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BOUNDEDNESS OF g-FUNCTIONS ON TRIEBEL-LIZORKIN SPACES

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Abstract. We prove that the g-function operator g_ϕ, where $\phi(x) = h(|x|)\Omega(x)$ with $\Omega(x) = \Omega(x') \in H^1(S^{n-1})$ and $h(s)$ satisfing certain continuity hypothesis, is bounded on Triebel-Lizorkin space $F_p^{\alpha,q}(R^n)$ when $0 < \alpha < 1$ and $1 < p, q < \infty$. In particular, we get that the Marcinkiewicz integral operator μ_Ω with H^1-kernel is bounded on $F_p^{\alpha,q}$.

1. INTRODUCTION

Recently, the boundedness of singular integral operators on Sobolev sapces and Triebel-Lizorkin spaces is investigated widely, for example, see [2], [3], [4] and [7]. It was proved in [3] that

$$T_{\Omega,\alpha}f(x) = P.\ V.\ \int_{R^n} b(|y|)\Omega(y)|y|^{-n-\alpha}f(x-y)dy$$

is bounded on $F_p^{\beta,q}$ for $\Omega \in L^r(S^{n-1})$ and $\alpha = 0$, and bounded from $F_p^{\beta+\alpha,q}$ to $F_p^{\beta,q}$ for $\alpha > 0$ and $\Omega \in H^s(S^{n-1})$ where $r > 1$, $s = (n-1)/(n-1+\alpha)$, $n \geq 2$, $\beta \in R$, $1 < p, q < +\infty$, $b \in L^\infty(R^+)$ and Ω satisfies some cancellation condition. When Ω satisfies

$$\sup_{\theta \in S^{n-1}} \int_{S^{n-1}} |\Omega(y')|(\ln|y' \cdot \theta|^{-1})^{1+\gamma} d\sigma(y') < \infty,\ \forall\ \gamma > 0,$$

a condition firstly introduced by Grafakos and Stefanov in [8], we proved in [6] that T_Ω (with $b(|y|) \equiv 1$ and $\alpha = 0$) is bounded on $F_p^{\beta,q}$.

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In this paper we shall consider the g-function and the Marcinkiewicz integral. Recall that the g-function is defined by

$$g_\phi(f)(x) = \left(\int_0^{+\infty} |\phi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} \tag{1}$$

where $\phi_t(x) = t^{-n}\phi(x/t)$. For $\phi(x) = \Omega(x)|x|^{1-n}\chi_{|x|<1}(x)$ with $\Omega \in L^1(S^{n-1})$ and $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$, g_ϕ is just the Marcinkiewicz integral μ_Ω. The most common assumptions imposed on $\phi(x)$ to ensure the L^p boundedness of g_ϕ are the following:

(i) $|\phi(x)| \le C(1+|x|)^{-n-\epsilon}$,

(ii) $\int_{R^n} \phi(x)dx = 0$,

(iii) $\int_{R^n} |\phi(x+y) - \phi(x)|dx \le C|y|^\epsilon, \ \epsilon > 0$.

First, the above three conditions imply an L^2 estimate of g_ϕ. In addition, one can obtain a Hörmander-type condition:

$$\int_{|x|>2|y|} \left(\int_0^\infty |\phi_t(x+y) - \phi_t(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx \le C, \ \forall \ y \ne 0. \tag{2}$$

And at last the L^p boundedness of g_ϕ is reached by using the famous BCP method.

Let us recall the fundamental result of BCP. Let F be a Banach space and $K(x)$ be a strongly measurable, locally integrable function from $R^n \backslash \{0\}$ to F. An operator U is called a BCP operator if

(1) there is a $p_0 \in (1, +\infty)$ such that for any $f \in L^{p_0}(R^n)$,

$$\int_{R^n} \|U(f)\|_F^{p_0} dx \le C \int_{R^n} |f|^{p_0} dx,$$

(2) for every continuous function with compact support supp$(f) \subset R^n$, $U(f)(x)$ coincides with $K * f(x)$ outside supp(f) and there exists a constant $M \ge 0$ such that

$$\int_{|x|>2|y|} \|K(x-y) - K(x)\|_F dx \le M, \ \forall y \ne 0.$$

Under these assumptions, U can be extended to a bounded operator from $L^p(R^n)$ to $L^p(R^n, F)$ for every $p \in (1, +\infty)$, see [1].

Note that g_ϕ commutes with translation (i.e. $\tau_y g_\phi = \phi \tau_y$ for each $y \in R^n$), but it is not linear, which brings difficulties in studying its boundedness on $F_p^{\beta,q}$.

In [9], using the BCP method, Korry proved that g_ϕ is bouned on Sobolev space $H_p^\alpha(R^n)$, $0<\alpha<1$ with ϕ satisfying (i)-(iii). In fact, by Korry's method, the space H_p^α can easily be replaced by the more general space $F_p^{\alpha,q}$ with $0<\alpha<1$ and $1<p,q<+\infty$. As a corollary, we get the $F_p^{\alpha,q}$ boundedness of μ_Ω when $\Omega\in Lip_r(S^{n-1})$ $(0<r\leq 1)$, $0<\alpha<1$ and $1<p,q<\infty$. It is natural to ask whether μ_Ω is bounded on $F_p^{\alpha,q}$ for a rough Ω. In this paper, we shall search for some rough conditions on $\phi(x)$ such that the g-function operator is bounded on $F_p^{\alpha,q}$.

Below we always assume $\phi(x)=h(|x|)\Omega(x)$ where $\Omega(x')\in L^1(S^{n-1})$ and $h(s)$ is a function on R^+. The method we shall use is the BCP method developed in [9] and the rotation method developed in [5] and [11]. Our main result is

Theorem 1. *Let $\Omega\in H^1(S^{n-1})$ which satisfies $\int\Omega(x')d\sigma(x')=0$. Suppose there exists an $\epsilon>0$ such that $|h(s)|\leq C\frac{s^{-n+\epsilon}}{(1+s)^{2\epsilon}}$, and a $\gamma>0$ such that*

$$\int_R \left|(s+m)^{n-1}h(s+m)-s^{n-1}h(s)\right| ds\leq C|m|^\gamma.$$

Then g_ϕ is bounded on $F_p^{\alpha,q}$, $0<\alpha<1$, $1<p,q<+\infty$.

Corollary 2. *When $\Omega\in H^1(S^{n-1})$ satisfies $\int_{S^{n-1}}\Omega(x')d\sigma(x')=0$, μ_Ω is bounded on $F_p^{\alpha,q}$, $0<\alpha<1$, $1<p,q<+\infty$.*

It is unknown whether μ_Ω is bounded on $F_p^{\alpha,q}(R^n)$ for $\alpha\geq 1$ and $1<p,q<+\infty$.

2. Some Lemmas

We first recall the definition of $F_p^{\alpha,q}$. Let $\Phi\in C_c^\infty(R^n)$ which satisfies supp$(f)\subset\{\xi : 1/2<|\xi|<2\}$ and $\Phi(\xi)>1$ if $3/5<|\xi|<5/3$. Denote $\hat{\Psi}_k(\xi)=\Phi(2^k\xi)$. We say that f is in the non-homogenous Triebel-Lizorkin space $F_p^{\alpha,q}$, $\alpha>0$, $1<p,q<\infty$ if

$$\|f\|_{F_p^{\alpha,q}}=\|f\|_{L^p}+\left\|\left(\sum_{k\in Z}|2^{-k\alpha}\Psi_k*f|^q\right)^{1/q}\right\|_{L^p}<+\infty.$$

In this paper, we shall use an equivalent norm of $F_p^{\alpha,q}$. Define

$$S_r(f)(x)=\left(\int_0^{+\infty}\left(\int_{B_n}\left|\frac{f(x+l\xi)-f(x)}{l^\alpha}\right|^r d\xi\right)^{q/r}\frac{dl}{l}\right)^{1/q}$$

where B_n is the unit ball of R^n.

Lemma 1. *If* $0<\alpha<1$, $1<p,q<+\infty$ *and* $1\le r<\min(p,q)$, *then*

$$N_r(f)=\|f\|_{L^p}+\|S_r(f)\|_{L^p}$$

is an equivalent norm of $F_q^{\alpha,p}$.

This result can be found on page 194 of [10], or in Lemma 1 of [9].

Let $H=L^2(R^+,dt/t)$, we have

$$g_\phi(f)(x)=|\phi_t*f(x)|_H.$$

Set $E_k=R^k\times(0,+\infty)\times B_n$ with $k=1$ or n, and $\vec{p}=(p_1,p_2,p_3)$ with $1<p_i<+\infty$ for $i=1,2,3$. Denote by $L^{\vec{p}}(E_k)$ the space of all measurable functions $F(x,l,\xi)$ defined on E_k such that

$$\|F\|_{L^{\vec{p}}(E_k)}=\left(\int_{R^k}\left(\int_0^{+\infty}\left(\int_{B_n}|F(x,l,\xi)|^{p_3}d\xi\right)^{p_2/p_3}dl\right)^{p_1/p_2}dx\right)^{1/p_1}<+\infty.$$

Similarly, we define the H-valued function space $L^{\vec{p}}(E_k,H)$. We have

Lemma 2. *Let* $\sigma(s)\in L^1(R)$, $\int_R\sigma(s)ds=0$ *and* $|\sigma_t(s)|_H\le B|s|^{-1}$. *If* $\sigma(s)$ *satisfies*

$$\int_{|s|>2|m|}|\sigma_t(s+m)-\sigma_t(s)|_H ds\le B,\ \forall\ m\ne 0, \tag{3}$$

then for each $F(s,l,\xi)\in L^{\vec{p}}(E_1)$, *we have*

$$\|g_\sigma(F(\cdot,l,\xi))(s)\|_{L^{\vec{p}}(E_1)}=\|U_t(F)\|_{L^{\vec{p}}(E_1,H)}\le C(\|\sigma\|_{L^1}+B)\|F\|_{L^{\vec{p}}(E_1)}$$

where $U_t(F)(s,l,\xi)=\sigma_t*(F(\cdot\ ,l,\xi))(s)$.

Proof. Note the given conditions imply that g_σ is bounded on L^p, see [11], Lemma 1. By Hörmander's condition (3) and the BCP method, we get the Lemma by iteration. See [9], Lemma 2 for details. ∎

Now taking $\lambda\in C_c^\infty(R)$ with supp$(\lambda)\subset[1,2]$, defining

$$\sigma^{(u')}(s)=|s|^{n-1}\mathcal{R}\left(\lambda(|\cdot|)|\cdot|^{1-n}\Omega(\cdot)\right)(su')$$

where $\mathcal{R}=(\mathcal{R}_1,\mathcal{R}_2,\cdots,\mathcal{R}_n)$ and $\mathcal{R}_j$ denotes the j-th Riesz transform on R^n and setting

$$\Omega^*(u')=\sup_{s\in R,j=0,1}(1+|s|)^{2+j}\left|\partial_s^j\sigma^{(u')}(s)\right|,$$

we have the following Lemma 3.

Lemma 3. *Let* $\Omega \in H^1(S^{n-1})$ *be an even kernel. Then* $\sigma^{(u')}(s) \in L^1(R)$ *for a.e.* $u' \in S^{n-1}$, $|\sigma_t^{(u')}(s)|_H \le C\Omega^*(u')|s|^{-1}$ *and*

$$\int_{|s|>2|m|} \left|\sigma_t^{(u')}(s+m) - \sigma_t^{(u')}(s)\right|_H ds \le C\Omega^*(u').$$

Furthermore,

$$\int_{S^{n-1}} \Omega^*(u')d\sigma(u') \le C(1 + \|\Omega\|_{H^1}),$$

see [*11*].

3. Proof of Theorem

Proof of Theorem 1. Denote Ω^o, Ω^e the odd and even part of Ω. Then

$$\phi_t * f(x) = \int_{R^n} t^{-n} h(\frac{|u|}{t})\Omega^o(u) f(x-u)du + \int_{R^n} t^{-n} h(\frac{|u|}{t})\Omega^e(u) f(x-u)du.$$

The first integral equals

$$\int_{S^{n-1}} \Omega^o(u') \int_0^{+\infty} t^{-n} s^{n-1} h(\frac{s}{t}) f(x - su')dsd\sigma(u')$$
$$= \int_{S^{n-1}} \Omega^o(u') \int_0^{+\infty} g_t(s) f(x - su')dsd\sigma(u')$$

where $g(s) = s^{n-1}h(s)$. Extend $h(s)$ to the whole real line such that $h(s) = 0$ when $s < 0$. Then if we set $\tilde{g}(s) = g(s) - g(-s)$, the first integral can be written as

$$\frac{1}{2}\int_{S^{n-1}} \Omega^o(u') \int_R \tilde{g}_t(s) f(x - su')dsd\sigma(u'). \tag{5}$$

To rewrite the second integral, we first set $\rho(s) = g(s) - \lambda(s)$ where $\lambda(s)$ is as in the definition of $\sigma^{(u')}$ and further, we require that $\int_R \lambda(s)ds = \int_R g(s)ds$. Then the second integral equals

$$\int_{R^n} \left(t^{-n} h(\frac{|u|}{t}) - t^{-1}|u|^{1-n}\lambda(\frac{|u|}{t}) \right) \Omega^e(u) f(x-u)du$$
$$+ \int_{R^n} t^{-1}|u|^{1-n}\lambda(\frac{|u|}{t})\Omega^e(u) f(x-u)du.$$

Denote the above two terms by I_1 and I_2 respectively. Obviously

$$I_1 = \int_{S^{n-1}} \Omega^e(u') \int_R \rho_t(s) f(x - su')dsd\sigma(u'). \tag{6}$$

Noting that $\mathcal{R}^2 = -\mathrm{Id}$, we have

$$\begin{aligned} I_2 &= \int_{R^n} -\mathcal{R}^2 \left[\Omega^e(\cdot)|\cdot|^{1-n}\lambda_t(|\cdot|)\right](u)f(x-u)du \\ &= \int_{R^n} \mathcal{R}\left[\Omega^e(\cdot)|\cdot|^{1-n}\lambda_t(|\cdot|)\right](u)\mathcal{R}f(x-u)du \\ &= \int_{R^n} t^{1-n}\mathcal{R}\left[(\lambda(|\cdot|)|\cdot|^{1-n})_t\Omega^e(\cdot)\right](u)\mathcal{R}f(x-u)du \\ &= \int_{S^{n-1}}\int_0^{+\infty}(\frac{s}{t})^{n-1}\mathcal{R}\left[(\lambda(|\cdot|)|\cdot|^{1-n})_t\Omega^e(\cdot)\right](su')\mathcal{R}f(x-su')dsd\sigma(u') \\ &= \frac{1}{2}\int_{S^{n-1}}\int_R(\frac{|s|}{t})^{n-1}\mathcal{R}\left[(\lambda(|\cdot|)|\cdot|^{1-n})_t\Omega^e(\cdot)\right](su')\mathcal{R}f(x-su')dsd\sigma(u'). \end{aligned}$$

Recalling the definition of $\sigma^{(u')}(s)$, we have

$$I_2 = \frac{1}{2}\int_{S^{n-1}}\int_R \sigma_t^{(u')}(s)\mathcal{R}f(x-su')dsd\sigma(u'). \tag{7}$$

Below we shall always write f instead of $\mathcal{R}f$ in (7) because we have already known, for example by [3], that $\mathcal{R}f$ is bounded on $F_q^{\alpha,p}$.

Decompose $R^n = L(u') + L(u')^\perp$ where $L(u') = \{au' : \ a \in R\}$, then $x = \tilde{x} + su'$ with $\tilde{x} \in L(u')^\perp$. Setting $f_{\tilde{x}}^{u'}(s) = f(\tilde{x}+su')$, we have

$$\begin{aligned} g_\phi(f)(x) = |\phi_t * f(x)|_H &\le \frac{1}{2}\int_{S^{n-1}}\left|\sigma_t^{(u')} * f_{\tilde{x}}^{u'}(s)\right|_H du' \\ &+\frac{1}{2}\int_{S^{n-1}}|\Omega^o(u')|\left|\tilde{g}_t * f_{\tilde{x}}^{u'}(s)\right|_H d\sigma(u') + \int_{S^{n-1}}|\Omega^o(u')|\left|\rho_t * f_{\tilde{x}}^{u'}(s)\right|_H d\sigma(u'). \end{aligned} \tag{8}$$

By Lemma 2 and Lemma 3, $|\sigma_t^{(u')} * f_{\tilde{x}}^{u'}(s)|_H$ is bounded on $L^p(R)$. Later we shall prove that both $\tilde{g}(s)$ and $\rho(s)$ satisfy the hypothesis of Lemma 2. So we can use Minkowski's inequality to get the L^p boundedness of g_ϕ. Thus by Lemma 1, we only need to show $\|S_r(g_\phi(f))\|_{L^p(R^n)} \le C\|S_rf\|_{L^p(R^n)}$. Note that

$$\begin{aligned} \|S_r(g_\phi(f))(x)\|_{L^p(R^n)} &= \left\|\frac{g_\phi(f)(x+l\xi) - g_\phi(f)(x)}{l^{\alpha+1/q}}\right\|_{L^{\vec{p}}(E_n)} \\ &\le \left\|g_\phi\left(\frac{\tau_{l\xi}(f) - f}{l^{\alpha+1/q}}\right)(x)\right\|_{L^{\vec{p}}(E_n)}, \end{aligned}$$

where $\vec{p} = (p,q,r)$ and $0 < r < \min\{p,q\}$. Set $F(x,l,\xi) = \frac{\tau_{l\xi}(f)(x) - f(x)}{l^{\alpha+1/q}}$ and $F_{\tilde{x}}^{u'}(s,l,\xi) = F(\tilde{x}+su',l,\xi)$. Using (8) with $f(\cdot)$ being replaced by $F(\cdot,l,\xi)$, we get

$$\begin{aligned}\|S_r(g_\phi(f))(x)\|_{L^p(R^n)} \le\ & \frac{1}{2}\int_{S^{n-1}} \left\|\sigma_t^{(u')} * F_{\tilde{x}}^{u'}(s,l,\xi)\right\|_{L^{\vec{p}}(E_n,H)} d\sigma(u') \\ & +\frac{1}{2}\int_{S^{n-1}} |\Omega^o(u')| \left\|\tilde{g}_t * F_{\tilde{x}}^{u'}(s,l,\xi)\right\|_{L^{\vec{p}}(E_n,H)} d\sigma(u') \\ & +\int_{S^{n-1}} |\Omega^e(u')| \left\|\rho_t * F_{\tilde{x}}^{u'}(s,l,\xi)\right\|_{L^{\vec{p}}(E_n,H)} d\sigma(u')\end{aligned}$$

Now we may apply Lemma 2 directly to finish our proof because we can further write, for example, the last integral as

$$\int_{S^{n-1}} |\Omega^e(u')| \left(\int_{L(u')^\perp} \left\|\rho_t * F_{\tilde{x}}^{u'}(s,l,\xi)\right\|_{L^{\vec{p}}(E_1,H)}^p d\tilde{x}\right)^{1/p} d\sigma(u').$$

Finally we show that $\rho(s)$ and $\tilde{g}(s)$ satisfy the required condition of Lemma 2. We shall only compute for $\tilde{g}(s)$. It is trivial that $\int_R \tilde{g}(s)ds = 0$ and $\tilde{g}(s) \in L^1(R)$. If $s > 0$,

$$\begin{aligned}|\tilde{g}_t(s)|_H^2 &= \int_0^{+\infty} \left|t^{-1} g(\frac{s}{t})\right|^2 \frac{dt}{t} \\ &= \int_0^{+\infty} \left|t^{-1} (\frac{s}{t})^{n-1} h(\frac{s}{t})\right|^2 \frac{dt}{t} \\ &\le C\int_0^{s} \left|t^{-1} (\frac{s}{t})^{n-1} (\frac{s}{t})^{-n-\epsilon}\right|^2 \frac{dt}{t} + C\int_s^{+\infty} \left|t^{-1} (\frac{s}{t})^{n-1} (\frac{s}{t})^{-n+\epsilon}\right|^2 \frac{dt}{t} \\ &= Cs^{-2-2\epsilon}\int_0^s t^{-1+2\epsilon}dt + Cs^{-2+2\epsilon}\int_s^{+\infty} t^{-1-2\epsilon}dt \\ &= Cs^{-2}.\end{aligned}$$

The case $s < 0$ is similar. To prove the Hörmander's condition (3), we also assume $s > 0$. Note first that by the assumption of Theorem 1, $|h(s)| \le Cs^{-n-\beta}$ for all $s > 0$ if we take $0 < \beta < \min\{\epsilon, \gamma\}$. So when $s > 2|m|$,

$$\begin{aligned}\left|g(\frac{s+m}{t}) - g(\frac{s}{t})\right| &= \left|(\frac{s+m}{t})^{n-1} h(\frac{s+m}{t}) - (\frac{s}{t})^{n-1} h(\frac{s}{t})\right| \\ &\le C(\frac{s+m}{t})^{-1-\beta} + C(\frac{s}{t})^{-1-\beta} \\ &\le C(\frac{t}{s})^{1+\beta}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{s>2|m|} |g_t(s+m)-g_t(s)|_H ds \\
&= \int_{s>2|m|} s^{-\frac{1+\beta}{2}} \left(s^{1+\beta} \int_0^{+\infty} |g_t(s+m)-g_t(s)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} ds \\
&\le \left(\int_{s>2|m|} s^{-1-\beta} ds \right)^{\frac{1}{2}} \left(\int_{s>2|m|} s^{1+\beta} \int_0^{+\infty} |g_t(s+m)-g_t(s)|^2 \frac{dt}{t} ds \right)^{\frac{1}{2}}.
\end{aligned}$$

The first bracket equals $|m|^{-\beta}$, while the second one does not exceed

$$\begin{aligned}
&\int_{s>2|m|} s^{1+\beta} \int_0^{+\infty} t^{-2} \left| g(\frac{s+m}{t}) - g(\frac{s}{t}) \right| C(\frac{t}{s})^{1+\beta} \frac{dt}{t} ds \\
&= C \int_{s>2|m|} \int_0^{+\infty} \left| g(\frac{s+m}{t}) - g(\frac{s}{t}) \right| t^{-2+\beta} dt ds \\
&= C \int_0^{+\infty} t^{-1+\beta} \int_{s>2|m|} \left| g(\frac{s+m}{t}) - g(\frac{s}{t}) \right| \frac{ds}{t} dt \\
&\le C \int_0^{+\infty} t^{-1+\beta} \int_R \left| g(s+\frac{m}{t}) - g(s) \right| ds dt \\
&\le C \int_0^{|m|} \|g\|_{L^1} t^{-1+\beta} dt + C \int_{|m|}^{+\infty} |\frac{m}{t}|^{\gamma} t^{-1+\beta} dt \\
&= C|m|^{\beta}.
\end{aligned}$$

Therefore we have proved

$$\int_{|s|>2|m|} |g_t(s+m)-g_t(s)|_H ds \le C, \ \forall \ m \ne 0. \qquad \blacksquare$$

Proof of Corollary 2. We only need to show that $h(s) = \frac{\chi_{[0,1]}(s)}{s^{n-1}}$ satisfies the assumption in Theorem 1. It is obvious that $h(s)$ has the desired size at 0 with $\epsilon = 1$ and by simple calculation

$$\begin{aligned}
&\int_R |(s+m)^{n-1} h(s+m) - s^{n-1} h(s)| ds \\
= \ &\int_R \left| \chi_{[0,1]}(s+m) - \chi_{[0,1]}(s) \right| ds \\
\le \ & 2|m|.
\end{aligned} \qquad \blacksquare$$

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