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TRIPLE POSITIVE SOLUTIONS OF NONLINEAR THIRD ORDER BOUNDARY VALUE PROBLEMS

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Abstract. In this paper we consider the following nonlinear third order two-point boundary value problem

$$\begin{aligned}&x'''(t) + f(t, x(t)) = 0, \quad a < t < b,\\ &x(a) = x''(a) = x(b) = 0.\end{aligned}$$

By using the Leggett-Williams and Krasnosel'skii fixed-point theorems, we offer criteria for the existence of three positive solutions to the boundary value problem. Examples are also included to illustrate the results obtained.

1. Introduction

The boundary value problems of differential, integral and difference equations have received a vast amount of attention in the recent literature and a lot of researchers have discussed the existence of single, double, and triple positive solutions for various boundary value problems, see for example [1-13,15,17-21] and the references therein. By means of the Leggett-Williams fixed point theorem, Agarwal and O'Regan [1] presented criteria which guarantee the existence of three nonnegative solutions to a class of second order impulsive equations, and Anderson [4] established the existence at least three positive solutions to a third order three-point boundary value problem. Using the Krasnosel'skii fixed-point theorem, Anderson and Davis [7] studied the existence of multiple positive solutions for a third order three-point right focal boundary value problem, Wong and Agarwal [19] considered the existence of multiple positive solutions for a two-point right focal boundary value

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problem and Yao [21] obtained the existence and multiplicity of positive solutions for a third order three-point boundary value problem.

Motivated by the results mentioned, in this paper we derive criteria for the existence of three positive solutions to the following nonlinear third order two-point boundary value problem

$$x'''(t) + f(t, x(t)) = 0, \quad a < t < b, \tag{1.1}$$

$$x(a) = x''(a) = x(b) = 0, \tag{1.2}$$

where $f \in C([a, b] \times \mathbb{R}, [0, +\infty))$. Our arguments are based upon the positivity of the Green's function $G(t, s)$ and the Leggett-Williams and Krasnosel'skii fixed-point theorems.

This paper is organized as follows. Section 2 contains the necessary definitions, notation, properties of the Green's function $G(t, s)$ and fixed point theorems, which play key roles in this paper. The existence criteria of three positive solutions for equation (1.1), (1.2) are discussed in Section 3. Finally, three examples are presented in Second 4 to illustrate the importance of the results obtained.

2. Preliminaries

Let X be a Banach space and Y be a cone in X. A mapping α is said to be a nonnegative continuous concave functional on Y if $\alpha : Y \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y), \quad x, y \in Y,\ t \in [0, 1].$$

x is said to be a *positive solution* of equation (1.1), (1.2) if x is a solution of equation (1.1), (1.2) and $x(t) > 0$ for each $t \in (a, b)$. Throughout this paper, we assume that $C[a, b]$ denotes the Banach space of all continuous functions on $[a, b]$ with the supremum norm $\|u\| =: \sup_{t\in[a,b]} |u(t)|$ for each $u \in C[a, b]$, p and q are constants with $a < p < q < b$,

$$\underline{f}_0 = \liminf_{s\to 0^+} \frac{1}{s} \min\{f(t, s) : t \in [p, q]\},$$

$$\underline{f}_\infty = \liminf_{s\to +\infty} \frac{1}{s} \min\{f(t, s) : t \in [p, q]\},$$

$$\overline{f}_0 = \limsup_{s\to 0^+} \frac{1}{s} \max\{f(t, s) : t \in [a, b]\},$$

$$\overline{f}_\infty = \limsup_{s\to +\infty} \frac{1}{s} \max\{f(t, s) : t \in [a, b]\},$$

$$g(t) = \frac{(b-t)^2(b+t-2a)}{2(b-a)}, \quad h(t) = \frac{(b-t)^2(t-a)}{2(b-a)^3}, \quad t \in [a, b],$$

$$
\begin{aligned}
k^{-1} &= \int_a^b g(s)ds = \frac{5(b-a)^3}{24}, \\
m^{-1} &= \min_{t\in[p,q]} h(t) \int_p^q g(s)ds \\
&= \left[\frac{(b-p)^3-(b-q)^3}{3} - \frac{(b-p)^4-(b-q)^4}{8(b-a)} \right] \min\{h(p), h(q)\}, \\
P &= \{x \in C[a,b] : x \text{ is concave on } [a,b],\, x(t) \geq h(t)\|x\|,\, t\in[a,b]\}, \\
P_r &= \{x \in P : \|x\| < r\}, \quad \partial P_r = \{x \in P : \|x\| = r\}, \quad r > 0, \\
\overline{P}_r &= \{x \in P : \|x\| \leq r\}, \quad P(\alpha, r, s) = \{x \in P : r \leq \alpha(x), \|x\| \leq s\}, \quad s > r > 0,
\end{aligned}
$$

where α is a nonnegative continuous concave functional on P and

$$
G(t,s) = \begin{cases} \dfrac{(t-a)(b-s)^2}{2(b-a)} - \dfrac{(t-s)^2}{2}, & a \leq s \leq t \leq b, \\ \dfrac{(t-a)(b-s)^2}{2(b-a)}, & a \leq t \leq s \leq b, \end{cases}
$$

is the Green's function of the homogeneous problem $x'''(t) = 0$ satisfying the boundary condition (1.2). It is easy to verify that P is a cone of $C[a,b]$.

Lemma 2.1. (i) *Equation* (1.1), (1.2) *has a solution* $y \in C[a,b]$ *if and only if the operator* $T : C[a,b] \to C[a,b]$ *defined by*

$$
Tx(t) = \int_a^b G(t,s) f(s, x(s))ds, \quad t \in [a,b],\, x \in C[a,b]
$$

has a fixed point $y \in C[a,b]$;

(ii) *Suppose that the following condition*

$$
f(t_0, 0) > 0 \quad \textit{for some } t_0 \in [a,b] \tag{2.1}
$$

is fulfilled, then each solution $y \in C[a,b]$ *of equation* (1.1), (1.2) *satisfies that* $\|y\| > 0$.

Lemma 2.2.

(*i*) $0 \leq h(t) \leq \frac{1}{2},\ 0 \leq g(t) \leq g(a) = \frac{(b-a)^2}{2},\ t \in [a,b];$

(*ii*) $h(t)g(s) \leq G(t,s) \leq g(s),\ t, s \in [a,b].$

(*iii*) *For each* $s \in [a,b]$, *the function* $G(\cdot, s)$ *is concave in the first argument on* $[a,b]$.

Proof. (i) is clear. Now we show that (ii) holds. For $a \le t \le s \le b$, by (i) we infer that

$$G(t,s) = \frac{(t-a)(b-s)^2}{2(b-a)} \le \frac{(s-a)(b-s)^2}{2(b-a)} \le g(s)$$

and

$$\begin{aligned} G(t,s) &= \frac{(t-a)(b-s)^2}{2(b-a)} = \frac{t-a}{b+s-2a} g(s) \\ &\ge \frac{t-a}{2(b-a)} g(s) \ge h(t)g(s). \end{aligned}$$

For $a \le s < t \le b$, by (i) we deduce that

$$\begin{aligned} G(t,s) &= \frac{(t-a)(b-s)^2}{2(b-a)} - \frac{(t-s)^2}{2} \\ &= \frac{1}{2(b-a)}[(b^2-2bs+s^2)(t-a)-(t^2-2ts+s^2)(b-a)] \\ &= \frac{b-t}{2(b-a)}(2as-s^2+bt-ab-at) \\ &= \frac{b-t}{2(b-a)}[-(a-s)^2+a(a-b)+t(b-a)] \\ &= \frac{b-t}{2(b-a)}[(b-a)(t-a)-(a-s)^2] \\ &\le \frac{b-s}{2(b-a)}[(b-a)^2-(a-s)2] \\ &= g(s) \end{aligned}$$

and

$$\begin{aligned} G(t,s) &= \frac{(t-a)(b-s)^2}{2(b-a)} - \frac{(t-s)^2}{2} \\ &= \frac{(b-t)[(b-a)(t-a)-(a-s)^2]}{(b-s)^2(b+s-2a)} g(s) \\ &= \frac{(b-t)[(b-a)(t-a)-(t-a)^2+(t-a)^2-(a-s)^2]}{(b-s)^2(b+s-2a)} g(s) \\ &= \frac{(b-t)[(t-a)(b-t)+(t-s)(t+s-2a)]}{(b-s)^2(b+s-2a)} g(s) \\ &\ge \frac{(b-t)^2(t-a)}{(b-s)^2(b+s-2a)} g(s) \\ &\ge \frac{(b-t)^2(t-a)}{2(b-a)(b-s)^2} g(s) \\ &\ge h(t)g(s). \end{aligned}$$

That is, (ii) holds. Let $c \in [0,1]$ and $t, r, s \in [a,b]$ with $t \leq r$. In order to show (iii), we have to consider the following possible cases:

Case 1. Suppose that $s \leq t$. Notice that $ct + (1-c)r \geq s$. It follows that

$$\begin{aligned} & G(ct+(1-c)r, s) - cG(t,s) - (1-c)G(r,s) \\ &= \frac{(ct+(1-c)r-a)(b-s)^2}{2(b-a)} - \frac{(ct+(1-c)r-s)^2}{2} - \frac{c(t-a)(b-s)^2}{2(b-a)} \\ &\quad + \frac{c(t-s)^2}{2} - \frac{(1-c)(r-a)(b-s)^2}{2(b-a)} + \frac{(1-c)(r-s)^2}{2} \\ &= \frac{c(t-s)^2}{2} + \frac{(1-c)(r-s)^2}{2} - \frac{[c(t-s)+(1-c)(r-s)]^2}{2} \\ &\geq 0. \end{aligned}$$

Case 2. Suppose that $r \leq s$. Since $ct + (1-c)r \leq s$, it follows that

$$\begin{aligned} & G(ct+(1-c)r, s) - cG(t,s) - (1-c)G(r,s) \\ &= \frac{(ct+(1-c)r-a)(b-s)^2}{2(b-a)} - \frac{c(t-a)(b-s)^2}{2(b-a)} - \frac{(1-c)(r-a)(b-s)^2}{2(b-a)} \\ &= 0. \end{aligned}$$

Case 3. Suppose that $t < s < r$ and $ct + (1-c)r \leq s$. It is easy to verify that

$$\begin{aligned} & G(ct+(1-c)r, s) - cG(t,s) - (1-c)G(r,s) \\ &= \frac{(ct+(1-c)r-a)(b-s)^2}{2(b-a)} - \frac{c(t-a)(b-s)^2}{2(b-a)} \\ &\quad - \frac{(1-c)(r-a)(b-s)^2}{2(b-a)} + \frac{(1-c)(r-s)^2}{2} \\ &= \frac{(1-c)(r-s)^2}{2} \\ &\geq 0. \end{aligned}$$

Case 4. Suppose that $t < s < r$ and $ct + (1-c)r > s$. Then

$$\begin{aligned} & G(ct+(1-c)r, s) - cG(t,s) - (1-c)G(r,s) \\ &= \frac{(ct+(1-c)r-a)(b-s)^2}{2(b-a)} - \frac{(ct+(1-c)r-s)^2}{2} - \frac{c(t-a)(b-s)^2}{2(b-a)} \\ &\quad - \frac{(1-c)(r-a)(b-s)^2}{2(b-a)} + \frac{(1-c)(r-s)^2}{2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-c)(r-s)^2}{2} - \frac{[(1-c)(r-s)-c(s-t)]^2}{2} \\
&\geq \frac{(1-c)(r-s)^2}{2} - \frac{(1-c)^2(r-s)^2}{2} \\
&= \frac{c(1-c)(r-s)^2}{2} \\
&\geq 0.
\end{aligned}
$$

Hence (iii) holds. This completes the proof. ■

Lemma 2.3. (Leggett-Williams Fixed-Point Theorem [14]). *Let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and α be a nonnegative continuous concave functional on P such that $\alpha(x) \leq \|x\|$ for all $x \in \overline{P}_c$. Suppose that there exist $0 < d_0 < a_0 < b_0 \leq c$ such that*

(i) $\{x \in P(\alpha, a_0, b_0) : \alpha(x) > a_0\} \neq \emptyset$ *and* $\alpha(Tx) > a_0$ *for* $x \in P(\alpha, a_0, b_0)$;

(ii) $\|Tx\| < d_0$ *for* $\|x\| \leq d_0$;

(iii) $\alpha(Tx) > a_0$ *for* $x \in P(\alpha, a_0, c)$ *with* $\|Tx\| > b_0$.

Then T has at least three fixed points x_1, x_2, x_3 in $\overline{P}_c$ satisfying

$$\|x_1\| < d_0, \quad a_0 < \alpha(x_2), \quad \|x_3\| > d_0 \quad \text{and} \quad \alpha(x_3) < a_0.$$

Lemma 2.4. (Krasnosel'skii Fixed-Point Theorem [16]). *Let $(X, \|\cdot\|)$ be a Banach space and let $Y \subset X$ be a cone in X. Assume that A and B are open subsets of X with $0 \in A$, $\overline{A} \subset B$ and $T : Y \cap (\overline{B} \setminus A) \to Y$ is a completely continuous operator such that, either*

(i) $\|Tu\| \leq \|u\|$ *for* $u \in Y \cap \partial A$, *and* $\|Tu\| \geq \|u\|$ *for* $u \in Y \cap \partial B$, *or*

(ii) $\|Tu\| \geq \|u\|$ *for* $u \in Y \cap \partial A$, *and* $\|Tu\| \leq \|u\|$ *for* $u \in Y \cap \partial B$.

Then T has at least one fixed point in $Y \cap (\overline{B} \setminus A)$.

3. Existence of Three Positive Solutions

Now we are ready to establish sufficient conditions for the existence of at least three positive solutions of equation (1.1), (1.2) under certain conditions by applying the positivity of the Green's function $G(t, s)$ and the Leggett-Williams and Krasnosel'skii fixed-point theorems, respectively. Our first result employs Lemma 2.3.

Theorem 3.1. *Suppose that there exist four constants d_0, a_0, b_0 and c_0 satisfying*

$$0 < d_0 < a_0, \quad \frac{a_0}{\min\{h(p), h(q)\}} \le b_0 \le c_0; \tag{3.1}$$

$$f(t,s) < kd_0, \quad t \in [a,b],\, s \in [0,d_0]; \tag{3.2}$$

$$f(t,s) > ma_0, \quad t \in [p,q],\, s \in [a_0,b_0]; \tag{3.3}$$

$$f(t,s) < kc_0, \quad t \in [a,b],\, s \in [0,c_0], \tag{3.4}$$

respectively. If (2.1) *holds, then equation* (1.1), (1.2) *possesses at least three positive solutions* $x_1, x_2, x_3 \in \overline{P}_{c_0}$ *such that*

$$\begin{gathered} 0 < \|x_1\| < d_0, \qquad a_0 < \min\{x_2(p), x_2(q)\}, \quad \|x_3\| > d_0, \\ \min\{x_3(p), x_3(q)\} < a_0. \end{gathered} \tag{3.5}$$

Proof. Define the operator $T : P \to C[a,b]$ by

$$Tx(t) = \int_a^b G(t,s)f(s,x(s))ds, \quad t \in [a,b],\, x \in C[a,b] \tag{3.6}$$

and put

$$\alpha(x) = \min_{t\in[p,q]} |x(t)|, \quad x \in P.$$

It is not difficult to see that α is a nonnegative continuous concave functional on P and $\alpha(x) = \min\{x(p), x(q)\} \le \|x\|$, $x \in P$. By virtue of Lemma 2.2 and (3.6), we infer that

$$\begin{aligned} Tx(ct + (1-c)r) &= \int_a^b G(ct + (1-c)r, s)f(s,x(s))ds \\ &\ge \int_a^b [cG(t,s) + (1-c)G(r,s)]f(s,x(s))ds \\ &= cTx(t) + (1-c)Tx(r), \quad x \in P,\, t, r \in [a,b],\, c \in [0,1], \\ \|Tx\| &= \sup_{t\in[a,b]} \int_a^b G(t,s)f(s,x(s))ds \\ &\le \int_a^b g(s)f(s,x(s))ds, \quad x \in P \end{aligned}$$

and

$$\begin{aligned} Tx(t) &= \int_a^b G(t,s)f(s,x(s))ds \\ &\ge h(t)\int_a^b g(s)f(s,x(s))ds \\ &\ge h(t)\|Tx\|, \quad t \in [a,b],\, x \in P. \end{aligned}$$

That is, $T : P \to P$. Furthermore, T is completely continuous by an application of Arzela-Ascoli Theorem. Now we assert that $T(\overline{P}_{c_0}) \subset P_{c_0}$. Let $x \in \overline{P}_{c_0}$. It is easy to see that

$$0 \le h(t)\|x\| \le x(t) \le c_0, \quad t \in [a, b].$$

This together with Lemma 2.2 and (3.4) yield that

$$\begin{aligned}\|Tx\| &= \sup_{t\in[a,b]} \int_a^b G(t, s)f(s, x(s))ds \le \int_a^b g(s)f(s, x(s))ds \\ &< kc_0 \int_a^b g(s)ds = c_0,\end{aligned}$$

which implies that $T(\overline{P}_{c_0}) \subset P_{c_0}$. Similarly we could deduce that $\|Tx\| < d_0$ for $\|x\| \le d_0$ by Lemma 2.2 and (3.2). Choose

$$x_0(t) = \frac{3}{4}a_0 + \frac{1}{4}b_0, \quad t \in [a, b].$$

This together with (3.1) guarantee that $x_0 \in \{x \in P(\alpha, a_0, b_0) : \alpha(x) > a_0\} \ne \emptyset$. For any $x \in P(\alpha, a_0, b_0)$, Lemma 2.2 and (3.3) ensure that

$$\begin{aligned}\alpha(Tx) &= \min_{t\in[p,q]} \int_a^b G(t, s)f(s, x(s))ds \ge \min_{t\in[p,q]} \int_a^b h(t)g(s)f(s, x(s))ds \\ &\ge \min\{h(p), h(q)\} \int_p^q g(s)f(s, x(s))ds \\ &> \min\{h(p), h(q)\}ma_0 \int_p^q g(s)ds = a_0.\end{aligned}$$

For any $x \in P(\alpha, a_0, c_0)$ and $\|Tx\| > b_0$, Lemma 2.2, (3.1) and (3.6) guarantee that

$$\begin{aligned}\|Tx\| &= \max_{t\in[a,b]} \int_a^b G(t, s)f(s, x(s))ds \\ &\le \int_a^b g(s)f(s, x(s))ds\end{aligned}$$

and

$$\begin{aligned}\alpha(Tx) &= \min_{t\in[p,q]} \int_a^b G(t, s)f(s, x(s))ds \\ &\ge \min\nolimits_{t\in[p,q]} \textstyle\int_a^b h(t)g(s)f(s, x(s))ds \\ &= \min\{h(p), h(q)\} \int_a^b g(s)f(s, x(s))ds \\ &\ge \min\{h(p), h(q)\}\|Tx\| > \min\{h(p), h(q)\}b_0 \ge a_0.\end{aligned}$$

Lemma 2.3 gives that the operator T has at least three fixed points $x_1, x_2, x_3 \in \overline{P}_{c_0}$ with

$$\|x_1\| < d_0, \quad a_0 < \alpha(x_2), \quad \|x_3\| > d_0 \quad \text{and} \quad \alpha(x_3) < a_0. \tag{3.7}$$

Lemmas 2.1 and 2.2, (2.1) and (3.7) imply that equation (1.1), (1.2) possesses at least three positive solutions $x_1, x_2, x_3 \in \overline{P}_{c_0}$ satisfying (3.5). This completes the proof. ∎

Next we continue to use Lemma 2.3 to give other existence criteria of three positive solutions for equation $(1.1), (1.2)$.

Theorem 3.2 *Suppose that there exist four constants d_0, a_0, b_0 and c satisfying $(3.2), (3.3)$,*

$$0 < d_0 < a_0, \quad \frac{a_0}{\min\{h(p), h(q)\}} \leq b_0, \tag{3.8}$$

$$f(t, s) \leq ks, \quad t \in [a, b],\ s \in [c, +\infty), \tag{3.9}$$

respectively. If (2.1) holds, then there exists $c_0 > \max\{c, b_0\}$ such that equation (1.1), (1.2) possesses at least three positive solutions $x_1, x_2, x_3 \in \overline{P}_{c_0}$ satisfying (3.5).

Proof. In order to prove Theorem 3.2, we need only to show that there exists $c_0 > \max\{b_0, c\}$ satisfying (3.4). Suppose that f is bounded on $[a, b] \times [0, +\infty)$. It follows that there exists $c_0 > \max\{b_0, c\}$ such that

$$f(t, s) < kc_0, \quad t \in [a, b],\ s \in [0, +\infty).$$

Suppose that f is unbounded on $[a, b] \times [0, +\infty)$. The continuity of f and (3.9) yield that there exist $c_0 > \max\{b_0, c\}$ and $s_0 \in (c, c_0)$such that

$$f(t, s) \leq f(t, s_0) \leq ks_0 < kc_0, \quad t \in [a, b],\ s \in [0, c_0].$$

That is, (3.4) holds. Thus Theorem 3.2 follows from Theorem 3.1. This completes the proof. ∎

Theorem 3.3. *Suppose that there exist two constants a_0 and b_0 with $0 < \frac{a_0}{\min\{h(p),h(q)\}} \leq b_0$ satisfying (3.3) and*

$$\max\{\overline{f}_0, \overline{f}_\infty\} < k. \tag{3.10}$$

If (2.1) holds, then there exist two constants d_0 and c_0 with $0 < d_0 < a_0$ and $c_0 > b_0$ such that equation (1.1), (1.2) possesses at least three positive solutions $x_1, x_2, x_3 \in \overline{P}_{c_0}$ satisfying (3.5).

Proof. Notice that (3.10) implies that there exist $d_0 \in (0, a_0)$ and $c > b_0$ satisfying

$$\frac{1}{s}\max\{f(t,s) : t \in [a,b]\} < \frac{\overline{f}_0 + k}{2}, \quad s \in (0, d_0],$$
$$\frac{1}{s}\max\{f(t,s) : t \in [a,b]\} < \frac{\overline{f}_\infty + k}{2}, \quad s \in [c, +\infty),$$

which give that

$$f(t,s) < kd_0, \quad t \in [a,b],\, s \in [0, d_0],$$
$$f(t,s) < ks, \quad t \in [a,b],\, s \in [c, +\infty).$$

Thus Theorem 3.3 follows from Theorem 3.2. This completes the proof. ■

Remark 3.1. Theorems (3.1)-(3.3) guarantee only that equation (1.1), (1.2) possesses at least two positive solutions and a nonnegative solution provided that (2.1) is omitted.

Now we use Lemma 2.4 to provide a few existence criteria of triple positive solutions for equation (1.1), (1.2).

Theorem 3.4. *Suppose that there exist four constants d_0, a_0, b_0 and c_0 with $0 < d_0 < a_0 < b_0 < c_0$ satisfying* (3.2)*,*

$$f(t,s) > ma_0, \quad t \in [p,q],\, s \in [\min\{h(p), h(q)\}a_0, a_0]; \tag{3.11}$$

$$f(t,s) < kb_0, \quad t \in [a,b],\, s \in [0, b_0]; \tag{3.12}$$

$$f(t,s) > mc_0, \quad t \in [p,q],\, s \in [\min\{h(p), h(q)\}c_0, c_0]. \tag{3.13}$$

Then equation (1.1)*,* (1.2) *possesses at least three positive solutions $x_1, x_2, x_3 \in \overline{P}_{c_0}$ satisfying*

$$d_0 < \|x_1\| < a_0 < \|x_2\| < b_0 < \|x_3\| < c_0. \tag{3.14}$$

Proof. Define the operator $T : P \to C[a,b]$ by (3.6). Then $T : P \to P$ is completely continuous. We assert that (3.2) and (3.11) imply that there exist two positive constants d_1 and a_1 with $d_0 < d_1 < a_1 < a_0$ satisfying

$$f(t,s) \le kd_1, \quad t \in [a,b],\, s \in [0, d_1]; \tag{3.15}$$

$$f(t,s) \ge ma_1, \quad t \in [p,q],\, s \in [\min\{h(p), h(q)\}a_1, a_1]. \tag{3.16}$$

Set

$$\psi(s) = \min\{f(t,x) : t \in [p,q], x \in [\min\{h(p),h(q)\}s, s]\}, \quad s \geq 0,$$
$$\varphi(s) = \max\{f(t,x) : t \in [a,b], x \in [0,s]\}, \quad s \geq 0.$$

It is clear that (3.2) and (3.11) are equivalent to $\frac{\varphi(d_0)}{d_0} < k$ and $\frac{\psi(a_0)}{a_0} > m$, respectively. Note that

$$\frac{\psi(d_0)}{d_0} \leq \frac{\varphi(d_0)}{d_0} < k < m < \frac{\psi(a_0)}{a_0}.$$

The continuity of ψ yields that there exists some $a_1 \in (d_0, a_0)$ satisfying $\frac{\psi(a_1)}{a_1} = m$, which implies that (3.16) holds. Because

$$\frac{\varphi(d_0)}{d_0} < k < m = \frac{\psi(a_1)}{a_1} \leq \frac{\varphi(a_1)}{a_1},$$

it follows from the continuity of φ that there exists some $d_1 \in (d_0, a_1)$ with $\frac{\varphi(d_1)}{a_1} = k$, which means that (3.15) holds.

Now we claim that equation (1.1), (1.2) possesses at least one positive solution $x_1 \in P$ with $d_0 < d_1 \leq \|x_1\| \leq a_1 < a_0$. Let x be in ∂P_{d_1}. It is easy to verify that $\|x\| = d_1$ and

$$0 \leq h(t)d_1 \leq x(t) \leq d_1, \quad t \in [a,b].$$

Thus (3.15) yields that

$$f(t, x(t)) \leq kd_1, \quad t \in [a,b]. \tag{3.17}$$

In light of Lemma 2.2, (3.6) and (3.17), we get that

$$\begin{aligned}
\|Tx\| &= \sup_{t\in[a,b]} \int_a^b G(t,s)f(s,x(s))ds \\
&\leq \int_a^b g(s)f(s,x(s))ds \\
&\leq kd_1 \int_a^b g(s)ds \\
&= d_1,
\end{aligned}$$

which yields that

$$\|Tx\| \leq \|x\|, \quad x \in \partial P_{d_1}. \tag{3.18}$$

Let x be in ∂P_{a_1}. It follows that $\|x\| = a_1$ and

$$\min\{h(p), h(q)\}a_1 \leq h(t)\|x\| \leq x(t) \leq a_1, \quad t \in [p,q].$$

Note that (3.16) yields that

$$f(t,x(t)) \geq ma_1, \quad t \in [p,q]. \tag{3.19}$$

By virtue of Lemma 2.2, (3.6) and (3.19), we observe that

$$\begin{aligned} Tx(t) &= \int_a^b G(t,s)f(s,x(s))ds \\ &\geq h(t)\int_a^b g(s)f(s,x(s))ds \\ &\geq \min\{h(p),h(q)\}\int_p^q g(s)f(s,x(s))ds \\ &\geq \min\{h(p),h(q)\}ma_1\int_p^q g(s)ds \\ &= a_1, \quad t \in [p,q], \end{aligned}$$

which implies that

$$\|Tx\| \geq \|x\|, \quad x \in \partial P_{a_1}. \tag{3.20}$$

Thus Lemma 2.4, (3.18) and (3.20) imply that equation (1.1), (1.2) possesses at least one solution $x_1 \in P$ with $d_0 < d_1 \leq \|x_1\| \leq a_1 < a_0$. Note that

$$x_1(t) \geq h(t)\|x_1\| \geq d_1 h(t) > 0, \quad t \in (a,b).$$

That is, the solution x_1 of equation (1.1), (1.2) is positive. Similarly, by (3.11)-(3.13) we could conclude that there exist a_2, b_2, b_3, c_3 with $a_0 < a_2 < b_2 < b_0 < b_3 < c_3 < c_0$ such that equation (1.1), (1.2) possesses at least two positive solutions $x_2, x_3 \in P$ with

$$a_0 < a_2 \leq \|x_2\| \leq b_2 < b_0 < b_3 \leq \|x_3\| \leq c_3 < c_0.$$

This completes the proof. ■

Theorem 3.5. *Suppose that there exist two constants* a_0 *and* b_0 *with* $0 < a_0 < b_0$ *satisfying* (3.11) *and* (3.12)*, respectively. If the function* f *satisfies*

$$\overline{f}_0 < k \quad \textit{and} \quad \underline{f}_\infty > \frac{m}{\min\{h(p),h(q)\}}, \tag{3.21}$$

then there exists constants d_0 *and* c_0 *with* $0 < d_0 < a_0$ *and* $b_0 < c_0$ *such that equation* (1.1), (1.2) *possesses at least three positive solutions* $x_1, x_2, x_3 \in \overline{P}_{c_0}$ *satisfying* (3.14).

Proof. Notice that (3.21) implies that there exist $d_0 \in (0, a_0)$ and c_0 with $\min\{h(p), h(q)\}c_0 > b_0$ satisfying

$$\frac{1}{s}\max\{f(t,s) : t \in [a,b]\} < \frac{\overline{f}_0 + k}{2}, \quad s \in (0, d_0],$$
$$\frac{1}{s}\min\{f(t,s) : t \in [p,q]\} > \frac{m}{\min\{h(p), h(q)\}}, \quad s \in [\min\{h(p), h(q)\}c_0, +\infty),$$

which yield that

$$f(t,s) < kd_0, \quad t \in [a,b],\ s \in [0, d_0],$$
$$f(t,s) > \frac{m}{\min\{h(p), h(q)\}} s \geq mc_0, \quad t \in [p,q],\ s \in [\min\{h(p), h(q)\}c_0, c_0].$$

Hence Theorem 3.5 follows from Theorem 3.4. This completes the proof. ■

By combining arguments used in Theorems 3.1-3.5, we have the following results.

Theorem 3.6. *Assume that there exist four constants* d_0, a_0, b_0 *and* c_0 *with* $0 < d_0 < a_0 < b_0 < c_0$ *satisfying* (3.4),

$$f(t,s) > md_0, \quad t \in [p,q],\ s \in [\min\{h(p), h(q)\}d_0, d_0]; \tag{3.22}$$

$$f(t,s) < ka_0, \quad t \in [a,b],\ s \in [0, a_0]; \tag{3.23}$$

$$f(t,s) > mb_0, \quad t \in [p,q],\ s \in [\min\{h(p), h(q)\}b_0, b_0]. \tag{3.24}$$

Then equation (1.1), (1.2) *possesses at least three positive solutions* $x_1, x_2, x_3 \in \overline{P}_{c_0}$ *satisfying* (3.14).

Theorem 3.7. *Assume that there exist three constants* d_0, a_0 *and* b_0 *with* $0 < d_0 < a_0 < b_0$ *satisfying* (3.22), (3.23) *and* (3.24), *respectively. If the function* f *satisfies* (3.9) *for some constant* $c > 0$, *then there exists* $c_0 > \max\{c, b_0\}$ *such that equation* (1.1), (1.2) *possesses at least three positive solutions* $x_1, x_2, x_3 \in \overline{P}_{c_0}$ *satisfying* (3.14).

Theorem 3.8. *Assume that there exist two constants* a_0 *and* b_0 *with* $0 < a_0 < b_0$ *satisfying* (3.23) *and* (3.24), *respectively. If the function* f *satisfies*

$$\overline{f}_\infty < k \quad \text{and} \quad \underline{f}_0 > \frac{m}{\min\{h(p), h(q)\}}, \tag{3.25}$$

then there exist constants d_0 *and* c_0 *with* $0 < d_0 < a_0$ *and* $< b_0 < c_0$ *such that equation* (1.1), (1.2) *possesses at least three positive solutions* $x_1, x_2, x_3 \in \overline{P}_{c_0}$ *satisfying* (3.14).

4. Examples

In this section, we construct three examples to illustrate the usefulness of the results obtained in Section 3.

Example 4.1. Let $a = 0$, $b = 3$, $p = 1$, $q = 2$, $d_0 = 1$, $a_0 = 7$, $b_0 = 190$, $c_0 = 303807105000$ and

$$f(t,s) = \begin{cases} \dfrac{1-\frac{ts}{3}}{45+t^2}, & t\in[a,b],\ s\in(-\infty,d_0], \\ \dfrac{1-\frac{t}{3}}{45+t^2} + \dfrac{3}{41}(t^2+s^2)(s-1)^3, & t\in[a,b],\ s\in(d_0,b_0], \\ \dfrac{1-\frac{t}{3}}{45+t^2} + \dfrac{20253807}{41}(t^2+36100) + \dfrac{s-190}{9}, & t\in[a,b],\ s\in(b_0,+\infty). \end{cases}$$

It is clear that $k = \frac{8}{45}$, $h(p) = \frac{2}{27}$, $h(q) = \frac{1}{27}$, $m = \frac{648}{41}$, and

$$\begin{aligned} &f(t,s) \le \frac{1}{45} < kd_0, \quad t\in[a,b],\ s\in[0,d_0], \\ &f(t,s) > \frac{3}{41}s(s-1)^3 > ma_0, \quad t\in[p,q],\ s\in[a_0,b_0], \\ &f(t,s) \le 1 + \frac{731344716963}{41} + \frac{303807104810}{9} < kc_0, \quad t\in[a,b],\ s\in[0,c_0]. \end{aligned}$$

Theorem 3.1 ensures that equation (1.1), (1.2) has at least three positive solutions $x_1, x_2, x_3 \in \overline{P}_{c_0}$ satisfying (3.5).

Example 4.2. Let $a = 1$, $b = 4$, $p = 2$, $q = 3$, $d_0 = 1$, $a_0 = 40$, $b_0 = 629600$, $c_0 = 17628800$ and

$$f(t,s) = \begin{cases} \dfrac{1-s}{45+t^2(s-1)^2}, & t\in[a,b],\ s\in(-\infty,d_0], \\ (2079+27t^2)(s-1), & t\in[a,b],\ s\in(d_0,a_0], \\ 81081+1053t^2+\dfrac{s-40}{45+ts}, & t\in[a,b],\ s\in(a_0,b_0], \\ 81081+1053t^2+\dfrac{629560}{45+629600t}+729t^2(s-629600)^2, & t\in[a,b],\ s\in(b_0,+\infty). \end{cases}$$

Note that $k = \frac{8}{45}$, $h(p) = \frac{2}{27}$, $h(q) = \frac{1}{27}$, $m = \frac{1296}{67}$. It is not difficult to verify that

$$\begin{aligned} &f(t,s) < \frac{1}{45} < kd_0, \quad t\in[a,b],\ s\in[0,d_0], \\ &f(t,s) > 1053 > ma_0, \quad t\in[p,q],\ s\in[\min\{h(p),h(q)\}a_0,a_0], \\ &f(t,s) < 97929 + \frac{629560}{45} < kb_0, \quad t\in[a,b],\ s\in[0,b_0], \\ &f(t,s) > 85293 + 2916\left(\frac{c_0}{27} - 629600\right)^2 > mc_0, \quad t\in[p,q],\ s\in[\min\{h(p),h(q)\}c_0,c_0]. \end{aligned}$$

Theorem 3.4 guarantees that equation (1.1), (1.2) possesses at least three positive solutions $x_1, x_2,\ x_3 \in \overline{P}_{c_0}$ satisfying (3.14).

Example 4.3. Let $a = 0, b = 5, p = 1, q = 2, d_0 = 2, a_0 = 605, b_0 = 515320, c_0 = 4103820000$ and

$$f(t,s) = \begin{cases} \dfrac{751+(1+t)\sqrt{2-s}}{191}, & t\in[a,b],\ s\in(-\infty,d_0], \\ \dfrac{751}{191} + \dfrac{4(s-2)}{125}, & t\in[a,b],\ s\in(d_0,a_0], \\ \dfrac{554367}{23875} + \dfrac{(124+t^2)(s-605)}{4}, & t\in[a,b],\ s\in(a_0,b_0], \\ \dfrac{554367}{23875} + \dfrac{514715(124+t^2)}{4} + \dfrac{(1+4t)(s-515320)}{625}, & t\in[a,b],\ s\in(b_0,+\infty). \end{cases}$$

It is easy to verify that $k = \frac{24}{625}$, $m = \frac{375}{191}$, $h(p) = \frac{8}{125}$, $h(q) = \frac{9}{125}$,

$$\begin{aligned} &f(t,s) > \frac{751}{191} > md_0, \quad t\in[p,q],\ s\in[\min\{h(p),h(q)\}d_0, d_0], \\ &f(t,s) \le \frac{751}{191} + \frac{2412}{125} < ka_0, \quad t\in[a,b],\ s\in[0,a_0], \\ &f(t,s) \ge 1011754 > mb_0, \quad t\in[p,q],\ s\in[\min\{h(p),h(q)\}b_0, b_0], \\ &f(t,s) < 19559194 + \frac{21}{625}(c_0 - 515320) < kc_0, \quad t\in[a,b],\ s\in[0,c_0]. \end{aligned}$$

That is, the conditions of Theorem 3.6 are fulfilled. Consequently, Theorem 3.6 yields that equation (1.1), (1.2) has at least three positive solutions $x_1, x_2, x_3 \in \overline{P}_{c_0}$ satisfying (3.14).

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