TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 3, pp. 997-1005, June 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

NON LIMIT-POINT DIFFERENTIAL EXPRESSIONS WITH ESSENTIAL SPECTRUM

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Abstract. It shall be shown that there exist real symmetric differential expressions of every even order with a nonempty essential spectrum which are not in the limit-point case.

1. Main Result

Theorem. *Let* $n > 1$. *The symmetric differential expression*

$$My = (-1)^n y^{(2n)} + (x^2 y')' \tag{1}$$

satisfies the following properties:

1. *Its deficiency index is* $d(M) = n + 1$.
2. *Its essential spectrum is nonempty.*

2. Theoretical and Historical Background

First we give a brief review of some basic facts concerning linear operators generated by ordinary differential expressions and their spectral properties which can be found in [2] and [8]. Given an ordinary differential expression of the form

$$My = \sum_{k=0}^{m} r_k y^{(k)},$$

Received January 31, 2007, accepted October 17, 2007.
Communicated by Sen-Yen Shaw.
2000 *Mathematics Subject Classification*: 47E05.
Key words and phrases: Deficiency index, Symmetric ordinary differential expressions, Essential spectrum.
This research was completed while the first author was a visiting researcher at the University of Duisburg-Essen, Essen, Germany under the German Academic Exchange Program.

where r_k are functions on an interval I, $r_k \in C^k(I, \mathbb{C})$ for every $k = 0, ..., m$ and for all $x \in I, r_m(x) \neq 0$. We define the maximal operator $T_1(M)$ generated by M in $\mathcal{L}^2(I)$ as:

$$\mathcal{D}(T_1(M)) = \{\, y \mid y \in AC^n(I) \cap \mathcal{L}^2(I) \,\text{and}\, My \in \mathcal{L}^2(I)\}$$
$$T_1(M)y = My = \sum_{k=0}^{m} r_k y^{(k)}.$$

The operator $T_R(M)$ in $\mathcal{L}^2(I)$ is the restriction of $T_1(M)$ to those $y \in \mathcal{D}(T_1(M))$ with compact support in the interior of I and are infinitely differentiable. Since $T_R(M) \subseteq T_1(M)$ and the maximal operator $T_1(M)$ is closed, it follows that $T_R(M)$ is closable and a unique minimal closed extension for $T_R(M)$ exists. This operator is called the minimal operator associated with M in $\mathcal{L}^2(I)$ and will be denoted by $T_0(M)$.

The spectral properties of M we shall be considering consist of the essential spectrum of M, denoted by $\sigma_e(M)$, and the nullities of M, denoted by nul$(M - \lambda)$ which are defined as follows :

$$\sigma_e(M) = \{\lambda \in \mathbb{C} \mid \text{range } T_0(M - \lambda I_{\mathcal{L}^2}) \text{ is not closed}\}$$
$$\text{nul}(M - \lambda) = \dim \ker(T_1(M - \lambda I_{\mathcal{L}^2})) \quad \text{for } \lambda \in \mathbb{C},$$

where $I_{\mathcal{L}^2}$ is the identity on $\mathcal{L}^2(I)$. It is well-known that the essential spectrum as defined here with the minimal operator is the same one as that of the one defined with the maximal operator and all linear operators in between generated by M. Therefore, it should be noted that the essential spectrum defined above depends only on the expression M and not on the operators generated by M. In [1] and [10], conditions for the existence of the essential spectrum have been determined.For these notions the asymptotic behaviour of the coefficients r_k at infinity is crucial in the case $I = [1, \infty)$.

Let us now consider real symmetric expressions of order $m = 2n$ which are of the form

$$My = \sum_{k=0}^{n} (-1)^k (p_k y^{(k)})^{(k)}. \tag{2}$$

where $p_k \in C^k(I, \mathbb{R})$ for $k = 0, \ldots, n$ and $p_n > 0$ on $I = [1, \infty)$. For these expressions, $\sigma_e(M) \subseteq \mathbb{R}$. Since the nullities of M are constant in $\mathbb{C} \setminus \mathbb{R}$, we consider the nullity only for $\lambda = i$. We refer to this nullity as the deficiency index of M and denote it by $d(M)$. Expressions of the form (2) on $\mathcal{L}^2[1, \infty)$ satisfy the following properties:

1. $n \leq d(M) \leq 2n$
2. If $d(M) = 2n$, then $\sigma_e(M) = \emptyset$.

Expression M is said to be in the limit-point case if d(M) $= n$ and in the limit circle case when d(M) $= 2n$. The above property means that if an expression is in the limit circle case, then it must have an empty essential spectrum. Hence, if the expression M has a nonempty essential spectrum, its deficiency index should be less than $2n$. Glazman[3] and Kodaira[6] were able to show that an expression need not be in the limit-point or limit circle case. Glazman gave explicit examples of expressions M with $n < d(M) < 2n$. In fact, he showed that for every integer k with $n \leq k \leq 2n$, there is an expression M such that $d(M) = k$. However, he did not consider the existence of the essential spectrum in his results. For a long time, there was no known example of an expression with a nonempty essential spectrum which is not in the limit point case. Schultze [9] gave the first example of such an expression for $n = 2$. It is of the form

$$My = (y'')'' + (xy')',$$

and has $d(M) = 3$. In [11], an example for $n = 3$ was given and it is of the form

$$My = -(y^{(3)})^{(3)} - (xy'')''$$

and has $d(M) = 4$. Although exact evaluation of the essential spectra of the above expressions has not been achieved, these expressions have nonempty essential spectra according to [1] and [10].

In [11, p. 499], Schultze made the following conjecture: For all n, $k \in \mathbb{N}$ with $n \leq k < 2n$, there exist expressions M of order $2n$ with nonempty essential spectrum such that $d(M) = k$.

Our main result partially confirms this conjecture for arbitrary n with $k = n+1$. The proof is based on the theory of asymptotic expansions of solutions and the above mentioned existence result of the essential spectrum.

3. Proof of the Theorem

First we prove a lemma for the differential expression

$$My = (-1)^n y^{(2n)} + (x^2 y')'.$$

The method of showing that its deficiency index is $n+1$ consists of determining a fundamental system of formal solutions for the equation

$$My - iy = (-1)^n y^{(2n)} + x^2 y'' + 2xy' - iy = 0.$$

According to the Poincaré, Birkhoff, Trjitzinsky, Turrittin theory ([7, p.95],[4, Satz A], [12-14), there exists a fundamental system of formal solutions which are not necessarily convergent series satisfying the above equation algebraically. Here they are of the form $e^{ax^r}\sum_{s=0}^{\infty} a_s x^{j-\frac{r}{2}s}$. To each of these formal solutions there exists a holomorphic solution having this formal solution as asymptotic expansion.This last property then enables us to determine the deficiency index of M.

Lemma. *Let*

$$(-1)^n y^{(2n)} + (x^2 y')' - iy = 0, \tag{3}$$

where $n > 1$. Then (3) has formal solutions of the following forms:

(i) $y = \sum_{s=0}^{\infty} a_s x^{j-s}$ *where* $j^2 + j - i = 0$ *and defining the empty product as* 1,

$$a_s = \begin{cases} \dfrac{(-1)^{kn}\prod_{l=0}^{2nk-1}(j-l)}{\prod_{l=1}^{k} 2nl(2j-2nl+1)}, & s = 2nk \text{ for some } k \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

and

(ii) $y = e^{ax^r}\sum_{s=0}^{\infty} a_s x^{j-\frac{r}{2}s}$ *where* $r = \frac{n}{n-1}$, $a = \frac{1}{r}(-1)^{\frac{n}{2(n-1)}}$, $j = \frac{2n-1}{-2n+2}$, $a_0 = 1$, *and for some function* f

$$a_s = \begin{cases} f(s, a, j, r, n, a_{s-2}, \ldots, a_2), & s \text{ even} \\ 0, & s \text{ odd.} \end{cases} \tag{5}$$

Proof. We show that there is a solvable recursion for the constants a_s such that the given formal series satisfies (3) formally. In (i) we are even able to determine the a_s explicitly.

Let y be given by (i). Then

$$\begin{aligned}(-1)^n (y^{(n)})^{(n)} + (x^2 y')' - iy &= (-1)^n y^{(2n)} + x^2 y'' + 2xy' - iy \\ &= (-1)^n \sum_{s=0}^{\infty} a_s \prod_{l=0}^{2n-1}(j-s-l) x^{j-s-2n} \\ &\quad + \sum_{s=0}^{\infty} a_s((j-s)(j-s-1) + 2(j-s) - i) x^{j-s}.\end{aligned}$$

We show that the coefficient of x^{j-s} is zero for any $s \in \mathbb{N}_0$.

If $s = 0, \ldots, 2n-1$, the coefficient of x^{j-s} is given by

$$a_s((j-s)(j-s-1) + 2(j-s) - i) = -a_s s(2j-s+1) = 0$$

since $j^2+j-i=0$ and $a_s=0$ for $0<s\leq 2n-1$.

For $s\geq 2n$, the coefficient of x^{j-s} is now given by

$$\begin{aligned}(-1)^n a_{s-2n}\prod_{l=0}^{2n-1}(j-(s-2n)-l)+a_s((j-s)(j-s-1)+2(j-s)-i)\\ =(-1)^n a_{s-2n}\prod_{l=s-2n}^{s-1}(j-l)-a_s s(2j-s+1).\end{aligned} \tag{6}$$

If $s\neq 0(\mathrm{mod}\ 2n)$, then $a_s=a_{s-2n}=0$.

If $s=2nk$ for some $k\in\mathbb{N}$ then $s-2n=2n(k-1)$ and (6) can be expressed as

$$(-1)^n a_{2n(k-1)}\prod_{l=2n(k-1)}^{2nk-1}(j-l)-a_{2nk}(2nk)(2j-2nk+1)$$

and this is equal to 0 since

$$a_{2n(k-1)}=\frac{(-1)^{(k-1)n}\prod_{l=0}^{2n(k-1)-1}(j-l)}{\prod_{l=1}^{k-1}2nl(2j-2nl+1)}$$

and

$$a_{2nk}=\frac{(-1)^{kn}\prod_{l=0}^{2nk-1}(j-l)}{\prod_{l=1}^{k}2nl(2j-2nl+1)}.$$

Hence, for y satisfying (i), $(-1)^n(y^{(n)})^{(n)}-(-x^2y')'-iy=0$.

Suppose y is of the form (ii), then for any $n\in\mathbb{N}$ and $u=\sum_{s=0}^{\infty}a_s x^{j-\frac{r}{2}s}$

$$\begin{aligned}y^{(n)} &= \sum_{m=0}^{n}\binom{n}{m}(e^{ax^r})^{(n-m)}u^{(m)}\\ &= e^{ax^r}\sum_{m=0}^{n}\sum_{l=0}^{n-m}\binom{n}{m}c_{l,n-m}a^l r^l x^{lr-(n-m)}\sum_{s=0}^{\infty}a_s J_{s,m}x^{j-\frac{r}{2}s-m}\\ &= e^{ax^r}\sum_{s=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{n-m}a_s a^l r^l C_{l,n,m}J_{s,m}x^{j+lr-n-\frac{r}{2}s}\\ &= e^{ax^r}\sum_{s=0}^{\infty}\sum_{l=0}^{n}\sum_{m=0}^{n-l}a_s a^l r^l C_{l,n,m}J_{s,m}x^{j+lr-n-\frac{r}{2}s}\end{aligned}$$

where $c_{l,n-m}$ are constants, $C_{l,n,m}=\binom{n}{m}c_{l,n-m}$, $J_{s,0}=1$ and for $m>0$, $J_{s,m}=\prod_{\lambda=0}^{m-1}(j-\frac{r}{2}s-\lambda)$. In particular, $c_{k,k}=1$, $c_{k-1,k}=\dfrac{k(k-1)}{2}(r-1)$ and $c_{0,k}=0$ for $k>0$. Then

$$
\begin{aligned}
&(-1)^n(y^{(n)})^{(n)} - (-x^2y')' - iy \\
&= e^{ax^r}[(-1)^n \sum_{s=0}^{\infty}\sum_{l=0}^{2n}\sum_{m=0}^{2n-l} a_s a^l r^l C_{l,2n,m} J_{s,m} x^{j+lr-2n-\frac{r}{2}s} \\
&+ a^2r^2 \sum_{s=0}^{\infty} a_s x^{j+2r-\frac{r}{2}s} + ar(r-1)\sum_{s=0}^{\infty} a_s x^{j+r-\frac{r}{2}s} \\
&+ 2ar\sum_{s=0}^{\infty} a_s(j-\frac{r}{2}s)x^{j+r-\frac{r}{2}s} + \sum_{s=0}^{\infty} a_s(j-\frac{r}{2}s)(j-\frac{r}{2}s-1)x^{j-\frac{r}{2}s} \\
&+ 2ar\sum_{s=0}^{\infty} a_s x^{j+r-\frac{r}{2}s} + 2\sum_{s=0}^{\infty} a_s(j-\frac{r}{2}s)x^{j-\frac{r}{2}s} - i\sum_{s=0}^{\infty} a_s x^{j-\frac{r}{2}s}].
\end{aligned} \tag{7}
$$

The powers of x are of the form $j + lr - 2n - \frac{r}{2}s$ where $0 \le l \le 2n$. The highest power for a fixed s is $j + 2r - \frac{r}{2}s$ occurring when $l = 2n$. If we let $k = 2n - l$, then we can now express $j + lr - 2n - \frac{r}{2}s$ as $j + r(2 - k - \frac{s}{2})$, $0 \le k \le 2n$ and (7) can now be expressed as

$$\sum_{s=0}^{\infty}\sum_{k=0}^{2n} a_s F_k(s) x^{j+r(2-k-s/2)}$$

where $F_k(s)$ comes from the terms $(-1)^n \sum_{m=0}^{2n-l} a^l r^l C_{l,2n,m} J_{s,m}$ from $y^{(2n)}$ and the other terms, from $2xy'$ and x^2y''. In particular,

$$F_0(s) = (-1)^n a^{2n} r^{2n} + a^2 r^2 \tag{8}$$

$$
\begin{aligned}
F_1(s) &= (-1)^n a^{2n-1} r^{2n-1} n[(2n-1)(r-1) + 2j - rs)] \\
&\quad + ar(2j - r(s-1) + 1) \\
F_2(s) &= (-1)^n \sum_{m=0}^{2} a^{2n-2} r^{2n-2} C_{2n-2,2n-m} J_{s,m} \\
&\quad + \left(j - \frac{r}{2}s\right)\left(j - \frac{r}{2}s - 1\right) + 2 - i
\end{aligned} \tag{9}
$$

and for $2 \le k \le 2n$

$$F_k(s) = (-1)^n \sum_{m=0}^{k} a^{2n-k} r^{2n-k} C_{2n-k,2n-m} J_{s,m}.$$

Since $a = r^{-1}(-1)^{\frac{n}{2(n-1)}}$, $F_0(s) = 0$ for any s. Moreover, since $j = \frac{2n-1}{-2n+2}$, $F_1(0) = 0$ and $F_1(s) \neq 0$ for any $s > 0$.

Let $a_0 = 1$. We define $a_s = 0$ for $s < 0$ and s odd. For s even and $s \geq 2$, we define a_s as

$$a_s = -[F_1(s)]^{-1} \sum_{i=2}^{2n} a_{s+2-2i} F_i(s+2-2i). \tag{10}$$

We now show that the coefficient $\sum_{k=0}^{2n} a_s F_k(s)$ of $x^{j+r(2-k-s/2)}$ is 0 for $0 \leq k \leq 2n$ and $s \geq 0$.

The highest power of x is $j+2r$ and its coefficient is $F_0(0) = 0$. The powers of x can also be expressed as $x^{j+r(2-l/2)}$, where $l = 2k+s$, and its coefficient is given by

$$\sum_{i=0}^{2n} a_{l-2i} F_i(l-2i). \tag{11}$$

If l is odd, then $a_{l-2i} = 0$ for every i and (11) is equal to 0.

Suppose l is even. Since $F_0(s) = 0$ and

$$a_{l-2} = -[F_1(l-2)]^{-1} \sum_{i=2}^{2n} F_i(l-2i) a_{l-2i}$$

then (11) is also equal to 0 for l even. This proves the assertion. ■

As mentioned above, to these formal solutions

$$y = e^{ax^r} \sum_{s=0}^{\infty} a_s x^{j-\frac{r}{2}s}, \tag{12}$$

there exist holomorphic solutions of (3) having these formal solutions as asymptotic expansions. ¿From these asymptotic expansions, it follows that the holomorphic solutions are linearly independent and we can determine if they are in $\mathcal{L}^2$ or not. We now prove the Theorem.

Proof of the Theorem. First note that M is of the form (2) hence, it is symmetric. Moreover, M has a nonempty essential spectrum since it satisfies condition $[E]$ in [10] namely:

E. If M is symmetric and $r_k(x) = O(x^k)$ as $x \to \infty$ for $k = 0, \ldots, m$, then $\sigma_e(M) \neq \emptyset$.

Let $My = (-1)^n y^{(2n)} + (x^2 y')'$. The Lemma gives the formal solutions of $My - iy = 0$ which are of the form (12). The first form, where $r = 2$, $a = 0$, and $j^2 + j - i = 0$, gives two holomorphic solutions and only one of these is in $\mathcal{L}^2$.

In the second form, $r = \frac{n}{n-1}$, and we have

$$a = r^{-1}(-1)^{\frac{1}{2(n-1)}}.$$

This gives $2n-2$ formal solutions, two of which are purely imaginary. Among the $2n-4$ solutions, $n-2$ have negative real parts, hence, the corresponding holomorphic solutions are in $\mathcal{L}^2$. For $a = i$ and $a = -i$, we consider the value of j. Since

$$j = \frac{2n-1}{-2n+2} < -1/2,$$

we get two more holomorphic solutions in $\mathcal{L}^2$.Then among the so obtained linearly independent $2n$ holomorphic solutions of (3), exactly $1 + (n-2) + 2$ of these are in $\mathcal{L}^2$. This implies $d(M) = n+1$.

Hence, for every $n > 1$, there exists an expression M which is not in the limit-point case. ∎.

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