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BOUNDEDNESS OF SUBLINEAR OPERATORS IN HERZ-TYPE HARDY SPACES

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Abstract. Let $p \in (0,1]$, $q \in (1,\infty)$, $\alpha \in [n(1-1/q),\infty)$ and $w_1,\, w_2 \in A_1$. The author proves that the norms in weighted Herz-type Hardy spaces $H\dot{K}_q^{\alpha,\,p}(w_1,\, w_2)$ and $HK_q^{\alpha,\,p}(w_1,\, w_2)$ can be achieved by finite central atomic decompositions in some dense subspaces of them. As an application, the author proves that if T is a sublinear operator and maps all central $(\alpha,\, q,\, s;\, w_1,\, w_2)_0$-atoms (resp. central $(\alpha,\, q,\, s;\, w_1,\, w_2)$-atoms of restrict type) into uniformly bounded elements of certain quasi-Banach space $\mathcal{B}$ for certain nonnegative integer s no less than the integer part of $\alpha - n(1-1/q)$, then T uniquely extends to a bounded operator from $H\dot{K}_q^{\alpha,\,p}(w_1,\, w_2)$ (resp. $HK_q^{\alpha,\,p}(w_1,\, w_2)$) to $\mathcal{B}$.

1. Introduction

The theory of Hardy spaces associated to Herz spaces obtained a great development in the past few years and played important roles in Harmonic analysis; see [4, 1, 14, 15, 16, 17, 7, 8, 11].

To establish the boundedness of operators in Hardy type spaces on $\mathbb{R}^n$, one usually appeals to the atomic decomposition characterization (see [5, 13, 14]) of these spaces, which means that a function or distribution in Hardy type spaces can be represented as a linear combination of atoms. Then, the boundedness of linear operators in Hardy type spaces can be deduced from their behavior on atoms in principle.

However, Meyer [20, p. 513] (see also [6, 3]) gave an example of $f \in H^1(\mathbb{R}^n)$ whose norm cannot be achieved by its finite atomic decompositions via $(1,\,\infty,\,0)$-atoms. Based on this fact, Bownik [3, Theorem 2] constructed a surprising example of a linear functional defined on a dense subspace of $H^1(\mathbb{R}^n)$, which maps all

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$(1,\,\infty,\,0)$-atoms into bounded scalars, but yet cannot extend to a bounded linear functional on the whole $H^1(\mathbb{R}^n)$. This implies that it cannot guarantee the boundedness of linear operator T from $H^p(\mathbb{R}^n)$ with $p\in(0,\,1]$ to some quasi-Banach space $\mathcal{B}$ only proving that T maps all $(p,\,\infty,\,s)$-atoms into uniformly bounded elements of $\mathcal{B}$ for some $s\ge\lfloor n(1/p-1)\rfloor$. Here and in what follows $\lfloor t\rfloor$ means the integer part of real t. We should point out that this phenomenon has essentially already been observed by Y. Meyer in [19, p. 19]. Moreover, motivated by this, Yabuta [23] gave some very general sufficient conditions for the boundedness of a linear operator T from $H^p(\mathbb{R}^n)$ with $p\in(0,\,1]$ to $L^q(\mathbb{R}^n)$ with $q\ge 1$ or $H^q(\mathbb{R}^n)$ with $q\in[p,\,1]$. In [12], Yabuta's results were generalized to the setting of spaces of homogeneous type, and moreover, a sufficient condition for the boundedness of T from H^p with $p\in(0,\,1)$ to L^q with $q\in[p,\,1)$ is also provided. However, these conditions are not necessary.

Recently, in [24], a boundedness criterion was established via Lusin function characterizations of Hardy spaces on $\mathbb{R}^n$ as follows: a sublinear operator T extends to a bounded sublinear operator from Hardy spaces $H^p(\mathbb{R}^n)$ with $p\in(0,\,1]$ to some quasi-Banach space $\mathcal{B}$ if and only if T maps all $(p,\,2,\,s)$-atoms into uniformly bounded elements of $\mathcal{B}$ for some $s\ge\lfloor n(1/p-1)\rfloor$, which was also generalized to Hardy spaces H^p with p close to 1 from below on spaces of homogeneous type having the reverse doubling property in [25]. This result shows the structural difference between atomic characterization of $H^p(\mathbb{R}^n)$ via $(p,\,2,\,s)$-atoms and $(p,\infty,\,s)$-atoms. On the other hand, Meda, Sjögren and Vallarino [18] independently obtained some similar results by grand maximal function characterizations of Hardy spaces on $\mathbb{R}^n$. In fact, let $p\in(0,\,1]$, $p<q\in[1,\infty]$ and integer $s\ge\lfloor n(1/p-1)\rfloor$, and let $H^{p,\,q,\,s}_{\rm fin}(\mathbb{R}^n)$ be the set of all finite linear combinations of $(p,\,q,\,s)$-atoms. Denote by $\mathcal{C}(\mathbb{R}^n)$ the set of all continuous functions. For any $f\in H^{p,\,q,\,s}_{\rm fin}(\mathbb{R}^n)$ when $q<\infty$ or $f\in H^{p,\,q,\,s}_{\rm fin}(\mathbb{R}^n)\cap\mathcal{C}(\mathbb{R}^n)$ when $q=\infty$, Meda, Sjögren and Vallarino [18] proved that $\|f\|_{H^p(\mathbb{R}^n)}$ can be achieved by a finite atomic decomposition via $(p,\,q,\,s)$-atom when $q<\infty$ or continuous $(p,\,q,\,s)$-atom when $q=\infty$; from this, they further deduced that if T is a linear operator and maps all $(1,\,q,\,0)$-atoms with $q\in(1,\,\infty)$ or all continuous $(1,\,q,\,0)$-atoms with $q=\infty$ into uniformly bounded elements of some Banach space $\mathcal{B}$, then T uniquely extends to a bounded linear operator from $H^1(\mathbb{R}^n)$ to $\mathcal{B}$ which coincides with T on these $(1,\,q,\,0)$-atoms. Grafokos, Liu and Yang [9] generalize these results to Hardy spaces H^p with p close to 1 from below on spaces of homogeneous type having the reverse doubling property.

In this paper, let $p\in(0,1]$, $q\in(1,\infty)$, $\alpha\in[n(1-1/q),\infty)$ and $w_1,\,w_2\in A_1$. We prove that the norms in weighted Herz-type Hardy spaces $H\dot{K}^{\alpha,\,p}_q(w_1,\,w_2)$ and $HK^{\alpha,\,p}_q(w_1,\,w_2)$ can be achieved by finite central atomic decompositions in some dense subspaces of them. As an application, we prove that if T is a sublinear operator and maps all central $(\alpha,\,q,\,s;\,w_1,\,w_2)_0$-atoms (resp. central $(\alpha,\,q,\,s;\,w_1,\,w_2)$-

atoms of restrict type) into uniformly bounded elements of certain quasi-Banach space $\mathcal{B}$ for certain nonnegative integer $s \geq \lfloor \alpha - n(1-1/q) \rfloor$, then T uniquely extends to a bounded sublinear operator from $H\dot{K}_q^{\alpha,p}(w_1, w_2)$ (resp. $HK_q^{\alpha,p}(w_1, w_2)$) to $\mathcal{B}$.

This paper is organized as follows. In Section 2, we recall notations of weighted Herz-type Hardy spaces, and their central atomic decomposition characterizations in [14] via central $(\alpha, q, s; w_1, w_2)$-atoms or central $(\alpha, q, s; w_1, w_2)$-atoms of restrict type. Moreover, we introduce central $(\alpha, q, s; w_1, w_2)_0$-atoms. In Section 3, we prove that the norms of weighted Herz-type Hardy spaces in some dense subspaces can be achieved by finite atomic decompositions via central $(\alpha, q, s; w_1, w_2)_0$-atoms and central $(\alpha, q, s; w_1, w_2)$-atoms of restrict type; see Theorem 1 below. Then as an application, we establish some criteria to obtain the boundedness of sublinear operators in weighted Herz-type Hardy spaces (see Theorem 2 below).

Finally, we make some conventions. Throughout this paper, let $\mathbb{N}$ be the set all positive integer and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We always use C to denote a positive constant that is independent of main parameters involved but whose value may differ from line to line. We use $f \lesssim g$ to denote $f \leq Cg$.

2. Preliminaries

We begin with recalling definitions of weight Herz spaces; see [4, 14]. In what follows, we always let $B(x, r) \equiv \{y \in \mathbb{R}^n : \ |x-y| < r\}$ for any $r > 0$ and $x \in \mathbb{R}^n$, $B_k \equiv B(0, 2^k)$, $C_k \equiv B_k \setminus B_{k-1}$, $R_k \equiv (C_k \cup C_{k+1})$ and $\chi_k \equiv \chi_{C_k}$ for all $k \in \mathbb{Z}$.

Definition 1. Let $p \in (0, \infty)$, $q \in (1, \infty)$ and $\alpha \in \mathbb{R}$. Let w_1 and w_2 be nonnegative weight functions.

(i) The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(w_1, w_2)$ is defined to be the set of all $f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}; w_2)$ such that

$$\|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)} \equiv \left\{ \sum_{k\in\mathbb{Z}} [w_1(B_k)]^{p\alpha/n} \|f\chi_k\|^p_{L^q(\mathbb{R}^n;\, w_2)} \right\}^{1/p} < \infty.$$

(ii) The non-homogeneous weighted Herz space $K_q^{\alpha,p}(w_1, w_2)$ is defined to be the set of all $f \in L^q_{\text{loc}}(\mathbb{R}^n; w_2)$ such that

$$\|f\|_{K_q^{\alpha,p}(w_1, w_2)} \equiv \left\{ \|f\chi_{B_0}\|^p_{L^q(\mathbb{R}^n;\, w_2)} + \sum_{k=1}^{\infty} [w_1(B_k)]^{p\alpha/n} \|f\chi_k\|^p_{L^q(\mathbb{R}^n;\, w_2)} \right\}^{1/p} < \infty.$$

If $w_1 \equiv w_2 \equiv 1$, then $\dot{K}_q^{\alpha,p}(w_1, w_2)$ and $K_q^{\alpha,p}(w_1, w_2)$ are the standard Herz spaces in [4] and also [14].

To define the corresponding Hardy spaces, we first recall some notation. Let $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$ and $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions endowed with the semi-norms $\{\|\cdot\|_{m,\beta}\}_{m\in\mathbb{N},\beta\in\mathbb{Z}_+^n}$, where $\|\phi\|_{m,\beta} \equiv \sup_{x\in\mathbb{R}^n}(1+|x|^m)|D^\beta\phi(x)|$, $\beta \equiv (\beta_1, \cdots \beta_n)$ and $D^\beta\phi \equiv (\frac{\partial}{\partial x_1})^{\beta_1}\cdots(\frac{\partial}{\partial x_n})^n\phi$. Denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$.

Let $N \in \mathbb{N}$ and $\mathcal{S}_N(\mathbb{R}^n) \equiv \{\phi \in \mathcal{S}(\mathbb{R}^n) : \ \|\phi\|_{m,\beta} \le 1, \ m \le n+N, \ |\beta| \le N\}$. For $f \in \mathcal{S}'(\mathbb{R}^n)$, in [10], the grand maximum function of f is defined by $G_N(f)(x) \equiv \sup_{\phi\in\mathcal{S}_N} M_\phi(f)(x)$, where $M_\phi(f)(x) \equiv \sup_{|y-x|<t} |\phi_t * f(y)|$ and $\phi_t(x) \equiv t^{-n}\phi(t^{-1}x)$ for all $t > 0$ and $x \in \mathbb{R}^n$.

Recall in [6] that a function w is said to be in the Muckenhoupt class A_1 if there exists a constant $C > 0$ such that $Mw(x) \le Cw(x)$ for almost everywhere $x \in \mathbb{R}^n$, where M is the Hardy-Littlewood maximal operator.

The Hardy spaces associated to the weighted Herz spaces in [14] are defined as below.

Definition 2. Let $p \in (0, \infty)$, $q \in (1, \infty)$, $\alpha \in (0, \infty)$ and $N \equiv \max\{\lfloor \alpha - n(1-1/q)\rfloor + 1, 1\}$. Let $w_1, w_2 \in A_1$.

(i) The homogeneous Hardy space $H\dot{K}_q^{\alpha,p}(w_1, w_2)$ associated to $\dot{K}_q^{\alpha,p}(w_1, w_2)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{H\dot{K}_q^{\alpha,p}(w_1,w_2)} \equiv \|G_N(f)\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)} < \infty.$$

(ii) The non-homogeneous Hardy space $HK_q^{\alpha,p}(w_1, w_2)$ associated to $K_q^{\alpha,p}(w_1, w_2)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{HK_q^{\alpha,p}(w_1,w_2)} \equiv \|G_N(f)\|_{K_q^{\alpha,p}(w_1,w_2)} < \infty.$$

Let $p \in (0,\infty)$, $q \in (1,\infty)$ and $w_1, w_2 \in A_1$. Notice that if $\alpha \in (0, n(1-1/q))$, then $(H\dot{K}_q^{\alpha,p}(w_1, w_2) \cap L^q(\mathbb{R}^n\setminus\{0\}; w_2)) = \dot{K}_q^{\alpha,p}(w_1, w_2)$ and $(HK_q^{\alpha,p}(w_1, w_2) \cap L^q(\mathbb{R}^n; w_2)) = K_q^{\alpha,p}(w_1, w_2)$; and if $\alpha \in [n(1-1/q), \infty)$, then $(H\dot{K}_q^{\alpha,p}(w_1, w_2) \cap L^q(\mathbb{R}^n \setminus \{0\}; w_2)) \subsetneq \dot{K}_q^{\alpha,p}(w_1, w_2)$ and $(HK_q^{\alpha,p}(w_1, w_2) \cap L^q(\mathbb{R}^n; w_2)) \subsetneq K_q^{\alpha,p}(w_1, w_2)$; see [14, 15]. Thus, in what follows, we always assume that $p \in (0,\infty)$, $q \in (1,\infty)$ and $\alpha \in [n(1-1/q), \infty)$.

Now we state the definition of central atoms. Lu and Yang [14] introduced central $(\alpha, q, s; w_1, w_2)$-atoms and central $(\alpha, q, s; w_1, w_2)$-atoms of restrict type, and use them to characterize the spaces $H\dot{K}_q^{\alpha,p}(w_1, w_2)$ and $HK_q^{\alpha,p}(w_1, w_2)$.

Definition 3. Let $q \in (1,\infty)$, $\alpha \in [n(1-1/q), \infty)$, $s \ge \lfloor \alpha - n(1-1/q)\rfloor$ and $w_1, \ w_2 \in A_1$.

(i) A function a on $\mathbb{R}^n$ is called a central $(\alpha,\, q,\, s;\, w_1,\, w_2)$-atom if it satisfies that

(A1) $\operatorname{supp} a \subset B(0,\, r)$ for some $r > 0$;

(A2) $\|a\|_{L^q(\mathbb{R}^n;\, w_2)} \le [w_1(B(0,\, r))]^{-\alpha/n}$;

(A3) $\int_{\mathbb{R}^n} a(x) x^\beta\, dx = 0$ for all $|\beta| \le s$.

(ii) A function a on $\mathbb{R}^n$ is called a central $(\alpha,\, q,\, s;\, w_1,\, w_2)_0$-atom if it satisfies (A1) through (A3), and $a(x) = 0$ on some neighborhood of 0.

(iii) A function a on $\mathbb{R}^n$ is called a central $(\alpha,\, q,\, s;\, w_1,\, w_2)$-atom of restrict type if it satisfies (A1) with $r \ge 1$, (A2) and (A3).

Theorem A . *Let* $p \in (0, \infty)$, $q \in (1, \infty)$, $\alpha \in [n(1 - 1/q), \infty)$ *and nonnegative integer* $s \ge \lfloor \alpha - n(1 - 1/q) \rfloor$.

(*i*) *Then* $f \in H\dot{K}_q^{\alpha,\, p}(w_1,\, w_2)$ *if and only if* $f = \sum_{k\in\mathbb{Z}} \lambda_k a_k$ *in* $\mathcal{S}'(\mathbb{R}^n)$, *where* a_k *is a central* $(\alpha,\, q,\, s;\, w_1,\, w_2)$*-atom supported in* B_k *and* $\sum_{k\in\mathbb{Z}} |\lambda_k|^p < \infty$. *Moreover,* $\|f\|_{H\dot{K}_q^{\alpha,\, p}(w_1,\, w_2)} \sim \inf\{(\sum_{k\in\mathbb{Z}} |\lambda_k|^p)^{1/p}\}$, *where the infimum is taken over all the above decompositions of* f.

(*ii*) *Then* $f \in HK_q^{\alpha, p}(w_1,\, w_2)$ *if and only if* $f = \sum_{k\in\mathbb{Z}_+} \lambda_k a_k$ *in* $\mathcal{S}'(\mathbb{R}^n)$, *where* a_k *is a central* $(\alpha,\, q,\, s;\, w_1,\, w_2)$*-atom of restrict type supported in* B_k *for* $k \in \mathbb{Z}_+$ *and* $\sum_{k\in\mathbb{Z}_+} |\lambda_k|^p < \infty$. *Moreover,* $\|f\|_{HK_q^{\alpha,\, p}(w_1,\, w_2)} \sim \inf\{(\sum_{k\in\mathbb{Z}_+} |\lambda_k|^p)^{1/p}\}$, *where the infimum is taken over all the above decompositions of* f.

3. Main Results and Their Proofs

In this section, we first prove that the norms in $H\dot{K}_q^{\alpha,\, p}(w_1,\, w_2)$ and $HK_q^{\alpha, p}(w_1,\, w_2)$ can be achieved by finite central atomic decomposition in some dense subspaces of them.

To this end, let $p \in (0, \infty)$, $q \in (1, \infty)$, $\alpha \in [n(1 - 1/q), \infty)$, $s \ge \lfloor \alpha - n(1 - 1/q) \rfloor$ and $w_1,\ w_2 \in A_1$. Denote by $\dot{F}_p^{a,\, q,\, s}(w_1,\, w_2)$ the collection of all finite linear combinations of central $(\alpha,\, q,\, s;\, w_1,\, w_2)$-atoms, and for $f \in F_p^{a,\, q,\, s}(w_1,\, w_2)$, define

$$\|f\|_{\dot{F}_p^{a,\, q,\, s}(w_1,\, w_2)} \equiv \inf\Big\{\Big(\sum_{j=1}^m |\lambda_j|^p\Big)^{1/p} :\ m \in \mathbb{N},\, f = \sum_{j=1}^m \lambda_j a_j,\ \{a_j\}_{j=1}^m \text{ are central } (\alpha,\, q,\, s;\, w_1,\, w_2)_0\text{−atoms}\Big\}. \tag{3.1}$$

Let $C\dot{F}_p^{a,\, q,\, s}(w_1,\, w_2)$ be the collection of all finite linear combinations of $\mathcal{C}^\infty(\mathbb{R}^n)$ central $(\alpha, q, s; w_1, w_2)_0$-atoms, and for $f \in C\dot{F}_p^{a,q,s}(w_1,w_2)$, define $\|f\|_{C\dot{F}_p^{a,q,s}(w_1,w_2)}$

as in (3.1) just replacing central $(\alpha, q, s;\ w_1, w_2)_0$-atoms by $\mathcal{C}^\infty(\mathbb{R}^n)$ central $(\alpha, q, s;\ w_1, w_2)_0$-atoms.

We also denote by $F_p^{a,q,s}(w_1, w_2)$ (resp. $CF_p^{a,q,s}(w_1, w_2)$) the collection of all finite linear combinations of central $(\alpha, q, s;\ w_1, w_2)$-atoms of restrict type (resp. $\mathcal{C}^\infty(\mathbb{R}^n)$ central $(\alpha, q, s;\ w_1, w_2)$-atoms of restrict type) and for $f \in F_p^{a,q,s}(w_1, w_2)$ (resp. $f \in CF_p^{a,q,s}(w_1, w_2)$), define $\|f\|_{F_p^{a,q,s}(w_1,w_2)}$ (resp. $\|f\|_{CF_p^{a,q,s}(w_1,w_2)}$) as (3.1) just replacing central $(\alpha, q, s;\ w_1, w_2)_0$-atom by central $(\alpha, q, s;\ w_1, w_2)$-atom of restrict type (resp. $\mathcal{C}^\infty(\mathbb{R}^n)$ central $(\alpha, q, s;\ w_1, w_2)$-atom of restrict type).

For $s \in \mathbb{Z}_+$, let $\mathcal{D}_s(\mathbb{R}^n)$ be the collection of all functions $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} f(x)x^\beta\,dx = 0$ for all $|\beta| \le s$, and let $\dot{\mathcal{D}}_s(\mathbb{R}^n)$ be the set of all functions $f \in \mathcal{D}_s(\mathbb{R}^n)$ with $0 \notin \operatorname{supp} f$.

One of our main results is as follows.

Theorem 1. *Let* $p \in (0,\infty)$, $q \in (1,\infty)$, $\alpha \in [n(1-1/q),\infty)$ *and nonnegative integer* $s \ge \lfloor \alpha - n(1-1/q) \rfloor$.

- (i) *Then* $\|\cdot\|_{H\dot{K}_q^{\alpha,p}(w_1,w_2)}$ *and* $\|\cdot\|_{\dot{F}_p^{\alpha,q,s}(w_1,w_2)}$ *(resp.* $\|\cdot\|_{C\dot{F}_p^{\alpha,q,s}(w_1,w_2)}$*) are equivalent on* $\dot{F}_p^{\alpha,q,s}(w_1, w_2)$ *(resp.* $C\dot{F}_p^{\alpha,q,s}(w_1, w_2)$*).*
- (ii) *Then* $\|\cdot\|_{HK_q^{\alpha,p}(w_1,w_2)}$ *and* $\|\cdot\|_{F_p^{\alpha,q,s}(w_1,w_2)}$ *(resp.* $\|\cdot\|_{CF_p^{\alpha,q,s}(w_1,w_2)}$*) are equivalent on* $F_p^{\alpha,q,s}(w_1, w_2)$ *(resp.* $CF_p^{\alpha,q,s}(w_1, w_2)$*).*

Proof. Since the proof of (ii) is similar to that of (i), we only prove (i) here. Moreover, we only prove the equivalence between $\|\cdot\|_{H\dot{K}_q^{\alpha,p}(w_1,w_2)}$ and $\|\cdot\|_{C\dot{F}_p^{\alpha,q,s}(w_1,w_2)}$ on $C\dot{F}_p^{\alpha,q,s}(w_1, w_2)$ since the equivalence between $\|\cdot\|_{H\dot{K}_q^{\alpha,p}(w_1,w_2)}$ and $\|\cdot\|_{\dot{F}_p^{\alpha,q,s}(w_1,w_2)}$ on $\dot{F}_p^{\alpha,q,s}(w_1, w_2)$ can be obtained by a slight modification.

It is easy to see that for any $f \in C\dot{F}_p^{\alpha,q,s}(w_1, w_2)$, $\|f\|_{C\dot{F}_p^{\alpha,q,s}(w_1,w_2)} \le \|f\|_{H\dot{K}_q^{\alpha,p}(w_1,w_2)}$. To prove the converse, we use some ideas in [14].

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $0 \le \psi(x) \le 1$ for all $x \in \mathbb{R}^n$, $\psi(x) = 1$ if $|x| \le 1/2 + 1/10$ and $\psi(x) = 0$ if $|x| \ge 1 - 1/10$. Let $\varphi(x) \equiv \psi(x/2) - \psi(x)$ for all $x \in \mathbb{R}^n$. Then $\operatorname{supp}\varphi \subset \{x \in \mathbb{R}^n :\ 1/2 \le |x| < 2\} = R_0$. Let $\Phi_k(x) \equiv \varphi(2^{-k}x)$ for $x \in \mathbb{R}^n$. Then $\sum_{k\in\mathbb{Z}} \Phi_k(x) = 1$ for $x \in \mathbb{R}^n \setminus \{0\}$ and $\operatorname{supp}\Phi_k \subset R_k$.

Let $\{\widetilde{\psi}_\beta :\ |\beta| \le s\} \subset \mathcal{S}(\mathbb{R}^n)$ being a dual basis of $\{x^\beta :\ |\beta| \le s\}$ with respect to the weight $|R_0|^{-1}\varphi$, namely,

$$\frac{1}{|R_0|}\int_{\mathbb{R}^n} x^\beta \widetilde{\psi}_\gamma(x)\varphi(x)\,dx = \delta_{\beta\gamma},$$

where $\delta_{\beta\gamma} = 0$ if $\beta \ne \gamma$ and $\delta_{\beta\gamma} = 1$ if $\beta = \gamma$. By changing of the variable, we have

$$2^{-k(n+|\beta|)}\frac{1}{|R_0|}\int_{\mathbb{R}^n} x^\beta \widetilde{\psi}_\gamma(2^{-k}x)\varphi(2^{-k}x)\,dx = \delta_{\beta\gamma}.$$

For all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $|\beta| \le s$, set

$$\psi_{k,\beta}(x) \equiv 2^{-k(n+|\beta|)}|R_0|^{-1}\widetilde{\psi}_\beta(2^{-k}x)\Phi_k(x).$$

Then $\psi_{k,\beta} \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}\psi_{k,\beta} \subset R_k$,

$$\|\psi_{k,\beta}\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^{-k(n+|\beta|)} \tag{3.2}$$

and

$$\int_{\mathbb{R}^n} \psi_{k,\beta}(x)x^\gamma\,dx = \delta_{\beta\gamma}. \tag{3.3}$$

Let $f \in \mathcal{D}_s(\mathbb{R}^n)$. For $k \in \mathbb{Z}$, put $f_k \equiv f\Phi_k$ and

$$P_k \equiv \sum_{|\beta|\le s} \psi_{k,\beta} \int_{\mathbb{R}^n} f_k(y)y^\beta\,dy.$$

It is easy to see that $f_k - P_k \in \dot{\mathcal{D}}_s(\mathbb{R}^n)$ and $\operatorname{supp}(f_k - P_k) \subset R_k$ for $k \in \mathbb{Z}$. Now we decompose f as follows,

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}}[f_k(x)-P_k(x)]+\sum_{k\in\mathbb{Z}}P_k(x) \\ &= \sum_{k\in\mathbb{Z}}[f_k(x)-P_k(x)]+\sum_{k\in\mathbb{Z}}\sum_{|\beta|\le s}[\psi_{k,\beta}(x)-\psi_{k+1,\beta}(x)]\int_{\mathbb{R}^n}\sum_{\ell=-\infty}^{k} f_\ell(y)y^\beta\,dy, \end{aligned}$$

where the equality holds for $x \in \mathbb{R}^n \setminus \{0\}$.

Notice that $|f(x)| \le G_N(f)(x)$ for all $x \in \mathbb{R}^n$. Then by (3.2), the Hölder inequality and the property of the A_1 weight (see [6]), we have

$$\begin{aligned} \|P_k\|_{L^q(\mathbb{R}^n;\,w_2)} &\le \sum_{|\beta|\le s} 2^{-kn}\left\{\int_{B_{k+1}} w_2(x)^q\,dx\right\}^{1/q}\int_{\mathbb{R}^n} G_N(f)(x)\chi_{R_{k+1}}(x)\,dx \\ &\le \sum_{|\beta|\le s}\|G_N(f)\chi_{R_k}\|_{L^q(\mathbb{R}^n;\,w_2)}\left\{\frac{1}{|B_{k+1}|}\int_{B_{k+1}}[w_2(x)]^q\,dx\right\}^{1/q} \\ &\quad\times\left\{\frac{1}{|B_{k+1}|}\int_{B_{k+1}}[w_2(x)]^{1/(q-1)}\,dx\right\}^{1-1/q} \lesssim \|\chi_{R_k}G_N(f)\|_{L^q(\mathbb{R}^n;\,w_2)}. \end{aligned}$$

This implies that

$$\|f_k - P_k\|_{L^q(\mathbb{R}^n;\,w_2)} \lesssim \|\chi_{R_k}G_N(f)\|_{L^q(\mathbb{R}^n;\,w_2)}.$$

Let $\lambda_k \equiv [w_1(B_{k+1})]^{\alpha/n}\|\chi_{R_k} G_N(f)\|_{L^q(\mathbb{R}^n;\, w_2)}$ and $a_k \equiv (\lambda_k)^{-1}(f_k - P_k)$. Then there exists a constant $C > 0$ such that Ca_k is a central $(\alpha,\, q,\, s;\, w_1,\, w_2)$-atom supported in $B_{k+1} \setminus B_{k-1}$. Notice that $w_1 \in A_1$ implies that there exist constant $C > 0$ and $\delta > 0$ such that $w_1(B(x,\, \lambda r)) \le C\lambda^\delta w_1(B(x,\, r))$ for all $x \in \mathbb{R}^n$, $r > 0$ and $\lambda > 1$; see [6]. We then have

$$\left\{\sum_{k\in\mathbb{Z}} |\lambda_k|^p\right\}^{1/p} \le \left\{\sum_{k\in\mathbb{Z}} [w_1(B_{k+1})]^{pk\alpha/n}\|\chi_{R_k} G_N(f)\|^p_{L^q(\mathbb{R}^n;\, w_2)}\right\}^{1/p} \lesssim \|f\|_{H\dot{K}^{\alpha,\,p}_q(w_1,\, w_2)}. \tag{3.4}$$

To estimate the second summation, for $|\beta| \le s$, we set $\phi^{(\beta)}(y) \equiv \sum_{\ell=-\infty}^{0} \varphi(2^{-\ell}y)y^\beta$ for $y \ne 0$, and $\phi^{(\beta)}(0) = 0$ if $|\beta| > 0$, $\phi^{(\beta)}(0) = 1$ if $|\beta| = 0$. Then there exists a constant $C > 0$ such that $C\phi^{(\beta)} \in \mathcal{S}_N(\mathbb{R}^n)$ for all $|\beta| \le s$. Moreover, it is easy to see that

$$\left|\int_{\mathbb{R}^n} \sum_{\ell=-\infty}^{k} f_\ell(y) y^\beta\, dy\right| = 2^{k(n+|\beta|)}|\phi^{(\beta)}_{2^k} * f(0)| \lesssim 2^{k(n+|\beta|)}\chi_{B_{k+1}}(x) G_N(f)(x). \tag{3.5}$$

Notice that (3.2) and (3.3) imply that $\psi_{k,\,\beta} - \psi_{k+1,\,\beta} \in \dot{\mathcal{D}}_s(\mathbb{R}^n)$ and

$$|\psi_{k,\,\beta} - \psi_{k+1,\,\beta}| \lesssim 2^{-k(n+|\beta|)}\chi_{R_k \cup R_{k+1}}. \tag{3.6}$$

Let $\mu_k \equiv C[w_1(B_{k+2})]^{\alpha/n}\|\chi_{R_k} G_N(f)\|_{L^q(\mathbb{R}^n;\, w_2)}$ and

$$b_k \equiv (\mu_k)^{-1} \sum_{|\beta|\le s} (\psi_{k,\,\beta} - \psi_{k+1,\,\beta}) \int_{\mathbb{R}^n} \sum_{\ell=-\infty}^{k} f_\ell(y) y^\beta\, dy.$$

Then by (3.5) and (3.6), we have that $b_k \in \dot{\mathcal{D}}_s(\mathbb{R}^n)$ supported in $B_{k+2} \setminus B_{k-1}$ and $\|b_k\|_{L^q(\mathbb{R}^n;\, w_2)} \lesssim [w_1(B_{k+2})]^{-\alpha/n}$, which implies that there exists a constant $C > 0$ such that $Cb_k \in \dot{\mathcal{D}}_s(\mathbb{R}^n)$ is a central $(\alpha,\, q,\, s : w_1,\, w_2)$-atom supported in $B_{k+2} \setminus B_{k-1}$ for all $k \in \mathbb{Z}$. By (3.5) again, we have

$$\left\{\sum_{k\in\mathbb{Z}} |\mu_k|^p\right\}^{1/p} \lesssim \left\{\sum_{k\in\mathbb{Z}} [w_1(B_{k+2})]^{pk\alpha/n}\left\|\chi_{R_k \cup R_{k+1}} G_N(f)\right\|^p_{L^q(\mathbb{R}^n;\, w_2)}\right\}^{1/p} \lesssim \|f\|_{H\dot{K}^{\alpha,\,p}_q(w_1,\, w_2)}. \tag{3.7}$$

Notice that if $|k - \ell| > 2$, then $\operatorname{supp} a_k \cap \operatorname{supp} a_\ell = \emptyset$ and $\operatorname{supp} b_k \cap \operatorname{supp} b_\ell = \emptyset$. Then we have $f(x) = \sum_{k\in\mathbb{Z}} \lambda_k a_k(x) + \sum_{k\in\mathbb{Z}} \mu_k b_k(x)$ pointwise for all $x \in \mathbb{R}^n \setminus$

$\{0\}$ and in $\mathcal{S}'(\mathbb{R}^n)$, which together with (3.4), (3.7) and the facts that $\{Ca_k,\, Cb_k\}_{k\in\mathbb{Z}}$ are central $(p,\, q,\, s;\, w_1,\, w_2)_0$-atoms gives the central atomic decomposition of f.

Moreover, for $f \in \dot{\mathcal{D}}_s(\mathbb{R}^n)$, assume $\operatorname{supp} f \subset B_{k_0-1} \setminus B_{k_1+1}$. If $k > k_0$ or $k < k_1$, then $f_k = 0$ and

$$\int_{\mathbb{R}^n} \sum_{\ell=-\infty}^{k} f_\ell(y) y^\beta \, dy = \int_{\mathbb{R}^n} f(y) y^\beta \, dy = 0,$$

which implies $a_k = 0$ and $b_k = 0$. We have a finite atomic decomposition of f, i. e., $f(x) = \sum_{k=k_1}^{k_0} (\lambda_k a_k(x) + \mu_k b_k(x))$, and by (3.4) and (3.7), $\|f\|_{C\dot{F}^{\alpha,\,q,\,s}(w_1,\,w_2)} \lesssim \|f\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)}$, which is desired and thus completes the proof of Theorem 1.

Remark 1. For any $f \in \mathcal{D}_s(\mathbb{R}^n)$, in the proof of Theorem 1 (i), we in fact prove that $f(x) = \sum_{k\in\mathbb{Z}} \lambda_k a_k(x)$ pointwise for all $x \in \mathbb{R}^n \setminus \{0\}$ and in $\mathcal{S}'(\mathbb{R}^n)$, where $\{a_k\}_{k\in\mathbb{Z}}$ are central $(\alpha,\, q,\, s;\, w_1,\, w_2)_0$-atoms with $\operatorname{supp} a_k \subset B_{k+2} \setminus B_{k-1}$ in $\mathcal{D}_s(\mathbb{R}^n)$ and $\left\{\sum_{k\in\mathbb{Z}} |\lambda_k|^p\right\}^{1/p} \lesssim \|f\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)}$. Moreover, if $\operatorname{supp} f \subset B_{k_0-1} \setminus B_{k_1+1}$ for some $k_1 \in \mathbb{Z}$, then $\lambda_k = 0$ for $k > k_0$ and $k < k_1$.

Based on Theorem 1 and Remark 1, we have the following conclusion.

Lemma 1. *Let* $p \in (0, \infty)$, $q \in (1, \infty)$, $\alpha \in [n(1 - 1/q), \infty)$ *and nonnegative integer* $s \geq \lfloor \alpha - n(1 - 1/q) \rfloor$. *Then,*

(i) $C\dot{F}_p^{\alpha,\,q,\,s}(w_1,\, w_2)$ *and* $\dot{F}_p^{\alpha,\,q,\,s}(w_1,\, w_2)$ *are both dense in* $H\dot{K}_q^{\alpha,\,p}(w_1,\, w_2)$*;*

(ii) $CF_p^{\alpha,\,q,\,s}(w_1,\, w_2)$ *and* $F_p^{\alpha,\,q,\,s}(w_1,\, w_2)$ *are both dense in* $HK_q^{\alpha,\,p}(w_1,\, w_2)$.

Proof . Observing that $C\dot{F}_p^{\alpha,\,q,\,s}(w_1,\, w_2) \subset \dot{F}_p^{\alpha,\,q,\,s}(w_1,\, w_2)$, to prove (i), we only need to prove the density of $C\dot{F}_p^{\alpha,\,q,\,s}(w_1,\, w_2)$ in $H\dot{K}_q^{\alpha,p}(w_1,\, w_2)$.

To this end, let $f \in H\dot{K}_q^{\alpha,p}(w_1,\, w_2)$. By Theorem A (i), there exist $\{\lambda_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ and central $(\alpha,\, q,\, s;\, w_1,\, w_2)$-atoms $\{a_k\}_{k\in\mathbb{Z}}$ with $\operatorname{supp} a_k \subset B_k$ such that $f = \sum_{k\in\mathbb{Z}} \lambda_k a_k$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\left\{\sum_{k\in\mathbb{Z}} |\lambda_k|^p\right\}^{1/p} \lesssim \|f\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)}$. Let $f_L = \sum_{|k|<L} \lambda_k a_k$ for $L \in \mathbb{N}$. Then by Theorem A (i), $f_L \in H\dot{K}_q^{\alpha,p}(w_1,\, w_2)$ and

$$\|f - f_L\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)} = \left\| \sum_{|k|\geq L} \lambda_k a_k \right\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)} \lesssim \left\{ \sum_{|k|\geq L} |\lambda_k|^p \right\}^{1/p} \to 0.$$

Notice that $f_L \in L^q(\mathbb{R}^n) \cap H\dot{K}_q^{\alpha,p}(w_1,\, w_2)$ and $\operatorname{supp} f \subset B_{L-1}$. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. Then $\varphi_t * f_L \in \mathcal{D}_s(\mathbb{R}^n)$. We further claim that $\|\varphi_t * f_L - f_L\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)} \to 0$ as $t \to 0$. To see this, since for any $t \in (0, 2^{-L})$ and $|k| < L$, $\|a_k\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)} \leq 1$ and $[w_2(B_{k+1})]^{-(k+1)\alpha/n} \|\varphi_t * a_k -$

$a_k\|^{-1}_{L^q(\mathbb{R}^n;\,w_2)}(\varphi_t * a_k - a_k)$ is a central $(\alpha,\, q,\, s;\, w_1,\, w_2)$-atom supported in B_{k+1}, by Theorem A (i) and the property of $\{\varphi_t\}_{t>0}$, we then have

$$\begin{aligned}&\|\varphi_t * f_L - f_L\|^p_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)}\\&=\left\|\sum_{|k|<L}\lambda_k(\varphi_t * a_k - a_k)\right\|^p_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)}\\&\le\sum_{|k|<L}|\lambda_k|^p[w_1(B_{k+1})]^{p(k+1)\alpha/n}\|\varphi_t * a_k - a_k\|^p_{L^q(\mathbb{R}^n;\,w_2)},\end{aligned}$$

which converges to 0 as $t\to 0$. This verifies the claim.

Moreover, for any $f_L\in\mathcal{D}_s(\mathbb{R}^n)$, by Remark 1, we have an atomic decomposition $f_L=\sum_{k\in\mathbb{Z}}\lambda_{L,\,k}a_{L,\,k}$ pointwise for all $x\in\mathbb{R}^n\setminus\{0\}$ and in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where $a_{L,\,k}\in\mathcal{C}_c^\infty(\mathbb{R}^n)$ is supported in $B_{k+2}\setminus B_{k-1}$ and $\{\sum_k|\lambda_{L,\,k}|^p\}^{1/p}\lesssim\|f_L\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)}$. Let $f_{L,\,J}=\sum_{|k|\le L}\lambda_{L,\,k}a_{L,\,k}$. Then $f_{L,\,J}\in\dot{\mathcal{D}}_s(\mathbb{R}^n)$ and $\|f_{L,\,J}-f_L\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)}\to 0$ as $J\to\infty$. This implies that $C\dot{F}_p^{\alpha,\,q,\,s}(w_1,\,w_2)$ is dense in $H\dot{K}_q^{\alpha,p}(w_1,\,w_2)$.

Applying Theorem A (ii), Theorem 1 (ii) and an argument similar to (i), we can also prove (ii). This completes the proof of Lemma 1.

As an corollary of Lemma 1, we have the following result.

Corollary 1. *Let* $p\in(0,\infty)$, $q\in(1,\infty)$, $\alpha\in[n(1-1/q),\infty)$ *and nonnegative integer* $s\ge\lfloor\alpha-n(1-1/q)\rfloor$. *Then* $\dot{\mathcal{D}}_s(\mathbb{R}^n)$ *is dense in* $H\dot{K}_q^{\alpha,p}(w_1,\,w_2)$ *and* $\mathcal{D}_s(\mathbb{R}^n)$ *is dense in* $HK_q^{\alpha,p}(w_1,\,w_2)$.

As an application of Theorem 1, we give some criteria on the boundedness of sublinear operators in $H\dot{K}_q^{\alpha,p}(w_1,\,w_2)$ and $HK_q^{\alpha,p}(w_1,\,w_2)$.

To this end, recall that a quasi-Banach space $\mathcal{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathcal{B}}$ which is nonnegative, non-degenerate ($\|f\|_{\mathcal{B}}=0$ if and only if $f=0$), homogeneous, and obeys the quasi-triangle inequality $\|f+g\|_{\mathcal{B}}\le K(\|f\|_{\mathcal{B}}+\|g\|_{\mathcal{B}})$ for certain constant $K\ge 1$ and any $f,\,g\in\mathcal{B}$. The notion of p-quasi-Banach space is given in [24].

Definition 4. Let $p\in(0,1]$. A quasi-Banach spaces $\mathcal{B}_p$ with a quasi-norm $\|\cdot\|_{\mathcal{B}_p}$ is said to be a p-quasi-Banach space if $\|f+g\|^p_{\mathcal{B}_p}\le\|f\|^p_{\mathcal{B}_p}+\|g\|^p_{\mathcal{B}_p}$ for any $f,\,g\in\mathcal{B}_p$.

Notice that all Banach spaces are 1-quasi-Banach spaces, and quasi-Banach spaces $L^p(\mathbb{R}^n)$, $H^p(\mathbb{R}^n)$, $\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)$, $H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)$, $K_q^{\alpha,\,p}(w_1,\,w_2)$ and $HK_q^{\alpha,\,p}$

$(w_1,\, w_2)$ with $p \in (0,1)$ are typically p-quasi-Banach spaces. Moreover, according to the Aoki-Rolewicz theorem (see [2] or [21]), any quasi-Banach space is, in essential, a p-quasi-Banach space, where $p = [\log_2(2K)]^{-1}$.

Recall that for any given r-quasi-Banach space $\mathcal{B}_r$ with $r \in (0,\, 1]$ and linear space $\mathcal{Y}$, an operator T from $\mathcal{Y}$ to $\mathcal{B}_r$ is called to be $\mathcal{B}_r$-sublinear if for any $f,\, g \in \mathcal{Y}$ and $\lambda,\, \nu \in \mathbb{C}$, we have

$$\|T(\lambda f + \nu g)\|_{\mathcal{B}_r} \le \left(|\lambda|^r \|T(f)\|^r_{\mathcal{B}_r} + |\nu|^r \|T(g)\|^r_{\mathcal{B}_r}\right)^{1/r}$$

and $\|T(f) - T(g)\|_{\mathcal{B}_r} \le \|T(f-g)\|_{\mathcal{B}_r}$.

Observe that if T is linear, then T is $\mathcal{B}_r$-sublinear. Moreover, if $\mathcal{B}_r = \dot{K}^{\alpha,r}_q(w_1, w_2)$, $K^{\alpha,r}_q(w_1, w_2)$ or $L^r(\mathbb{R}^n)$ and T is sublinear in the classical sense, then T is also $\mathcal{B}_r$-sublinear.

Another main result of this paper is as follows, which already has a lot of applications; see [11].

Theorem 2. *Let* $p \in (0,1]$, $r \in [p,1]$, $q \in (1,\infty)$, $\alpha \in [n(1-1/q),\infty)$ *and nonnegative integer* $s \ge \lfloor \alpha - n(1-1/q) \rfloor$.

(i) *If* T *is a* $\mathcal{B}_r$*-sublinear operator defined on* $\dot{F}^{a,\,q,\,s}_p(w_1,\, w_2)$ *such that*

$$S \equiv \sup\{\|Ta\|_{\mathcal{B}_r}:\ a \text{ is any central } (\alpha,\, q,\, s;\, w_1,\, w_2)_0\text{--atom}\} < \infty \tag{3.8}$$

or defined on $C\dot{F}^{a,\,q,\,s}_p(w_1,\, w_2)$ *such that*

$$S \equiv \sup\{\|Ta\|_{\mathcal{B}_r} : a \text{ is any } \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{\mathrm{n}}) \text{ central } (\alpha, q, s; w_1, w_2)_0\text{--atom}\} < \infty, \tag{3.9}$$

then T *uniquely extends to be a bounded* $\mathcal{B}_r$*-sublinear operator from* $H\dot{K}^{\alpha,\,p}_q$ $(w_1,\, w_2)$ *to* $\mathcal{B}_r$.

(ii) *If* T *is a* $\mathcal{B}_r$*-sublinear operator defined on* $F^{a,\,q,\,s}_p(w_1,\, w_2)$ *such that*

$$S \equiv \sup\{\|Ta\|_{\mathcal{B}_r}:\ a \text{ is any central } (\alpha,\, q,\, s;\, w_1,\, w_2) \text{--atom of restrict type}\} < \infty \tag{3.10}$$

or defined on $C\dot{F}^{a,\,q,\,s}_p(w_1,\, w_2)$ *such that*

$$S \equiv \sup\{\|Ta\|_{\mathcal{B}_r}:\ a \text{ is any } \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{\mathrm{n}}) \text{ central } (\alpha,\, q,\, s;\, w_1,\, w_2) \text{--atom of restrict type}\} < \infty, \tag{3.11}$$

then T *uniquely extends to be a bounded* $\mathcal{B}_r$*-sublinear operator from* $HK^{\alpha,\,p}_q$ $(w_1,\, w_2)$ *to* $\mathcal{B}_r$.

Proof . Assume that (3.9) holds, to prove (i), let $f \in C\dot{F}_p^{\alpha,\,q,\,s}(w_1,\,w_2)$. Then by Theorem 1 (i), there exist numbers $\{\lambda_k\}_{k=1}^m \subset \mathbb{C}$ and central $(\alpha,\,q,\,s;\,w_1,\,w_2)_0$-atoms $\{a_k\}_{k=0}^m \subset \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $f = \sum_{k=1}^m \lambda_k a_k$ and $\{\sum_{k=1}^m |\lambda_k|^p\}^{1/p} \lesssim \|f\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)}$. Since T is $\mathcal{B}_r$-sublinear, $\|Ta_k\|_{\mathcal{B}_r} \lesssim 1$ and $r \in [p, 1]$, we have

$$\|Tf\|_{\mathcal{B}_r} \le \left\{\sum_{k=1}^m |\lambda_k|^r \|Ta_k\|_{\mathcal{B}_r}^r\right\}^{1/r} \lesssim \left\{\sum_{k=1}^m |\lambda_k|^p\right\}^{1/p} \lesssim \|f\|_{H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)}.$$

Then using density of $C\dot{F}_p^{\alpha,\,q,\,s}(w_1,\,w_2)$ in $H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)$ given in Lemma 1 (i), we extend the $\mathcal{B}_r$-sublinear operator T uniquely to a bounded operator from $H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)$ to $\mathcal{B}_r$.

If (3.8) holds, by a slight modification of the above procedure, we can also uniquely and boundedly extend T to the whole $H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)$.

The proof of (ii) is similar to that of (i). We leave the details to the reader. This completes the proof of Theorem 2.

Remark 2.

(i) If T is $\mathcal{B}_r$-sublinear and (3.8) holds, then (3.9) also holds automatically and thus we have two extensions of T by Theorem 2 (i). Since both of the two extensions are unique and coincide on $\dot{F}_p^{\alpha,\,q,\,s}(w_1,\,w_2)$, by Lemma 1 (i), the two extensions of T coincide on $H\dot{K}_q^{\alpha,\,p}(w_1,\,w_2)$. Similarly, if (3.10) holds, then we have two extensions of T which coincide.

(ii) The conditions (3.8) or (3.9) and (3.10) or (3.11) are also necessary. Moreover, even when $\mathcal{B}_r \equiv \dot{K}_q^{\alpha,\,p}(w_1,\,w_2)$ (resp. $\mathcal{B}_r \equiv K_q^{\alpha,\,p}(w_1,\,w_2)$), Theorem 2 also makes an improvement of Theorem 2 with $p \in (0, 1]$ in [14] by removing the $L^q(\mathbb{R}^n;\,w_2)$-boundedness of T and some size conditions therein. In fact, in Theorem 2, we do not need the $L^q(\mathbb{R}^n;\,w_2)$-boundedness of T or the continuity of T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. This is convenient in applications of Herz-type Hardy spaces to the boundedness of sublinear operators.

(iii) Theorem 2 with $p \in (1, \infty)$ in [14] still holds by using Lemma 1 and Theorem 1 to seal a gap in the proof in [14]. We leave the details to the reader.

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