TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 3, pp. 1031-1042, June 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

ELEMENTS IN EXCHANGE QB_∞-RINGS

Huanyin Chen

Abstract. An element $u \in R$ is pseudo invertible if there exist $v, w \in R$ such that $R(1-uv)R(1-wu)R$ is a nilpotent ideal. A ring R is a QB_∞ ring provided whenever $aR+bR=R$ with $a, b \in R$, there exists $y \in R$ such that $a+by$ is pseudo invertible. We prove, in this paper, that an exchange ring R is a QB_∞-ring if and only if whenever $x = xyx$, there exists a pseudo invertible $u \in R$ such that $x = xyu = uyx$ if and only if whenever $x = xyx$, there exists $a \in R$ such that $y+a$ is pseudo invertible and $1+xa$ is invertible. Also we characterize exchange QB_∞-rings by virtue of pseudo unit-regularity. These generalize the main results of Wei (2004, Theorem 3, Theorem 7; 2005, Theorem 2.2, Theorem 2.4 and Theorem 3.6).

1. INTRODUCTION

A ring R has stable one provided that $aR+bR=R$ with $a, b \in R$ implies that there exists a $y \in R$ such that $a+by$ is invertible (cf. [4] and [9]). Replacing invertible elements with weakly invertible elements in the definition of stable range one, one introduced some other conditions. A ring R has weakly stable range one if whenever $aR+bR=R$ with $a, b \in R$, there exists $y \in R$ such that $a+by$ is right or left invertible (cf. [5] and [12-13]). In [2], Ara et al. discovered a new class of rings, i.e., QB-rings. They called a ring R is a QB-ring provided that whenever $aR+bR=R$ with $a, b \in R$, there exists $y \in R$ such that $a+by$ is quasi invertible, where $u \in R$ is quasi invertible provided that there exist $v, w \in R$ such that $(1-uv)R(1-wu) = (1-wu)R(1-uv) = 0$. The class of QB-rings gives a nice infinite analogoue of stable range one (see [2-3] and [6]). In [7], the author introduced a new class of rings, i.e., QB_∞-rings. A ring R is a QB_∞ ring provided whenever $aR+bR=R$ with $a, b \in R$, there exists $y \in R$ such that $a+by$ is pseudo invertible, where $u \in R$ is pseudo invertible if there exists $v, w \in R$

Received May 8, 2006, accepted October 20, 2007.
Communicated by Wen-Fong Ke.
2000 *Mathematics Subject Classification*: 16E50, 16D70, 19B10.
Key words and phrases: Exchange ring, QB_∞-Ring, Pseudo unit-regularity.

such that $R(1-uv)R(1-wu)R$ is a nilpotent ideal. Clearly, every QB-ring is a QB_∞-ring, while the converse is not true. For example, the ring $TM_2(R)$ of all 2×2 upper triangular matrices over a QB-ring R is a QB_∞-ring, but $TM_2(R)$ is not a QB-ring (see [7, Example 3.3]). Also we note that infinite analogoues of stable range one were also studied in the context of C^*-algebras (cf. [10]).

A ring R is called an exchange ring if for every right R-module A and any two decompositions $A = M\oplus N = \bigoplus_{i\in I} A_i$, where $M_R \cong R_R$ and I is a finite index set, there exist submodules $A_i' \subseteq A_i$ such that $A = M\oplus(\bigoplus_{i\in I} A_i')$. It is well known that regular rings, π-regular rings, unit C^*-algebras of real rank zero, semiperfect rings, left or right continuous rings and clean rings are all exchange rings (cf. [1], [6], [9] and [11]). We prove, in this paper, that an exchange ring R is a QB_∞-ring if and only if whenever $x = xyx$, there exists a pseudo invertible $u\in R$ such that $x = xyu = uyx$ if and only if whenever $x = xyx$, there exists $a\in R$ such that $y+a$ is pseudo invertible and $1+xa$ is invertible. Also we characterize exchange QB_∞-rings by virtue of pseudo unit-regularity. These generalize [12, Theorem 3], [12,Theorem 7], [13,Theorem 2.2], [13, Theorem 2.4] and [13, Theorem 3.6].

Throughout, R is an associative ring with nonzero identity 1_R. $U(R)$ denotes the set of all units of R. $x\in R$ is called pseudo unit-regular provided that there exists a $u\in R_\infty^{-1}$ such that $x = xux$. We always use R_∞^r to stand for the set of all pseudo unit-regular elements in R.

2. Pseudo Invertibility

Let $Q(0) = \{r\in R \mid RrR$ is an nilpotent ideal of $R\}$. Then $Q(0)$ is an ideal of R. We begin with a characterization of exchange QB_∞-rings by virtue of pseudo-invertible elements.

Theorem 2.1. *Let R be an exchange ring. Then the following are equivalent:*

(i) *R is a QB_∞-ring.*

(ii) *Every regular element in R is pseudo unit-regular.*

Proof. $(1)\Rightarrow(2)$ Given any regular $x\in R$, there exists a $y\in R$ such that $x = xyx$. Since $yx + (1-yx) = 1$, we have a $z\in R$ such that $y + (1-yx)z = u\in R_\infty^{-1}$. Hence, $x = xyx = x\big(y+(1-yx)z\big)x = xux$, as required.

$(2)\Rightarrow(1)$ Suppose that $ax+b = 1$ in R. In view of [11, Proposition 28.6], there exists an idempotent $e\in bR$ such that $1-e\in(1-b)R$. Assume that $e = bs$ and $1-e = axt$ for some $s,t\in R$. Then $axt+e = 1$; hence, $(1-e)a\in R$ is regular. By assumption, we can find a pseudo-invertible $u\in R$ such that $(1-e)a = (1-e)au(1-e)a$. Since $(1-e)axt+e = 1$, we have that $u(1-e)axt+ue = u$. Let $f = u(1-e)a$. Then $f = f^2\in R$. Clearly, $f(xt+ue)+(1-f)ue = u$, and so $(1-f)ue = (1-f)u$. Since $u \in R_\infty^{-1}$, it follows from [7, Lemma 2.1] that $u \equiv uvu\big(mod\ Q(0)\big)$ for a $v \in R$. Thus,

$$\begin{aligned}(1-f)uv(1-f)u &= (1-f)\big(uvu - uvu(1-e)au\big)\\ &\equiv (1-f)(u-fu)\\ &\equiv (1-f)u\big(mod\ Q(0)\big).\end{aligned}$$

Let $g = (1-f)uv(1-f)$. Then $g \equiv g^2\big(mod\, Q(0)\big)$. As a result, we get $f(xt+ue)+gu \equiv u\big(mod\ Q(0)\big)$. One easily checks that $fg = gf = 0$, and so $f(xt+ue) \equiv fu\big(mod\ Q(0)\big)$. One easily checks that

$$\begin{aligned}&u\big((1-e)a + ev(1-f)(1+fuev(1-f)\big)\big(1-fuev(1-f)\big)u\\ =\ &\big(f + uev(1-f)(1+fuev(1-f)\big)\big(1-fuev(1-f)\big)u\\ =\ &\big(f(1-fuev(1-f) + uev(1-f)\big)u\\ =\ &\big(f + (1-f)uev(1-f)\big)u\\ =\ &fu + (1-f)uv(1-f)u\\ =\ &fu + gu\\ \equiv\ &u\big(mod\ Q(0)\big).\end{aligned}$$

As $u \in R_\infty^{-1}$, it is easy to verify that

$$\begin{aligned}&a + bs\big(v(1-f) - a\big)\\ =\ &a - ea + ev(1-f)\\ =\ &(1-e)a + v(1-f)(1+fuv(1-f) \in R_\infty^{-1}.\end{aligned}$$

Therefore R is a QB_∞-ring. ∎

Corollary 2.2. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_∞-ring.*

(2) *For any regular $x \in R$, there exists $u \in R_\infty^{-1}$ such that ux is an idempotent.*

Proof. (1) $\Rightarrow$ (2) For any regular $x \in R$, it follows by Theorem 2.1 that there exists a $u \in R_\infty^{-1}$ such that $x = xux$. So $ux \in R$ is an idempotent.

(2) $\Rightarrow$ (1) For any regular $x \in R$, there exists a $u \in R_\infty^{-1}$ such that ux is an idempotent. Clearly, we have a $y \in R$ such that $x = xyx$ and $y = yxy$. From $yx + (1-yx) = 1$, we get $uyx + u(1-yx) = u$. As in the proof of Theorem 2.1, we can find a $z \in R$ such that $y + (1-yx)z = u \in R_\infty^{-1}$. Hence, $x = x\big(y+(1-yx)z\big)x = xux$. According to Theorem 2.1, we complete the proof. ∎

Lemma 2.3. *Let R be a ring and $x \in R$. Then the following are equivalent:*

(1) *There exists a $v \in R_{\infty}^{-1}$ such that $x = xvx$.*

(2) $x = xyx = xyu$*, where* $y \in R, u \in R_{\infty}^{-1}$.

(3) $x = xyx = uyx$*, where* $y \in R, u \in R_{\infty}^{-1}$.

Proof. $(1) \Rightarrow (2)$ Since $xy + (1 - xy) = 1$ with $y \in R_{\infty}^{-1}$, it follows by [7, Lemma 4.4] that $x+(1-xy)z \in R_{\infty}^{-1}$ for a $z \in R$. Hence $x = xy\big(x+(1-xy)z\big) = xyu$, where $u = x + (1 - xy)z \in R_{\infty}^{-1}$.

$(2) \Rightarrow (1)$ Suppose that $x = xyx = xyu$, where $y \in R, u \in R_{\infty}^{-1}$. Let $e = xy$. Then $e \in R$ is an idempotent. Since $xy + (1 - xy) = 1$, we have that $euy+(1-xy) = 1$, and so $euy(1-e)+(1-xy)(1-e) = 1-e$. This implies that $e+(1-xy)(1-e) = 1-euy(1-e) \in U(R)$. Therefore we get $x+(1-xy)(1-e) = \big(1 - euy(1 - e)\big)u \in R_{\infty}^{-1}$. In view of [7, Lemma 4.4], we can find a $z \in R$ such that $w := y + z(1 - xy) \in R_{\infty}^{-1}$. Thus, $x = x\big(y + z(1 - xy)\big)x = xwx$.

$(1) \Rightarrow (3)$ Since $yx + (1 - yx) = 1$ with $y \in R_{\infty}^{-1}$, it follows by [7, Lemma 4.4] that $x + z(1 - yx) \in R_{\infty}^{-1}$ for a $z \in R$. Then $x = \big(x + (1 - yx)z\big)yx = uyx$, where $u = x + z(1 - yx) \in R_{\infty}^{-1}$.

$(3) \Rightarrow (1)$ Suppose that $x = xyx = uyx$, where $y \in R, u \in R_{\infty}^{-1}$. Let $e = yx$. Then $e \in R$ is an idempotent. Since $yx + (1 - yx) = 1$, we have that $yue + (1 - yx) = 1$, and so $(1 - e)yue + (1 - e)(1 - yx) = 1 - e$. Hence, $e + (1 - e)yue = 1 - (1 - e)yue \in U(R)$. Thus, we get $x + u(1 - e)yue = u\big(1 - (1 - e)yue\big) \in R_{\infty}^{-1}$. By virtue of [7, Lemma 4.4], we have a $z \in R$ such that $w := y + (1 - yx)z \in R_{\infty}^{-1}$. Therefore $x = x\big(y + (1 - yx)z\big)x = xwx$, as asserted. ■

Theorem 2.4. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_{∞}-ring.*

(2) *Whenever $x = xyx$, there exists $u \in R_{\infty}^{-1}$ such that $x = xyu$.*

(3) *Whenever $x = xyx$, there exists $u \in R_{\infty}^{-1}$ such that $x = uyx$.*

Proof. $(1) \Rightarrow (2)$ Suppose that $x = xyx$. In view of Theorem 2.1, we can find a $v \in R_{\infty}^{-1}$ such that $x = xvx$. By Lemma 2.3, we have a $u \in R_{\infty}^{-1}$ such that $x = xvx = xvu$. Let $e = xv$. Then $e = e^2 \in R$. Since $xy + (1 - xy) = 1$, we have that $euy + (1 - xy) = 1$; hence, $euy(1 - e) + (1 - xy)(1 - e) = 1 - e$. This implies that $e + (1 - xy)(1 - e) = 1 - euy(1 - e) \in U(R)$, and so $x + (1 - xy)(1 - e) = \big(1 - euy(1 - e)\big)u \in R_{\infty}^{-1}$. Let $w = \big(1 - euy(1 - e)\big)u$. Then $x = xyx = xy\big(x + (1 - xy)(1 - e)\big) = xyw$.

$(2) \Rightarrow (1)$ is clear by Lemma 2.3 and Theorem 2.1.

$(1) \Leftrightarrow (3)$ is proved in the same manner. ■

Corollary 2.5. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_∞-ring.*

(2) *Whenever $x \in R$ is regular, there exist an idempotent $e \in R$ and a $u \in R_\infty^{-1}$ such that $x = eu$.*

(3) *Whenever $x \in R$ is regular, there exist an idempotent $e \in R$ and a $u \in R_\infty^{-1}$ such that $x = ue$.*

Proof. (1) $\Rightarrow$ (2) Since $x \in R$ is regular, there exists a $y \in R$ such that $x = xyx$. In view of Theorem 2.4, we have a $u \in R_\infty^{-1}$ such that $x = xyu$. Let $e = xy$. Then $e \in R$ is an idempotent and $x = eu$, as required.

(2) $\Rightarrow$ (1) Given regular $x \in R$, we have a $y \in R$ such that $x = xyx$. By assumption, we have a $u \in R_\infty^{-1}$ and an idempotent $e \in R$ such that $x = eu$. Since $xy + (1 - xy) = 1$, $euy + (1 - xy) = 1$. As in the proof of Theorem 2.4, we have that $x + (1 - xy)(1 - e) = \big(1 - euy(1 - e)\big)u \in R_\infty^{-1}$. This implies that $x = xyx = xyw$, where $w := \big(1 - euy(1 - e)\big)u \in R_\infty^{-1}$. In view of Lemma 2.3 and Theorem 2.4, we conclude that R is a QB_∞-ring.

(1) $\Leftrightarrow$ (3) is symmetric. ∎

Corollary 2.6. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_∞-ring.*

(2) *Whenever $\varphi : aR \cong bR$ with $a, b \in R$, there exists $u \in R_\infty^{-1}$ such that $b = \varphi(a)u$.*

(3) *Whenever $\varphi : Ra \cong Rb$ with $a, b \in R$, there exists $u \in R_\infty^{-1}$ such that $b = u\varphi(a)$.*

Proof. (1) $\Rightarrow$ (2) Whenever $\varphi : aR \cong bR$ with $a, b \in R$, we have $r, s \in R$ such that $b = \varphi(ar)$ and $a = \varphi^{-1}(bs)$. Thus, $a = \varphi^{-1}\big(\varphi(ar)s\big) = ars$. Since $rs + (1 - rs) = 1$, there exists a $z \in R$ such that $r + (1 - rs)z = u \in R_\infty^{-1}$. Hence, $au = a\big(r + (1 - rs)z\big) = ar = \varphi^{-1}(b)$, and therefore $b = \varphi(a)u$.

(2) $\Rightarrow$ (1) Given any regular $x \in R$, there exists a $y \in R$ such that $x = xyx$. Clearly, we have a R-isomorphism $\varphi : xyR \cong yxR$ given by $\varphi(xyr) = y(xyr)$ for any $r \in R$. By assumption, we have that $yx = \varphi(xy)u$ for a $u \in R_\infty^{-1}$, i.e., $yx = yxyu = yu$. Thus, $x = xyx = xyu$. As $yx \in R$ is an idempotent, it follows by Corollary 2.5 that R is a QB_∞-ring.

(1) $\Leftrightarrow$ (3) is symmetric. ∎

3. Extensions

The purpose of this section is to give extensions of Theorem 2.4. As shown below, we also obtain new characterizations of exchange QB-rings. Let R be a ring and $a, b \in R$. The symbol $a \natural b$ means that $RaRbR$ is a nilpotent ideal of R.

Theorem 3.1. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_∞-ring.*

(2) *Whenever $x = xyx$, there exists a $u \in R_\infty^{-1}$ such that $x = xyu = uyx$.*

(3) *Whenever $x = xyx$, there exists a $u \in R_\infty^{-1}$ such that $xyu = uyx$.*

Proof. (1) $\Rightarrow$ (2) Given any $x = xyx$, then we have $x = xzx, z = zxz$, where $z = yxy$. Since R is a QB_∞-ring, it follows by Theorem 2.1, there exists a $v \in R_\infty^{-1}$ such that $z = zvz$. Let $u = (1 - xz - vz)v(1 - zx - zv)$. One easily checks that $(1 - xz - vz)^2 = 1 = (1 - zx - zv)^2$. Hence $u \in R_\infty^{-1}$. Clearly,

$$\begin{aligned} xzu &= -xzv(1 - zx - zv) \\ &= -xzv + xzx + xzv \\ &= xzx \\ &= x. \end{aligned}$$

and

$$\begin{aligned} uzx &= (1 - xz - vz)v(-zvzx) \\ &= -(1 - xz - vz)vzx \\ &= -vzx + xzx + vzx \\ &= xzx \\ &= x. \end{aligned}$$

Thus, $x = xzu = x(yxy)u = xyu$ and $x = uzx = u(yxy)x = uyx$. As a result, we see that $x = xyu = uyx$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Given $x = xyx$, there exists a $u \in R_\infty^{-1}$ such that $xyu = uyx$. In view of [7, Lemma 2.1], we can find a $v \in R$ such that $(1 - uv)\natural(1 - vu)$ and $u \equiv uvu\big(mod\ Q(0)\big)$. Construct two maps

$$\varphi : xR \oplus (1 - xy)R \to yxR \oplus (1 - yx)R;$$
$$\varphi\big(xr + (1 - xy)s\big) = yxr + u(1 - xy)s \text{ for any } s, t \in R$$

and

$$\phi : yR \oplus (1 - yx)R \to xR \oplus (1 - xy)R,$$
$$\phi\big(yr + (1 - yx)s\big) = xyr + (1 - xy)v(1 - yx)s \text{ for any } s, t \in R.$$

One easily checks that $x\varphi(1)x = x\varphi(x) = xyx = x$. Furthermore, we see that

$$\begin{aligned} 1 - \phi(1)\varphi(1) &= 1 - \phi\big(\varphi(1)\big) \\ &= 1 - \phi\big(yxy + u(1 - xy)\big) \\ &= 1 - \big(xy + (1 - xy)vu(1 - xy)\big) \\ &= (1 - xy)(1 - vu)(1 - xy). \end{aligned}$$

Likewise, we have that $1-\varphi(1)\phi(1)=(1-yx)(1-uv)(1-yx)$. Thus, $R\big(1-\varphi(1)\phi(1)\big)R\big(1-\phi(1)\varphi(1)\big)R\subseteq R(1-uv)R(1-vu)R$. As $(1-uv)\natural(1-vu)$, we deduce that $\big(1-\varphi(1)\phi(1)\big)\natural\big(1-\phi(1)\varphi(1)\big)$. Hence, $\varphi(1)\in R_\infty^{-1}$. According to Theorem 2.1, we complete the proof. ∎

Let R be a ring and $a,b\in R$. We say that a and b are pseudo-similar, denoted by $a\overset{\sim}{\approx}b$, if there exist $x,y\in R$ such that $a=xby, b=yax, x=xyx$ and $y=yxy$. We now generalize [6, Theorem 13] and [13, Theorem 3.6] to exchange QB_∞-rings.

Corollary 3.2. *Let R be an exchange QB_∞-ring. Then $a\overset{\sim}{\approx}b$ with $a,b\in R$ implies that there exist $u,v\in R_\infty^{-1}$ such that $a=ubv$.*

Proof. Suppose that $a\overset{\sim}{\approx}b$ with $a,b\in R$. Then we have $x,y\in R$ such that $a=xby, b=yax, x=xyx$ and $y=yxy$. In view of Theorem 3.1, there exists a $u\in R_\infty^{-1}$ such that $x=xyu=uyx$. One easily checks that $ax=a(xyu)=(xby)xyu=(xby)u=au$ and $xb=(uyx)b=(uyx)(yax)=(uyxy)ax=u(yax)=ub$. In addition, $ax=(xby)x=x(yax)yx=x(yax)=xb$. Thus, we can find a $u\in R_\infty^{-1}$ such that $au=xb=ub$. Since $y=yxy$, it follows from Theorem 3.1 that $y=yxv$ for a $v\in R_\infty^{-1}$. Therefore $a=xby=xbyxv=xyaxyxv=xyaxv=xbv=ubv$, as asserted. ∎

Theorem 3.3. *Let R be an exchange ring. Then the following are equivalent :*

(1) *R is a QB_∞-ring.*

(2) *Whenever $x=xyx$, there exists some $a\in R$ such that $y+a\in R_\infty^{-1}$ and $1+xa\in U(R)$.*

Proof. $(1)\Rightarrow(2)$ Since $x=xyx$, it follows from $yx+(1-yx)=1$ that there exists a $z\in R$ such that $y+(1-yx)z\in R_\infty^{-1}$. Let $a=(1-yx)z$. Then $y+a\in R_\infty^{-1}$. In addition, we have $1+xa=1+x(1-yx)z=1\in U(R)$.

$(2)\Rightarrow(1)$ Given $x=xyx$, then $x=xzx$ and $z=zxz$, where $z=yxy$. By assumption, we have a $c\in R$ such that $x+c\in R_\infty^{-1}$ and $1+zc\in U(R)$. Thus, $1+z(u-x)\in U(R)$ for a $u\in R_\infty^{-1}$. Let $w=1+z(u-x)$. Then $zuw^{-1}+(1-zx)w^{-1}=1$. As $uw^{-1}\in R_\infty^{-1}$, it follows from [7, Lemma 4.4] that $v:=z+(1-zx)w^{-1}t\in R_\infty^{-1}$ for a $t\in R$. As a result, $x=xzx=x\big(z+(1-zx)w^{-1}t\big)x=xvx$. According to Theorem 2.1, R is a QB_∞-ring. ∎

Corollary 3.4. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_∞-ring.*

(2) *Whenever $x\in R$ is regular, there exist a $e\in r.ann(x)$ and a $u\in R_\infty^{-1}$ such that $y=e+u$.*

Proof. $(1) \Rightarrow (2)$ Given any regular $x \in R$, there exists a $y \in R$ such that $x = xyx$. Since $yx + (1 - yx) = 1$, we can find a $z \in R$ such that $u := y+(1-yx)z \in R_{\infty}^{-1}$. Thus, $y = (yx-1)z+u$. Let $e = (yx-1)z$. Then $y = e+u$, where $e \in r.ann(x)$ and $u \in R_{\infty}^{-1}$.

$(2) \Rightarrow (1)$ Given any regular $x \in R$, there exist a $e \in r.ann(x)$ and a $u \in R_{\infty}^{-1}$ such that $y = e + u$. Let $a = -e$. Then $y + a \in R_{\infty}^{-1}$ and $1 + xa = 1 \in U(R)$, as required. ■

Corollary 3.5. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_{∞}-ring.*

(2) *Whenever $x = xyx$, there exists $u \in R_{\infty}^{-1}$ such that $1 - x(y + u) \in U(R)$.*

Proof. $(1) \Rightarrow (2)$ Whenever $x = xyx$, then $-x = (-x)(-y)(-x)$. By Theorem 3.3., there exists $a \in R$ such that $-y + a \in R_{\infty}^{-1}$ and $1 - xa \in U(R)$. Let $-y + a = u$. Then $1 - x(y + u) \in U(R)$, as required.

$(2) \Rightarrow (1)$ Whenever $x = xyx$, there exists $u \in R_{\infty}^{-1}$ such that $1 - x(y + u) \in U(R)$. Let $a = -(y+u)$. then $1+xa \in U(R)$ and $y+a = -u \in R_{\infty}^{-1}$. According to Theorem 3.3, we complete the proof. ■

As in the proof of Theorem 3.1 and Theorem 3.3, we see that an exchange ring R is a QB-ring if and only if whenever $x = xyx$, there exists a quasi invertible $u \in R$ such that $x = xyu = uyx$ if and only if whenever $x = xyx$, there exists some $a \in R$ such that $y + a$ is quasi invertible and $1 + xa$ is invertible.

4. Pseudo Unit-regularity

In this section, we characterize exchange QB_{∞}-rings by virtue of pseudo unit-regularity.

Lemma 4.1. *Suppose that $ax + b = 1$ with $a = a^2, b, x \in R$. Then there exist a $z \in R$ and a $u \in U(R)$ such that $xu + zbu = 1$.*

Proof. Since $ax + b = 1$, we have $ax(1 - a) + b(1 - a) = 1 - a$; hence, $a + b(1 - a) = 1 - ax(1 - a) \in U(R)$. In view of [8, Lemma 3.1], there exists a $z \in R$ such that $x + zb \in U(R)$. Let $u = (x + zb)^{-1}$. Then $xu + zbu = 1$, as asserted. ■

Theorem 4.2. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_{∞}-ring.*

(2) *Whenever* $x = xyx$*, there exists some* $a \in R$ *such that* $x + a \in R^r_\infty$ *and* $1 + ya \in U(R)$.

Proof. $(1) \Rightarrow (2)$ Whenever $x = xyx$, we see that $x \in R^r_\infty$ from Theorem 2.1. Choose $a = 0$. Then $x + a \in R^r_\infty$ and $1 + ya = 1 \in U(R)$.

$(2) \Rightarrow (1)$ Assume that $x = xyx$. Then $x = xzx$ and $z = zxz$, where $z = yxy$. By assumption, we have a $c \in R$ such that $x + c \in R^r_\infty$ and $u := 1 + zc \in U(R)$. Let $a = xyc$. Then $1 + ya = 1 + yxyc = 1 + zc \in U(R)$. In addition, $x + a = x + xyc = xy(x + c)$. Also we see that

$$\begin{aligned} x + a &= x + xyc \\ &= x + xyxyc \\ &= x(1 + zc) \\ &= xu. \end{aligned}$$

This implies $x = (x + a)u^{-1} = xy(x + c)u^{-1}$. As $x + c \in R^r_\infty$, we see that $(x+c)u^{-1} \in R^r_\infty$. Thus, we have a $v \in R^{-1}_\infty$ such that $(x+c)u^{-1} = (x+c)u^{-1}v(x+c)u^{-1}$. Since $(x + c)u^{-1}v + \big(1 - (x + c)u^{-1}v\big) = 1$, it follows by [7, Lemma 4.4] that $w := (x+c)u^{-1} + \big(1-(x+c)u^{-1}v\big)t \in R^{-1}_\infty$ for a $t \in R$. Let $f = (x+c)u^{-1}v$. Then $(x + c)u^{-1} = (x + c)u^{-1}vw = fw$. Let $e = xy$. Then $x = efw$. Since $efwy + (1 - xy) = xy + (1 - xy) = 1$, by virtue of Lemma 4.1, we can find a $k \in U(R)$ and a $d_1 \in R$ such that $fwyk + d_1(1 - xy)k = 1$. By Lemma 4.1 again, we have a $l \in U(R)$ and a $d_2 \in R$ such that $wykl + d_2d_1(1-xy)kl = 1$. In view of [7, Lemma 4.4], there exists a $d \in R$ such that $ykl + dd_2d_1(1-xy)kl \in R^{-1}_\infty$. This implies that $q := y + dd_2d_1(1 - xy) \in R^{-1}_\infty$; hence, $x = xyx = xqx$. According to Theorem 2.1, R is a QB_∞-ring. ∎

Corollary 4.3. *Let* R *be an exchange ring. Then the following are equivalent:*

(1) R *is a* QB_∞*-ring.*

(2) *Whenever* $x = xyx$*, there exist a* $e \in r.ann(y)$ *and a* $u \in R^r_\infty$ *such that* $x = e + u$.

Proof. $(1) \Rightarrow (2)$ is trivial from Theorem 2.1.

$(2) \Rightarrow (1)$ Whenever $x = xyx$, there exist a $e \in r.ann(y)$ and a $u \in R^r_\infty$ such that $x = e + u$. This implies that $x - e = u \in R^r_\infty$ and $1 + ye = 1 \in U(R)$. Therefore we complete the proof by Theorem 4.2. ∎

In [4, Theorem 2.9], Canfell showed that R has stable range one if and only if whenever $aR + bR = dR$, there exists a $y \in R$ and a $u \in U(R)$ such that $a + by = du$. Wei extended this result to exchange rings having weakly stable range one (cf. [12, Theorem 7] and [13, Theorem 2.4]). Now we can generalize Canfell's result in case of exchange QB_∞-rings.

Theorem 4.4. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_∞-ring.*

(2) *Whenever $aR+bR=dR$, there exist $u,v\in R_\infty^r$ such that $au+bv=d$.*

(3) *Whenever $Ra+Rb=Rd$, there exist $u,v\in R_\infty^r$ such that $ua+vb=d$.*

Proof. $(1)\Rightarrow(2)$ Assume that $aR+bR=dR$. Then we can find some $s_1,s_2,x,y\in R$ such that $a=ds_1,b=ds_2$ and $ax+by=1$. Thus, $ds_1x+ds_2y=d$. Let $s_3=1-s_1x-s_2y$. Then $s_1x+s_2y+s_3=1$. This implies that $s_1R+s_2R+s_3R=R$. Since R is an exchange ring, by [11, Proposition 29.1], we can find orthogonal idempotents $e_1,e_2,e_3\in R$ such that $e_1=s_1z_1,e_2=s_2z_2,e_3=s_3$ for some $z_1,z_2,z_3\in R$, where $e_1+e_2+e_3=1$. Let $z_i'=z_ie_i$. Then $e_i=s_iz_i'$. One easily checks that $z_i's_iz_i'=z_i'e_i=z_i'$. That is, $z_i'\in R$ is regular. In view of Theorem 2.1, z_i' is pseudo unit-regular. Observing that $az_1'+az_2'=d(s_1z_1'+s_2z_2')=de_1+de_2=d(e_1+e_2+e_3)=d$, as required.

$(2)\Rightarrow(1)$ Given $ax+b=1$ in R, then there exist pseudo unit-regular $w_1,w_2\in R$ such that $aw_1+bw_2=1$. Assume that $w_1=w_1vw_1$ for a $v\in R_\infty^{-1}$. Since $vw_1+(1-vw_1)=1$, it follows from [7, Lemma 4.4] that $w_1+z(1-vw_1)=u\in R_\infty^{-1}$ for a $z\in R$. This implies that $w_1=\big(w_1+z(1-vw_1)\big)vw_1=ue$, where $e=vw_1\in R$ is an idempotent. Thus, $aue+bw_2=1$, and so $(1-e)aue+(1-e)bw_2=1-e$. As a result, we deduce that $w_1+u(1-e)bw_2=ue+u(1-e)bw_2=u\big(1-(1-e)aue\big)\in R_\infty^{-1}$. By [7, Lemma 4.4] again, $a+bw_2z\in R_\infty^{-1}$ for a $z\in R$. Therefore R is a QB_∞-ring.

$(1)\Leftrightarrow(3)$ is symmetric. ■

Corollary 4.5. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is a QB_∞-ring.*

(2) *Whenever $aR=bR$, there exists $u\in R_\infty^r$ such that $a=bu$.*

(3) *Whenever $Ra=Rb$, there exists $u\in R_\infty^r$ such that $a=ub$.*

Proof. $(1)\Rightarrow(2)$ is trivial by Theorem 4.4.

$(2)\Rightarrow(1)$ Whenever $x=xyx$, we have that $xR=xyR$. By assumption, there exists a $u\in R_\infty^r$ such that $x=(xy)u$. Thus, we can find a $v\in R_\infty^{-1}$ such that $u=uvu$. Since $uv+(1-uv)=1$, by [7, Lemma 4.4], there exists a $z\in R$ such that $w:=u+(1-uv)z\in R_\infty^{-1}$. This implies that $u=uvu=u\big(u+(1-uv)z\big)=uvw$. Let $e=xy$ and $f=uv$. Then $e,f\in R$ are idempotents and $x=efw$. As in the proof of Theorem 4.2, $x\in R$ is pseudo unit-regular. According to Theorem 2.1, R is a QB_∞-ring.

$(1)\Leftrightarrow(3)$ is proved in the same manner. ■

The class of exchange QB_∞-ring is very large. We end this paper by providing a class of such rings.

Example 4.6. Let R be an exchange QB-ring. Then the ring

$$T = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

is an exchange QB_∞-ring.

Proof. Clearly,

$$J(T) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in J(R), b \in R \right\}.$$

Then $T/J(T) \cong R/J(R)$, and so $T/J(T)$ is a QB-ring. One easily checks that idempotents lift modulo $J(T)$. Therefore T is an exchange ring by [11, Theorem 29.2]. For any $\begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + J(T) \in \big(T/J(T)\big)_\infty^{-1}$, then $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + J(T) \in \big(T/J(T)\big)_\infty^{-1}$. Thus, we can find some $c, d \in R$ and $m \in \mathbb{N}$ such that

$$\left(T \left(1 - \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} c & 0 & d \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \right) \right.$$
$$\left. T \left(1 - \begin{pmatrix} c & 0 & d \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \right) T \right)^m \subseteq J(T).$$

Hence, $\overline{(1-ac)}\natural\overline{(1-ca)}$ in $R/J(R)$, i.e., $\overline{a} \in \big(R/J(R)\big)_\infty^{-1}$. In view of [6, Lemma 4.1], we have a $d \in R_\infty^{-1}$ such that $a - d \in J(R)$. Write $(1-du)\natural(1-ud)$ for a $u \in R$. Then there exists some $m \in \mathbb{N}$ such that $\big(R(1-du)R(1-ud)R\big)^m = 0$. Hence,

$$\left(T \left(1_T - \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \right) \right.$$
$$\left. T \left(1_T - \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} \right) T \right)^{2m} = 0.$$

This implies that $\begin{pmatrix} d & 0 & \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} \in T_\infty^{-1}$. Obviously,

$$\begin{pmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} + J(T) = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix} + J(T).$$

Therefore, $\big(J(T)+T_{\infty}^{-1}\big)/J(T)=\big(T/J(T)\big)_{\infty}^{-1}$. By [6, Lemma 4.1] again, T is a QB_{∞}-ring, as asserted. ∎

References

1. P. Ara, The exchange property for purely infinite simple rings, *Proc. Amer. Math. Soc.*, **132** (2004), 2543-2547.
2. P. Ara, G. K. Pedersen and F. Perera, An infinite analogue of rings with stable rank one, *J. Algebra*, **230** (2000), 608-655.
3. P. Ara; G. K. Pedersen and F. Perera, Extensions and pullbacks in QB-rings, *Algebra Represent. Theory*, **8** (2005), 75-97.
4. M. J. Canfell, Completion of diagrams by automorphisms and Bass' first stable range condition, *J. Algebra*, **176** (1995), 480-503.
5. H. Chen, Elements in one-sided unit regular rings, *Comm. Algebra*, **25** (1997), 2517-2529.
6. H. Chen, On exchange QB-rings, *Comm. Algebra*, **31** (2003), 831-841.
7. H. Chen, On QB_{∞}-rings, *Comm. Algebra*, **34** (2006), 2057-2068.
8. K. R. Goodearl, Cancellation of low-rank vector bundles, *Pacific J. Math.*, **113** (1984), 289-302.
9. T. Y. Lam, A crash course on stable range, cancellation, substitution and exchange, *J. Algebra Apll.*, **3** (2004), 301-343.
10. G. K. Pedersen, The λ-function in operator algebras, *J. Operator Theory*, **26** (1991), 345-381.
11. A. A. Tuganbaev, *Rings Close to Regular*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.
12. J. Wei, Exchange rings with weakly stable range one, *Vietnam J. Math.*, **32** (2004), 441-449.
13. J. Wei, Unit-regularity and stable range conditions, *Comm. Algebra*, **33** (2005), 1937-1946.

Huanyin Chen
Department of Mathematics,
Hangzhou Normal University,
Hangzhou 310036,
P. R. China
E-mail: huanyinchen@yahoo.cn