectrum of (3.3) by

$$\sigma(\Delta) = \big\{k \in \mathbb{Z} : \ \ ikI - B_k - A \text{ is not invertible from } D(A) \text{ to } X\big\}. \tag{3.4}$$

Since A is closed, if $k \in \mathbb{Z} \setminus \sigma(\Delta)$, then $(ikI - B_k - A)^{-1}$ is a bounded linear operator on X. This is an easy consequence of the Closed Graph Theorem. We will also use the following notations: for $k \in \mathbb{Z}$

$$C_k := ikI - B_k; \; N_k := (C_k - A)^{-1}; \; M_k := ikN_k. \tag{3.5}$$

With these notations for (3.3), we remark that Lemma 2.4 remains true in the Triebel-Lizorkin spaces case. We are going to prove the following proposition, which is the analogue of Proposition 2.5 in the Triebel-Lizorkin spaces case.

Proposition 3.4. *Let A be a closed linear operator in a Banach space X. Suppose that $\sigma(\Delta) = \emptyset$ and $(k(B_{k+2} - 2B_{k+1} + B_k))_{k\in\mathbb{Z}}$, $(k^2(B_{k+3} - 3B_{k+2} + 3B_{k+1} - B_k)_{k\in\mathbb{Z}}$ are uniformly bounded. Then the following assertions are equivalent:*

- (i) $(M_k)_{k\in\mathbb{Z}}$ *is an $F^s_{p,q}$-multiplier for all* (*equivalently, for some*) $1 \le p < \infty$, $1 \le q \le \infty$ *and* $s \in \mathbb{R}$.
- (ii) $(M_k)_{k\in\mathbb{Z}}$ *is uniformly bounded.*

Proof. The implication (i)⇒(ii) follows from [6]. To show that the implication (ii)⇒(i) remains true, we assume that $(M_k)_{k\in\mathbb{Z}}$ is uniformly bounded. We are going to show that $(M_k)_{k\in\mathbb{Z}}$ satisfies the conditions (3.1) and (3.2). (3.1) is clearly true by the proof of Proposition 2.5. To show (3.2), we claim that

$$\sup_{k\in\mathbb{Z}} \|k^4(N_{k+3} - 3N_{k+2} + 3N_{k+1} - N_k)\| < \infty. \tag{3.6}$$

Indeed, by the proof of Lemma 2.4, for $k \in \mathbb{Z}$,

$$\begin{aligned} N_{k+2} - 2N_{k+1} + N_k &= (N_{k+2} - N_k)(B_{k+2} - B_{k+1} - i)N_{k+1} \\ &\quad + N_k(B_{k+2} - 2B_{k+1} + B_k)N_{k+1} := J_{1,k} + J_{2,k}. \end{aligned}$$

For $J_{2,k}$, we have

$$\begin{aligned} & J_{2,k+1} - J_{2,k} \\ = {} & N_{k+1}(B_{k+3} - 2B_{k+2} + B_{k+1})N_{k+2} - N_k(B_{k+2} - 2B_{k+1} + B_k)N_{k+1} \\ = {} & (N_{k+1} - N_k)(B_{k+3} - 2B_{k+2} + B_{k+1})N_{k+2} \\ & + N_k(B_{k+3} - 3B_{k+2} + 3B_{k+1} - B_k)N_{k+2} \\ & + N_k(B_{k+2} - 2B_{k+1} + B_k)(N_{k+2} - N_{k+1}). \end{aligned}$$

Therefore $\sup_{k\in\mathbb{Z}} \|k^4(J_{2,k+1} - J_{2,k})\| < \infty$ by assumption and Lemma 2.4. For

$J_{1,k}$, we have

$$\begin{aligned} J_{1,k+1}-J_{1,k} &= (N_{k+3}-N_{k+1})(B_{k+3}-B_{k+2}-i)N_{k+2} \\ &\quad -(N_{k+2}-N_k)(B_{k+2}-B_{k+1}-i)N_{k+1} \\ &= (N_{k+3}-N_{k+2}-N_{k+1}+N_k)(B_{k+3}-B_{k+2}-i)N_{k+2} \\ &\quad +(N_{k+2}-N_k)(B_{k+3}-2B_{k+2}+B_{k+1})N_{k+2} \\ &\quad +(N_{k+2}-N_k)(B_{k+2}-B_{k+1}-i)(N_{k+2}-N_{k+1}). \end{aligned}$$

We conclude that $\sup_{k\in\mathbb{Z}}\|k^4(J_{1,k+1}-J_{1,k})\|<\infty$ by assumption and Lemma 2.4, as $N_{k+3}-N_{k+2}-N_{k+1}+N_k=(N_{k+3}-2N_{k+2}+N_{k+1})+(N_{k+2}-2N_{k+1}+N_k)$. We have shown that (3.6) is valid. Now we are ready to show that $(M_k)_{k\in\mathbb{Z}}$ satisfies (3.2). For $k\in\mathbb{Z}$

$$\begin{aligned} & M_{k+3}-3M_{k+2}+3M_{k+1}-M_k \\ &= i(k+3)N_{k+3}-3i(k+2)N_{k+2}+3i(k+1)N_{k+1}-ikN_k \\ &= ik(N_{k+3}-3N_{k+2}+3N_{k+1}-N_k)+3i(N_{k+3}-2N_{k+2}+N_{k+1}). \end{aligned}$$

Thus $\sup_{k\in\mathbb{Z}}\|k^3(M_{k+3}-3M_{k+2}+3M_{k+1}-M_k)\|<\infty$ by Lemma 2.4 and (3.6). We have shown that $(M_k)_{k\in\mathbb{Z}}$ satisfies (3.1) and (3.2). Hence $(M_k)_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier by Theorem 3.2. The proof is completed. ∎

Now, we are ready to state the main result of this section.

Theorem 3.5. *Let A be a closed linear operator defined in a Banach space X. Assume that $(k(B_{k+2}-2B_{k+1}+B_k))_{k\in\mathbb{Z}}$ and $(k^2(B_{k+3}-3B_{k+2}+3B_{k+1}-B_k)_{k\in\mathbb{Z}}$ are uniformly bounded. Then the following assertions are equivalent:*

- (i) *the problem (3.3) has $F^s_{p,q}$-maximal regularity for all (equivalently, for some) $1\le p<\infty$, $1\le q\le\infty$ and $s>0$.*
- (ii) *$\sigma(\Delta)=\emptyset$ and $(M_k)_{k\in\mathbb{Z}}$ is uniformly bounded.*

Proof. The implication (i)⇒(ii) follows from the same argument used in the proof of [1, Theorem 2.3] or [11, Proposition 3.3]. We omit the details. To show that the implication (ii)⇒(i) is true, we assume that $\sigma(\Delta)=\emptyset$ and $(M_k)_{k\in\mathbb{Z}}$ is uniformly bounded. We claim that $(B_kN_k)_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier. Indeed, we know from the proof of Theorem 2.7 that $(B_kN_k)_{k\in\mathbb{Z}}$, $(k(B_{k+1}N_{k+1}-B_kN_k))_{k\in\mathbb{Z}}$ and $(k^2(B_{k+2}N_{k+2}-2B_{k+1}N_{k+1}+B_kN_k))_{k\in\mathbb{Z}}$ are uniformly bounded. It remains to show that $\sup_{k\in\mathbb{Z}}\|k^3(B_{k+3}N_{k+3}-3B_{k+2}N_{k+2}+3B_{k+1}N_{k+1}-B_kN_k)\|<\infty$. From the proof of Theorem 2.7, we have

$$\begin{aligned} & B_{k+2}N_{k+2}-2B_{k+1}N_{k+1}+B_kN_k \\ &= [B_{k+2}(N_{k+2}-N_{k+1})-B_k(N_{k+1}-N_k)]+(B_{k+2}-2B_{k+1} \\ &\quad +B_k)N_{k+1} := L_{1,k}+L_{2,k}. \end{aligned}$$

Then

$$\begin{aligned}
&L_{2,k+1} - L_{2,k} \\
&= (B_{k+3} - 2B_{k+2} + B_{k+1})N_{k+2} - (B_{k+2} - 2B_{k+1} + B_k)N_{k+1} \\
&= (B_{k+3} - 3B_{k+2} + 3B_{k+1} - B_k)N_{k+2} + (B_{k+2} \\
&\quad -2B_{k+1} + B_k)(N_{k+2} - N_{k+1}).
\end{aligned}$$

Hence $\sup_{k\in\mathbb{Z}} \|k^3(L_{2,k+1} - L_{2,k})\| < \infty$ by assumption and Lemma 2.4. For $L_{1,k}$, we have

$$\begin{aligned}
&L_{1,k+1} - L_{1,k} = B_{k+3}(N_{k+3} - N_{k+2}) - B_{k+1}(N_{k+2} - N_{k+1}) \\
&\quad -B_{k+2}(N_{k+2} - N_{k+1}) + B_k(N_{k+1} - N_k) \\
&= B_{k+3}(N_{k+3} - 2N_{k+2} + N_{k+1}) - B_{k+2}(N_{k+2} - 2N_{k+1} + N_k) \\
&\quad +(B_{k+3} - B_{k+1})(N_{k+2} - N_{k+1}) - (B_{k+2} - B_k)(N_{k+1} - N_k) \\
&= B_{k+3}(N_{k+3} - 2N_{k+2} + N_{k+1}) - B_{k+2}(N_{k+2} - 2N_{k+1} + N_k) \\
&\quad +(B_{k+3} - B_{k+2} - B_{k+1} + B_k)(N_{k+2} - N_{k+1}) \\
&\quad +(B_{k+2} - B_k)(N_{k+2} - 2N_{k+1} + N_k).
\end{aligned}$$

Thus $\sup_{k\in\mathbb{Z}} \|k^3(L_{1,k+1} - L_{1,k})\| < \infty$ by assumption and Lemma 2.4, as we have $B_{k+3} - B_{k+2} - B_{k+1} + B_k = (B_{k+3} - 2B_{k+2} + B_{k+1}) + (B_{k+2} - 2B_{k+1} + B_k)$.

We have shown that $(B_kN_k)_{k\in\mathbb{Z}}$ satisfies (3.1) and (3.2), thus it is an $F^s_{p,q}$-multiplier. The rest of the proof follows the same lines as the proof of Theorem 2.7. We omit the details. ∎

Remark 3.6.

(i) When $1 < p < \infty, 1 < q \le \infty$ and $s \in \mathbb{R}$, the Marcinkiewicz condition of order 2 is already sufficient for a sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ to be an $F^s_{p,q}$-multiplier by Theorem 3.2. This fact together with the proof of Theorem 2.7 implies that under the weaker assumption that $(k(B_{k+2} - 2B_{k+1} + B_k))_{k\in\mathbb{Z}}$ is bounded, the problem (3.3) has $F^s_{p,q}$-maximal regularity for some (equivalently, for all) $1 < p < \infty, 1 < q \le \infty$ and $s > 0$ if and only if $\sigma(\Delta) = \emptyset$ and $(M_k)_{k\in\mathbb{Z}}$ is uniformly bounded.

(ii) Examples of closed operators A and $F \in \mathcal{L}(B^s_{p,q}(\mathbb{T}, X), X)$ (resp. $F \in \mathcal{L}(F^s_{p,q}(\mathbb{T}, X), X)$) satisfying (ii) of Theorem 2.7 (resp. Theorem 3.5) can be found in [11, Example 3.8]. The same proof as in [11, Example 3.8] gives the following: let X be a Banach space, A be a closed linear operator and $F \in \mathcal{L}(B^s_{p,q}(\mathbb{T}, X), X)$ (resp. $F \in \mathcal{L}(F^s_{p,q}(\mathbb{T}, X), X)$), such that $i\mathbb{Z} \subset \rho(A)$ and $sup_{k\in\mathbb{Z}}\|A(ik - A)^{-1}\| =: \eta < \infty$. Assume that $(k(B_{k+2} - 2B_{k+1} + B_k))_{k\in\mathbb{Z}}$

is uniformly bounded (resp. $(k(B_{k+2} - 2B_{k+1} + B_k))_{k\in\mathbb{Z}}$ and $(k^2(B_{k+3} - 3B_{k+2} + 3B_{k+1} - B_k)_{k\in\mathbb{Z}}$ are uniformly bounded) and $\|F\| < \frac{1}{\|A^{-1}\|\eta}$. Then (2.1) (resp. (3.3)) has $B^s_{p,q}$-maximal regularity. We remark that the conditions $i\mathbb{Z} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}} \|A(ik - A)^{-1}\| < \infty$ characterizes $B^s_{p,q}$-maximal regularity (resp. $F^s_{p,q}$-maximal regularity) of the problem (2.1) [2] (resp. (3.3) [6]) in the special case when $F = 0$.

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