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PERTURBATION ANALYSIS FOR THE MATRIX EQUATIONS $X \pm A^* X^{-1} A = I$

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Abstract. The nonlinear matrix equations $X \pm A^* X^{-1} A = I$ are investigated, where A is an $n \times n$ nonsingular matrix and I is an $n \times n$ identity matrix. Some new perturbation bounds for Hermitian positive definite solutions of these equations are derived by using elementary calculus techniques developed in[Sun J G , *BIT*, 31(1991), pp.341-352] and [Barrlund A, *BIT*, 31(1991), pp.358-363]. The new results are illustrated by numerical examples.

1. Introduction

We consider the nonlinear matrix equations

$$X \pm A^* X^{-1} A = I, \tag{1.1}$$

where A is an $n \times n$ nonsingular matrix , I is the $n \times n$ identity matrix and A^* represents the conjugate transpose of the matrix A. This kind of equations arises in various areas of applications, including control theory, ladder networks, dynamic programming, stochastic filtering, statistic,and so on [3, 4]. It is often required to find the symmetric positive definite solutions of the matrix equations (1.1).

The matrix equations (1.1) have been investigated for the existence of symmetric positive definite solutions by many authors[2, 3, 4, 8, 13]. Liu and Gao[8] presented sensitivity analysis of the maximal solutions by using implicit function theorem; Using algebra methods and the Schauder fixed-point theorem, some perturbation bounds for the Hermitian positive definite solutions to the matrix equations (1.1) are derived in [5, 6, 7, 9].

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Recently, applying the differential methods Chen and Li[2] obtained the first order perturbation bound of the maximal solution for the matrix equation $X + A^*X^{-1}A = P$. This paper is a continuation of the paper [2]. Here we derive some new perturbation bounds for the Hermitian positive definite solutions to the matrix equations (1.1) by using elementary calculus techniques developed in [1] and [10]. The results are illustrated by using numerical examples.

In this paper, $\mathcal{C}^{n\times n}$ denotes the set of $n \times n$ complex matrices. We use $\|\cdot\|$ for any unitary invariant matrix norm, $\|\cdot\|_2$ is the spectral norm. For Hermitian matrices M and N, we write $M \geq N (M > N)$ if $M - N$ is a Hermitian positive semidefinite (definite) matrix and $[M,\ N] = \{X : M \leq X \leq N\}$.

2. A Perturbation Bound for $X + A^*X^{-1}A = I$

In this section, we shall apply elementary calculus to derive a perturbation bound of the maximal solution for the matrix equation

$$X + A^*X^{-1}A = I, \tag{2.1}$$

where $A \in \mathcal{C}^{n\times n}$ is nonsingular. Consider the following polynomial equations

$$x^2 - x + \lambda_{\min}(A^*A) \;=\; 0, \tag{2.2}$$

$$x^2 - x + \lambda_{\max}(A^*A) \;=\; 0. \tag{2.3}$$

If $\lambda_{\max}(A^*A) < \frac{1}{4}$, then Eq.(2.2) has two positive real roots $\alpha_2 < \beta_1$ and Eq.(2.3) also has two positive real roots $\alpha_1 < \beta_2$. Obviously, we have

$$0 < \alpha_1 \leq \alpha_2 < \frac{1}{2} < \beta_1 \leq \beta_2 < 1. \tag{2.4}$$

Liu and Gao [8] have proved that the following results.

Lemma 2.1. [8] *Suppose that* A *satisfies* $\|A\|_2 < \frac{1}{2}$. *Then Eq. (2.1) (1) has a Hermitian positive definite solution in* $[\alpha_1 I,\ \alpha_2 I]$; *(2) has a unique Hermitian positive definite solution in* $[\beta_1 I,\ \beta_2 I]$; *(3) has no Hermitian positive definite solution in* $[\alpha_2 I,\ \beta_1 I]$.

Remark 2.1. We call the unique solution of Eq.(2.1) in $[\beta_1 I,\ \beta_2 I]$ the maximal solution, which is denoted by X_L.

For a matrix $A = (a_{ij})$, we define the differential of A by $dA = (da_{ij})$. Next we give the differential bound of the maximal solution X_L to Eq.(2.1) .

Lemma 2.2. *Suppose that* $\|A\|_2 < \frac{1}{2}$. *Then the maximal solution* X_L *of Eq.* (2.1) *exists and for any unitary invariant norm* $\|\cdot\|$, *we have*

$$\|dX_L\| \leq \frac{4\|A\|_2}{1-4\|A\|_2^2}\|dA\|. \tag{2.5}$$

Proof. By Lemma 2.1, we know that there exsts a unique maximal solution X_L to the matrix equation (2.1) and $\|X_L^{-1}\|_2 < 2$. Hence $1 - \|A\|_2^2\|X_L^{-1}\|_2^2 > 1 - 4\|A\|_2^2 > 0$. It is known that the elements of X_L are differentiable functions of the elements of A. Differentiating $X_L + A^*X_L^{-1}A = I$, we get

$$dX_L + dA^*(X_L^{-1}A) - (A^*X_L^{-1})dX_L(X_L^{-1}A) + (A^*X_L^{-1})dA = 0. \tag{2.6}$$

By (2.6), we have

$$dX_L - (A^*X_L^{-1})dX_L(X_L^{-1}A) = -dA^*(X_L^{-1}A) - (A^*X_L^{-1})dA. \tag{2.7}$$

Taking any unitary invariant norm $\|\cdot\|$ in two sides of (2.7) we get

$$\begin{aligned} \|dX_L - (A^*X_L^{-1})dX_L(X_L^{-1}A)\| &= \|-dA^*(X_L^{-1}A) - (A^*X_L^{-1})dA\| \\ &\leq 2\|A\|_2\|X_L^{-1}\|_2\|dA\| \\ &\leq 4\|A\|_2\|dA\|. \end{aligned} \tag{2.8}$$

Combining (2.8) with

$$\begin{aligned} \|dX_L - (A^*X_L^{-1})dX_L(X_L^{-1}A)\| &\geq \|dX_L\| - \|A\|_2^2\|X_L^{-1}\|_2^2\|dX_L\| \\ &= (1 - \|A\|_2^2\|X_L^{-1}\|_2^2)\|dX_L\| \\ &\geq (1 - 4\|A\|_2^2)\|dX_L\|, \end{aligned}$$

we get the estimation (2.5). The proof is complete. ∎

Lemma 2.2 can now be used to derive the following perturbation bound for the maximal solution to Eq.(2.1).

Theorem 2.1. *Let* $A, \widetilde{A} \in \mathcal{C}^{n\times n}$ *and* $E = \widetilde{A} - A$. *If*

$$\|A\|_2 < \frac{1}{2}, \quad \|E\|_2 < \frac{1}{2}\left(\frac{1}{2} - \|A\|_2\right), \tag{2.9}$$

then the maximal solutions X_L *and* $\widetilde{X}_L$ *of the matrix equations*

$$X + A^*X^{-1}A = I \quad \textit{and} \quad \widetilde{X} + \widetilde{A}^*\widetilde{X}^{-1}\widetilde{A} = I \tag{2.10}$$

exist and satisfy that for any unitary invariant norm $\|\cdot\|$,

$$\|\widetilde{X}_L - X_L\| \le \frac{1}{2\|E\|_2} \ln\left(\frac{1-4\|A\|_2^2}{1-4(\|A\|_2+\|E\|_2)^2}\right)\|E\| \equiv C_{err}. \quad (2.11)$$

Proof. By the hypothesis (2.9) and Corollary 3.2 in Xu[12], there are the maximal solutions of the two matrix equations in (2.10). Let

$$A(t) = A + tE, \qquad 0 \le t \le 1.$$

Then we have $\|A(t)\|_2 < \frac{1}{2}$. Hence by Lemma 2.1, we know that for each $t \in [0,1]$, the matrix equation

$$X + A(t)^* X^{-1} A(t) = I \quad (2.12)$$

has the maximal solution, which is denoted by $X_L(t)$. In particular, we have

$$X_L(0) = X_L, \qquad X_L(1) = \widetilde{X}_L\ .$$

By Lemma 2.2,

$$\begin{aligned} \|\widetilde{X}_L - X_L\| &= \|X_L(1) - X_L(0)\| = \left\|\int_0^1 dX_L(t)\right\| \\ &\le \int_0^1 \|dX_L(t)\| \le \|E\| \int_0^1 \frac{4\|A(t)\|_2}{1-4\|A(t)\|_2^2} dt. \end{aligned} \quad (2.13)$$

Applying the perturbation theory for singular values, we get

$$\|A(t)\|_2 \le \|A\|_2 + \|E\|_2 t,$$

Consequently, from (2.13), we have

$$\begin{aligned} \|\widetilde{X}_L - X_L\| &\le \|E\| \int_0^1 \frac{4(\|A\|_2 + t\|E\|_2)}{1-4(\|A\|_2+t\|E\|_2)^2} dt \\ &= \frac{1}{2\|E\|_2} \ln\left(\frac{1-4\|A\|_2^2}{1-4(\|A\|_2+\|E\|_2)^2}\right)\|E\|. \end{aligned}$$

The proof is complete. ■

From Theorem 2.1, we have the following corollary 2.1.

Corollary 2.1. *Under the same assumption as theorem 2.1, then the maximal solutions* X_L *and* $\widetilde{X}_L$ *of the matrix equations (2.10) exist and satisfy that for the spectral norm* $\|\cdot\|_2$,

$$\|\widetilde{X}_L - X_L\|_2 \le \frac{4\|A\|_2}{1-4\|A\|_2^2}\|E\|_2 + O(\|E\|_2^2).$$

3. A Perturbation Bound for $X - A^* X^{-1} A = I$

In this section, we shall apply elementary calculus to derive a perturbation bound of the symmetric positive definite solution for the matrix equation

$$X - A^* X^{-1} A = I, \tag{3.1}$$

where $A \in \mathcal{C}^{n\times n}$ is nonsingular. The following result is due to Liu and Gao[8].

Lemma 3.1. [8]. *If* $\|A\|_2 < 1$, *then Eq.(3.1) has a unique Hermitian positive definite solution* X_0 *and* $X_0 > I$.

Next we give the differential bound of the unique Hermitian positive definite solution X_0 to Eq.(3.1).

Lemma 3.2. *Suppose that* $\|A\|_2 < 1$. *Then the unique Hermitian positive definite solution* X_0 *of the matrix equation (3.1) exists and for any unitary invariant norm* $\|\cdot\|$, *we have*

$$\|dX_0\| \le \frac{2\|A\|_2}{1 - \|A\|_2^2}\|dA\|. \tag{3.2}$$

Proof. By Lemma 3.1, we know that the unique Hermitian positive definite solution X_0 of Eq.(3.1) exists and $\|X_0^{-1}\|_2 < 1$. Hence $1 - \|A\|_2^2\|X_0^{-1}\|_2^2 > 0$. It is known that the elements of X_0 are differentiable functions of the elements of A. Differentiating $X_0 - A^* X_0^{-1} A = I$, we get

$$dX_0 - dA^*(X_0^{-1}A) + (A^* X_0^{-1})dX_0(X_0^{-1}A) - (A^* X_0^{-1})dA = 0. \tag{3.3}$$

From (3.3) we have

$$dX_0 + (A^* X_0^{-1})dX_0(X_0^{-1}A) = dA^*(X_0^{-1}A) + (A^* X_0^{-1})dA. \tag{3.4}$$

Taking unitary invariant norm $\|\cdot\|$ in the two sides of (3.4) we get

$$\begin{aligned} \|dX_0 + (A^* X_0^{-1})dX_0(X_0^{-1}A)\| &\le \|dA^*(X_0^{-1}A) + (A^* X_0^{-1})dA\| \\ &\le 2\|A\|_2\|X_0^{-1}\|_2\|dA\| \\ &\le 2\|A\|_2\|dA\|. \end{aligned} \tag{3.5}$$

Combining (3.5) with

$$\begin{aligned} \|dX_0 + (A^* X_0^{-1})dX_0(X_0^{-1}A)\| &\ge \|dX_0\| - \|A\|_2^2\|X_0^{-1}\|_2^2\|dX_0\| \\ &= (1 - \|A\|_2^2\|X_0^{-1}\|_2^2)\|dX_0\| \\ &\ge (1 - \|A\|_2^2)\|dX_0\|, \end{aligned}$$

we get the estimation (3.2). The proof is complete. ■

Next we give the main theorem in this section. Its proof is quite similar to the proof of Theorem 2.1 and so is omitted.

Theorem 3.1. *Let* $A, \widetilde{A} \in \mathcal{C}^{n\times n}$ *and* $E = \widetilde{A} - A$. *If*

$$\|A\|_2 < 1, \quad \|E\|_2 < 1 - \|A\|_2,$$

then the matrix equations

$$X - A^* X^{-1} A = I \quad \text{and} \quad \widetilde{X} - \widetilde{A}^* \widetilde{X}^{-1} \widetilde{A} = I \tag{3.6}$$

have unique Hermitian positive definite solutions X_0 *and* $\widetilde{X}_0$, *respectively, and satisfy that for any unitary invariant norm* $\|\cdot\|$,

$$\| \widetilde{X}_0 - X_0\| \leq \frac{1}{\|E\|_2} \ln \left(\frac{1 - \|A\|_2^2}{1 - (\|A\|_2 + \|E\|_2)^2} \right) \|E\| \equiv W_{err}. \tag{3.7}$$

From Theorem 3.1, we have the following corollary 3.1.

Corollary 3.1. *Under the same assumption as theorem 3.1, then the matrix equations (3.6) have unique Hermitian positive definite solutions* X_0 *and* $\widetilde{X}_0$, *respectively, and satisfy that for the spectral norm* $\|\cdot\|_2$,

$$\| \widetilde{X}_0 - X_0\|_2 \leq \frac{2\|A\|_2}{1 - \|A\|_2^2} \|E\|_2 + O(\|E\|_2^2).$$

4. Numerical Experiments

To illustrate the results of the previous sections, in this section several interesting numerical examples are given, which are carried out using MATLAB 7.0 with machine epsilon $\varepsilon = 2.2 \times 10^{-16}$. Firstly we describe the known result proposed by Xu[12] for $X + A^T X^{-1} A = I$. He proved the following theorem.

Theorem 4.1. [12, Corollary 3.2]. *Under the same assumptions as Theorem 2.1, then the maximal solutions* X_L *and* $\widetilde{X}_L$ *of Eqs.(2.10) exist and satisfy that*

$$\frac{\| \widetilde{X}_L - X_L\|_2}{\|X_L\|_2} \leq \frac{1}{\frac{1}{2} - \|A\|_2} \frac{\| \widetilde{A} - A\|_2}{\|A\|_2} \equiv X_{err}. \tag{4.1}$$

Secondly, applying the techniques developed by Sun and Xu [9], Hasanov and Ivanov [6] give perturbation bounds for the matrix equations $X \pm A^T X^{-1} A = Q$, which improve the corresponding results in [9, 11]. In particular, when $Q = I$, they obtained the following two theorems.

Theorem 4.2. [6, Theorem 2.1]. *Let* $A, \widetilde{A} = A + E \in \mathcal{C}^{n\times n}$ *and*

$$\begin{aligned} b_+ &= 1 - \|X_L^{-1}A\|_2^2 > 0 \\ c_+ &= 2\|X_L^{-1}A\|_2\|E\| + \|X_L^{-1}\|_2\|E\|^2, \end{aligned}$$

where X_L *is the maximal solution of Eq.(2.1). If*

$$\|X_L^{-1}A\|_2 < 1, \quad 2\|E\| \le \frac{(1 - \|X_L^{-1}A\|_2)^2}{\|X_L^{-1}\|_2},$$

then $D_+ = b_+^2 - 4c_+\|X_L^{-1}\|_2 \ge 0$*, the perturbed matrix equation* $\widetilde{X} + \widetilde{A}^* \widetilde{X}^{-1} \widetilde{A} = I$ *has the maximal solution* $\widetilde{X}_L$ *and*

$$\|\widetilde{X}_L - X_L\| \le \frac{b_+ - \sqrt{D_+}}{2\|X_L^{-1}\|_2} \equiv S_{err}^+. \tag{4.2}$$

Theorem 4.3. [6, Theorem 3.1]. *Let* $A, \widetilde{A} = A + E \in \mathcal{C}^{n\times n}$ *and*

$$\begin{aligned} b &= 1 - \|X_0^{-1}A\|_2^2 \\ c &= 2\|X_0^{-1}A\|_2\|E\| + \|X_0^{-1}\|_2\|E\|^2, \end{aligned}$$

where X_0 *is a unique Hermitian positive definite solution of Eq.(3.1). If*

$$\|X_0^{-1}A\|_2 < 1, \quad 2\|E\| \le \frac{(1 - \|X_0^{-1}A\|_2)^2}{\|X_0^{-1}\|_2},$$

then $D = b^2 - 4c\|X_0^{-1}\|_2 \ge 0$ *and the Hermitian positive definite solutions* X_0 *and* $\widetilde{X}_0$ *of the respective Eq.* (3.6) *satisfy*

$$\|\widetilde{X}_0 - X_0\| \le \frac{b - \sqrt{D}}{2\|X_0^{-1}\|_2} \equiv S_{err}. \tag{4.3}$$

Note that we use the spectral norm $\|\cdot\|_2$ in numerical experiments

Example 4.1. Consider the matrix equation

$$X + A^* X^{-1} A = I,$$

where

$$A = \frac{1}{10}\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{4.3}$$

Since A is a normal matrix, the maximal solution of the equation is given by

$$X_L = \frac{1}{2}(I + (I - 4A^*A)^{1/2}).$$

We now consider perturbation bounds for the maximal solution X_L when the coefficient matrix A is perturbed to $A_j = A + 10^{-2j}A_0$, where

$$A_0 = \frac{1}{\|C^* + C\|_2}(C^* + C),$$

C is a random matrix generated by MATLAB function **rand**. Let

$$X_L^{(j)} = \frac{1}{2}(I + (I - 4A_j^*A_j)^{1/2})$$

be the maximal solution to the perturbed matrix equation $X + A_j^*X^{-1}A_j = I$. Some results are listed in Table 1.

Table 1.

j	2	3	4	5	6
$\frac{\|X_L - X_L^{(j)}\|_2}{\|X_L\|_2}$	$1.09e-004$	$1.09e-006$	$1.09e-008$	$1.09e-010$	$1.09e-012$
X_{err}	$2.11e-003$	$2.11e-005$	$2.11e-007$	$2.11e-009$	$2.11e-011$
$\frac{C_{err}}{\|X_L\|_2}$	$3.37e-004$	$3.37e-006$	$3.37e-008$	$3.37e-010$	$3.37e-012$
$\frac{S_{err}^{+}}{\|X_L\|_2}$	$1.12e-004$	$1.12e-006$	$1.12e-008$	$1.12e-010$	$1.12e-012$

Remark 4.1. The results listed Table 1 show that the perturbation bounds (2.11) and (4.2) is fairly sharp and the bound (4.2) is slightly better than the one (2.11). But the bound in (2.11) doesn't involved with the maximal solution X_L to the matrix equation $X + A^*X^{-1}A = I$. Hence the estimate of the perturbation error in (2.11) could be computed easier than the one in (4.2). In fact, the accurate maximal solution X_L is unknown in general.

Remark 4.2. The bounds in (4.1) and (2.11) don't involved with the maximal symmetric positive definite solution X_L to Eq.(2.1). But the results listed Table 1 also show that the bound in (2.11) is better than the one in (4.1).

Example 4.2. Consider the equation $X - A^* X^{-1} A = I$ and its perturbed equation $X - A_j^* X^{-1} A_j = I$, where A and A_j are defined by Example 4.1. The unique Hermitian positive definite solutions of the above two equations are given by

$$X_0 = \frac{1}{2}\left(I + \sqrt{I + 4A^*A}\right) \quad \text{and} \quad X_0^{(j)} = \frac{1}{2}\left(I + \sqrt{I + 4A_j^*A_j}\right),$$

respectively. Some results are listed in Table 2.

Table 2.

j	2	3	4	5	6
$\|X_0^{(j)} - X_0\|_2$	$5.89e-005$	$5.880e-007$	$5.88e-009$	$5.88e-011$	$5.88e-013$
W_{err}	$8.67e-005$	$8.67e-007$	$8.67e-009$	$8.67e-011$	$8.67e-013$
S_{err}	$7.47e-005$	$7.46e-007$	$7.46e-009$	$7.46e-011$	$7.46e-013$

Remark 4.3. The results listed in Table 2 show that the perturbation bounds in (3.7) and (4.3) is fairly sharp and the bound in (4.3) is slightly better than the one in (3.7). But the bound in (3.7) doesn't involved with the unique Hermitian positive definite solution X_0 to the matrix equation $X - A^T X^{-1} A = I$. Hence the estimate of the perturbation error in (3.7) could be computed easier than the one in (4.3). In fact, the accurate Hermitian positive definite solution X_0 is unknown in general.

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