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CHARACTERIZATIONS OF PREINVEX AND PREQUASIINVEX SET-VALUED MAPS

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Abstract. The purpose of this paper is to characterize in terms of scalar preinvexity (resp. prequasiinvexity) the set-valued maps which are K-preinvex (resp. K-prequasiinvex) with respect to a closed convex cone K. Moreover, as applications of our results some conditions under which a local solution of set-valued scalar optimization for (VP) is a global supper efficient solution for (VP) are given.

1. Introduction

In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson [6]. His initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. In fact he has shown that the Kuhn-Tuker conditions are sufficient for optimality of nonlinear programming problems under invexity conditions. Kaul and Kaur [8] presented the notions of strictly pseudoinvex, pseudoinvex, and quasinvex functions, and investigated their applications in nonlinear programming. Weir and Mond [16], and Weir and Jeyakumar [15] have studied the basic properties of preinvex functions and their applications in optimization. Pini [13] introduced the concepts of prepseudoinvex and prequasiinvex functions and established the relationships between invexity and generalized invexity. Mohan and Neogy [12] showed that under certain assumptions, an invex function is preinvex and a quasiinvex function is prequasiinvex. More recently, characterizations and applications of preinvex functions, semistrictly

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preinvex functions, prequasiinvex functions, and semistrictly prequasiinvex functions were studied by Yang et al. [18], Yang and Li [17] and Jabarootian and Zafarani[7].

On the other hand, various generalizations of the classical notion of quasiconvex real-valued function have been given for vector-valued functions, their importance in vector optimization being nowadays recognized(see e.g. [10] and references therein). Among them, the concept of cone-quasiconvexity, introduced by Luc [9] is of special interest since it can be characterized in terms of convex level sets. The natural way to characterize cone-convexity via scalar quasiconvexity seems to be that indicated by Luc [10] in the particular case when the ordering cone is generated by an algebraic base of a finite-dimensional space, which consists to use the extreme directions of the nonnegative polar cone. In a series of the articles Benoist, Borwein and Popovic([2], [3] and [4]) obtained that similar characterization of cone convexity, cone-quaiconvexity, weak cone convexity and weak cone-quaiconvexity are still true for any closed convex cone with nonempty interior in a Banach space. Motivated, by their works, we characterize in terms of scalar preinvexity (resp. prequasiinvexity) the vector-valued functions which are K-preinvex (resp. K-prequasiinvex) with respect to a closed convex cone K. Moreover, as applications of our results some conditions under which a local solution of set-valued scalar optimization $(\mathrm{SP})_l$ is a global supper efficient solution for vector optimization problem (VP) are given. The paper is organized as follows. In section 2, we present some basic definitions and results that are required in the sequel. Some characterizations of K-preinvex and K-prequasiinvex are given in sections 3 and 4, respectively. Equivalence between weak K-preinvex (resp. weak K-prequasiinvex) with K-preinvex (resp. K-prequasiinvex) are obtained in section 5. Some applications of K-preinvex and K-prequasiinvex set-valued maps in optimization are mentioned in section 6.

2. Preliminaries

Let X be a vector space and $\eta : X \times X \to X$ a vector-valued function.

Definition 2.1. A subset U of X is said to be invex with respect to $\eta : X \times X \to X$ if, for any $x_1, x_2 \in U$ and $t \in [0, 1]$,

$$x_2 + t\eta(x_1, x_2) \in U.$$

A subset K of a topological vector space Y is said to be a cone if $\mathbb{R}_+ K \subseteq K$, cone K is said to be convex if $K + K \subseteq K$, and it is said to be pointed if $K \cap (-K) = \{0\}$. Let $F : U \longrightarrow 2^Y$ be a set-valued map defined on a nonempty invex subset U of vector space X with values in a topological vector space Y endowed with a convex cone $K \subset Y$.

Definition 2.2. A set-valued map $F : U \longrightarrow 2^Y$ is said to be:

(a) K-preinvex with respect to η on U, if for all $x_1, x_2 \in U$ and $t \in [0, 1]$, we have
$$tF(x_1) + (1 - t)F(x_2) \subset F(x_2 + t\eta(x_1, x_2)) + K;$$

(b) K-prequasiinvex with respect to η on U, if for all $x_1, x_2 \in U$ and $t \in [0, 1]$, we have
$$(F(x_1) + K) \cap (F(x_2) + K) \subset F(x_2 + t\eta(x_1, x_2)) + K.$$

(c) If int$K \neq \emptyset$ then F is said to be strictly K-prequasiinvex with respect to η on U, if for all $x_1, x_2 \in U$, $x_1 \neq x_2$ and $t \in (0, 1)$, we have
$$(F(x_1) + K) \cap (F(x_2) + K) \subset F(x_2 + t\eta(x_1, x_2)) + \text{int}K.$$

Example 2.1. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, and
$$U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 1, x_1 \geq 0, x_2 \geq 0\}.$$
we define $\eta : X \times X \longrightarrow X$, by $\eta((x_1, x_2), (y_1, y_2)) = (x_1, x_2)$ and $F_1 : X \longrightarrow 2^Y$ defined by
$$\begin{aligned} F_1(x_1, x_2) &= [0, x_1^2 + x_2^2] \quad &\text{if} \quad x_1^2 + x_2^2 \geq 1. \\ &= [4, 6] \quad &\text{if} \quad x_1^2 + x_2^2 < 1. \end{aligned}$$
Then the set U is invex with respect to η and the function F_1 is K-preinvex with respect to η on U. If we define $F_2 : X \longrightarrow 2^Y$, by $F_2(x_1, x_2) = [-x_1^2 - x_2^2, 0]$ then function F_2 is K- prequasiinvex with respect to η on U. If we define $F_3 : X \longrightarrow 2^Y$, by $F_3(x_1, x_2) = (0, x_1^2 + x_2^2)$ then function F_3 is strictly K- prequasiinvex with respect to η on U.

The following lemma gives a characterization of K-preinvex and K-prequasiinvex set valued maps in terms of their epigraphs and generalized level sets, respectively.

Lemma 2.1. *Let $F : U \longrightarrow 2^Y$ be a set-valued map*

(a) F is K-preinvex with respect to η on U, if and only if epigraph of F
$$epi(F) = \{(x, y) \in U \times Y : y \in F(x) + K\},$$
is invex with respect to η' where, $\eta'((x_1, y_1), (x_2, y_2)) = (\eta(x_1, x_2), y_1 - y_2)$.

(b) *F is K-prequasiinvex with respect to η on U, if and only if for each $y \in Y$ the following generalized level set is invex with respect to η,*

$$F^{-1}(y - K) = \{x \in U : y \in F(x) + K\}.$$

Proof. It is similar to the convex cases [3]. ■

Remark 2.1.

(1) Obviously, K-preinvex set-valued maps are K-prequasiinvex, since the cone K is convex.

(2) If F is K-preinvex with respect to η on U, then Dom(F)= $\{x \in U : F(x) \neq \emptyset\}$, is invex with respect to η, but if F is K-prequasiinvex then Dom(F) is not necessary invex with respect to η. However, if Y is directed with respect to K, i.e. $(y_1 + K) \cap (y_2 + K) \neq \emptyset$ for all $y_1, y_2 \in Y$, then Dom(F) is invex with respect to η whenever F is K-prequasiinvex with respect to η.

(3) Note that vector-valued functions may be studied in the same framework. Actually a function $f : U \longrightarrow Y$ defined on a nonempty invex subset U of X with respect to η is K-preinvex (resp. K-prequasiinvex, strictly K-prequasiinvex) with respect to η on U if the set-valued map $F : U \longrightarrow 2^Y$, defined for all $x \in U$ by $F(x) = \{f(x)\}$ is K-preinvex (resp. K-prequasiinvex, strictly K-prequasiinvex) with respect to η on U.

Let $\varphi : U \longrightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function. As usual in convex analysis, we adopt the following conventions:

$$(+\infty) + (-\infty) = +\infty, \quad 0 \,.\, (+\infty) = +\infty \quad and \quad 0 \,.\, (-\infty) = 0.$$

Recall that φ is said to be:

(a) preinvex with respect to η on U, if for all $x_1, x_2 \in U$ and $t \in [0, 1]$, we have

$$\varphi(x_2 + t\eta(x_1, x_2)) \leq t\varphi(x_1) + (1 - t)\varphi(x_2).$$

(b) prequasiinvex with respect to η on U, if for all $x_1, x_2 \in U$ and $t \in [0, 1]$, we have

$$\varphi(x_2 + t\eta(x_1, x_2)) \leq \max\{\varphi(x_1), \varphi(x_2)\}.$$

By a similar method as in Lemma 2.1, we can obtain the following characterization of preinvex and prequasiinvex functions.

Lemma 2.2. *Function $\varphi : U \longrightarrow \bar{\mathbb{R}}$ is:*

(a) preinvex with respect to η if and only if φ has a invex epigraph with respect to η' where

$$epi(\varphi) = \{(x, \lambda) \in U \times \mathbb{R} : \varphi(x) \leq \lambda\},$$

and if and only if its strict epigraph, i.e. the set epi $_s(\varphi) = \{(x, \lambda) \in U \times \mathbb{R} : \varphi(x) < \lambda\}$, is invex with respect to η'.

(b) prequasiinvex if and only if for every $\lambda \in \mathbb{R}$ the following level set is invex with respect to η:

$$\varphi^{-1}([-\infty, \lambda]) = \{x \in U : \varphi(x) \leq \lambda\}.$$

and if and only if the strict level set $\varphi^{-1}([-\infty, \lambda[) = \{x \in U : \varphi(x) < \lambda\}$, is invex with respect to η, for all $\lambda \in \mathbb{R}$.

In the next result we present a necessary and sufficient condition for being a real set valued $\mathbb{R}_+$-preinvex or $\mathbb{R}_+$-prequasiinvex in terms of preinvexity or prequasiinvexity of its marginal function.

Lemma 2.3. *Let $\phi : U \longrightarrow 2^{\mathbb{R}}$ be a set-valued map, defined on some vector space X, and let $\varphi : U \longrightarrow \mathbb{R}$ be its marginal function, defined for all $x \in U$ by $\varphi(x) = \inf \phi(x)$, where $\inf \emptyset = +\infty$. Then the following assertions hold:*

(1) If the map ϕ is $\mathbb{R}_+$-preinvex ($\mathbb{R}_+$-prequasiinvex) with respect to η on U, then its marginal function φ is preinvex (resp. prequasiinvex) with respect to η on U.

(2) if $\phi(x)$ is closed for all $x \in U$, then the map ϕ is $\mathbb{R}_+$-preinvex ($\mathbb{R}_+$-prequasiinvex) with respect to η on U, if and only if its marginal function φ is preinvex (resp. prequasiinvex) with respect to η on U.

Proof. It is similar to the proof of Lemma 1.1 of [3]. ∎

Remark 2.2. If $\phi(x)$ is not closed for all $x \in U$, assertion (2) in Lemma 2.3 fails to be true [3].

3. Characterization of K-Preinvex Set-valued Maps

In what follows we will characterize the K-pre(quasi)invex set-valued maps $F : U \longrightarrow 2^Y$ where U is a nonempty invex subset of X, in terms of pre(quasi)-invexity of certain extended real-valued functions. In order to get these characterizations, we have to endow the image space Y with a good enough linear topology, and we need to impose some additional hypotheses on the partial order induced by K and also on the structure of the values of F. Supposing that Y is a topological vector space partially ordered by a convex cone K. We denote by

$$K^+ = \{\ell \in Y^* : \ell(y) \geq 0,\ \forall y \in K\}$$

the nonnegative polar cone of K in the topological dual Y^* of Y. By extd K^+ we denote the set of extreme directions of K^+. Recall that $\ell \in$ extd K^+ if and only if $\ell \in K^+ \backslash \{0\}$ and for all $\ell_1, \ell_2 \in K^+$ such that $\ell = \ell_1 + \ell_2$ we actually have $\ell_1, \ell_2 \in \mathbb{R}_+ \ell$.
For any set-valued map $F : U \longrightarrow 2^Y$ and for every $\ell \in Y^*$, we denote by $\ell \circ F : U \longrightarrow 2^{\mathbb{R}}$ the composite set-valued map given for all $x \in U$ by $\ell \circ F(x) = \ell(F(x))$, and we denote by $\ell \circledcirc F : U \longrightarrow \bar{\mathbb{R}}$ the marginal function of $\ell \circ F$, defined for all $x \in U$ by $\ell \circledcirc F(x) = \inf \ell(F(x))$.

Theorem 3.1. *Let U be a nonempty invex subset of a vector space X and let Y be a locally convex space over real, partially ordered by a convex cone K. If $F : U \longrightarrow 2^Y$ is a set-valued map such that $F(x) + K$ is a closed convex set for all $x \in U$, then the following assertions are equivalent:*

(1) F is K-preinvex with respect to η on U.

(2) $\ell \circ F$ is $\mathbb{R}_+$-preinvex with respect to η on U, for every $\ell \in K^+$.

(3) $\ell \circledcirc F$ is preinvex with respect to η on U, for every $\ell \in K^+$.

Proof. With some modifications in the proof of Theorem 2.1 and Proposition 2.1 of [3], one can deduce the proof. ∎

Example 3.1. Consider X, Y, K, U, and $F = F_1$ of Example 2.1, if $\ell \in K^+ = \mathbb{R}_+$,then we have

$$\ell \circ F(x_1, x_2) = [0, \ell(x_1^2 + x_2^2)] \qquad \text{if} \quad x_1^2 + x_2^2 \geq 1;$$
$$\ell \circ F(x_1, x_2) = [4\ell, 6\ell] \qquad \text{if} \quad x_1^2 + x_2^2 < 1.$$

It is easy to see that $\ell \circ F$ is $\mathbb{R}_+$-preinvex with respect to η on U, and

$$\ell \circledcirc F(x_1, x_2) = 0 \qquad \text{if} \quad x_1^2 + x_2^2 \geq 1;$$

$$\ell \circledcirc F(x_1, x_2) = 4\ell \qquad \text{if} \quad x_1^2 + x_2^2 < 1.$$

Then $\ell \circledcirc F$ is preinvex with respect to η on U.

In order to obtain our next result, we need some assumptions:

Assumption A. Let $F : U \longrightarrow 2^Y$ is a set-valued map. Then

$$F(x) \subset F(y + \eta(x, y)) + K \quad \text{for all } x, y \in U.$$

Assumption C. Let $\eta : X \times X \to X$. Then, for any $x, y \in X$ and for any $\lambda \in [0, 1]$,

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y),$$

$$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

Remark 3.1. Let $\eta : X \times X \to X$ satisfy Assumption C. Then, it is shown that [19]

$$\eta(y + \lambda\eta(x, y), y) = \lambda\eta(x, y).$$

Example 3.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $F_1, F_2 : X \longrightarrow 2^Y$, $\eta_1, \eta_2, \eta_3 : X \times X \longrightarrow X$ be defined as follows:

$$\begin{aligned} F_1(x) &= \{0\} \times [0, -x] \quad \text{if} \quad x \le 0; \\ &= [0, x] \times [0, x] \quad \text{if} \quad x > 0; \\ F_2(x) &= \max\{0, x\} \times [0, |x|]; \end{aligned}$$

$$\begin{aligned} \eta_1(x, y) &= x - y \quad \text{if} \quad x \ge 0, y \ge 0; \\ &= x - y \quad \text{if} \quad x \le 0, y \le 0; \\ &= y - x \quad \text{if} \quad x > 0, y < 0; \\ &= y - x \quad \text{if} \quad x < 0, y > 0. \end{aligned}$$

$$\begin{aligned} \eta_2(x, y) &= x - y \quad \text{if} \quad x \ge 0, y \ge 0; \\ &= x - y \quad \text{if} \quad x \le 0, y \le 0; \\ &= 1 - y \quad \text{if} \quad x > 0, y < 0; \\ &= -1 - y \quad \text{if} \quad x < 0, y > 0; \end{aligned}$$

and

$$\begin{aligned} \eta_3(x, y) &= x - y \quad \text{if} \quad x \ge 0, y \ge 0; \\ &= x - y \quad \text{if} \quad x \le 0, y \le 0; \\ &= \frac{-y}{2} \quad \text{if} \quad x > 0, y < 0; \\ &= -1 - y \quad \text{if} \quad x < 0, y > 0. \end{aligned}$$

One can show that F_1 with respect to η_1, η_2 and η_3 is K-preinvex and therefore, it satisfies Assumption A. η_1 dose not satisfy Assumption C, while η_2 dose [18]. F_2 with respect to η_1 and η_2 is not K-prequasiinvex and therefore it is not K-preinvex. In fact it is enough to consider , $x = -1$, $y = \frac{1}{2}$, $t = 1$. F_2 with respect to η_3 is K-prequasiinvex but it is not K-preinvex, it is enough to consider , $x = -2$, $y = 2$, $t = \frac{1}{2}$. Let $U = [-7, -2] \cup [2, 10]$ and

$$\begin{aligned}\eta_4(x, y) &= x - y \qquad \text{if} \quad x \geq 0, y \geq 0;\\ &= x - y \qquad \text{if} \quad x \leq 0, y \leq 0;\\ &= 2 - y \qquad \text{if} \quad x > 0, y < 0;\\ &= -7 - y \qquad \text{if} \quad x < 0, y > 0.\end{aligned}$$

It is shown that U is invex with respect to η_4 which satisfying Assumption C [12]. F_1 is upper semicontinuous on U and K-preinvex with respect to η_4.

Definition 3.1. Let X and Y be normed linear spaces. A set-valued map $F : U \subset X \longrightarrow Y$ is upper semi-continuous at $x_0 \in U$ if

$$\forall \varepsilon > 0, \ \exists \delta > 0 \ \text{ such that } \forall x \in B_U(x_0, \delta), \ \ F(x) \subset F(x_0) + \varepsilon B_Y.$$

F is upper hemi-continuous at x_0 if

$$\forall p \in Y^*, \ x \mapsto \sigma(F(x), p) = \sup_{y \in F(x)} \langle p, y\rangle \quad \text{is upper semi-continuous at } x_o.$$

F is lower semi-continuous at x_0 if, for any sequence x_n converging to x_0 and for all $y_0 \in F(x_0)$, there exists a sequence of elements $y_n \in F(x_n)$ converging to y_0. F is lower hemi-continuous at x_0 if

$$\forall p \in Y^*, \ x \mapsto \sigma(F(x), p) = \sup_{y \in F(x)} \langle p, y\rangle \quad \text{is lower semi-continuous at } x_o.$$

Remark 3.2. It is trivial that any upper semi-continuous set-valued map is upper hemi-continuous, and any lower semi-continuous set-valued map is lower hemi-continuous [1].

We need the following infinite dimensional version of Theorem 3.2 of [17] for establishing our next result:

Lemma 3.1. *Suppose that X is a normed linear space and $U \subset X$ is a nonempty invex set with respect to η such that η satisfies Assumption C. Assume that $f : U \longrightarrow \mathbb{R}$ is a lower semi continuous real valued function and that satisfies*

$$f(y + \eta(x, y)) \leq f(x), \qquad \textit{for all } x, y \in U.$$

Then f is a preinvex function on U if and only if for all $x, y \in U$, there exists an $\alpha \in (0,1)$ such that

$$f(y + \alpha\eta(x,y)) \leq \alpha f(x) + (1-\alpha) f(y).$$

Proof. One can deduce easily from the proof of Theorem 3.2 of [16]. ∎

In the sequel we obtain a necessary and sufficient condition for an upper hemi-continuos set valued map to be K-preinvex.

Theorem 3.2. *Let U be a nonempty invex subset of a normed linear space X, Y a normed linear space partially ordered by a convex cone K and η satisfy Assumption C. Suppose that $F : U \longrightarrow 2^Y$ is an upper hemi-continuous set-valued map such that satisfies Assumption A and $F(x) + K$ is a closed convex set for all $x \in U$. Then F is K-preinvex on U if and only if for every $x, y \in U$, there exists an $\alpha \in (0,1)$ such that*

$$\alpha F(x) + (1-\alpha)F(y) \subset F(y + \alpha\eta(x,y)) + K.$$

Proof. The necessity follows directly from the definition of the K-preinvex function. Suppose that F is upper hemi-continuous, then for all $p \in Y^*$ we have $x \mapsto \sup_{y\in F(x)}\langle p, y\rangle$ is upper semi continuous. Hence if $x_0 \in U$,

$$\limsup_{x\to x_0} \sup_{z\in F(x)} \langle p, z\rangle \leq \sup_{z\in F(x_0)} \langle p, z\rangle, \quad \text{for all } p \in Y^*.$$

Now we consider $-\ell = p \in Y^*$ for every $\ell \in K^+$, then we have

$$-\liminf_{x\to x_0} \inf_{z\in F(x)} \langle \ell, z\rangle \leq -\inf_{z\in F(x_0)} \langle \ell, z\rangle, \quad \text{for all } \ell \in K^+.$$

Hence

$$\liminf_{x\to x_0} (\ell \circledcirc F)(x) \geq (\ell \circledcirc F)(x_0), \quad \text{for all } \ell \in K^+.$$

Therefore $\ell \circledcirc F$ is lower semi continuous for every $\ell \in K^+$. By our assumptions for every $x, y \in U$ there exist an $\alpha \in (0,1)$ such that

$$\alpha F(x) + (1-\alpha)F(y) \subset F(y + \alpha\eta(x,y)) + K.$$

If $\ell \in K^+$, we have

$$\inf_{z\in F(y+\alpha\eta(x,y))+K} \langle \ell, z\rangle \leq \inf_{z\in \alpha F(x)+(1-\alpha)F(y)} \langle \ell, z\rangle,$$

then

$$\inf_{z\in F(y+\alpha\eta(x,y))}\langle \ell, z\rangle + \inf_{k\in K}\langle \ell, k\rangle \le \alpha \inf_{z_1\in F(x)}\langle \ell, z_1\rangle + (1-\alpha)\inf_{z_2\in F(y)}\langle \ell, z_2\rangle.$$

Since $\ell \in K^+$, we deduce

$$(\ell \circledcirc F)(y+\alpha\eta(x,y)) \le \alpha(\ell \circledcirc F)(x) + (1-\alpha)(\ell \circledcirc F)(y).$$

On the other hand by Assumption A, we have

$$F(x) \subset F(y+\eta(x,y)) + K, \qquad \text{for all } x, y \in U.$$

Then in similar way we deduce

$$(\ell \circledcirc F)(y+\eta(x,y)) \le (\ell \circledcirc F)(x) \qquad \text{for all } x, y \in U.$$

Thus, the function $\ell \circledcirc F : U \longrightarrow \mathbb{R}$ satisfies all of the conditions of Lemma 3.1 for all $\ell \in K^+$ and therefore $\ell \circledcirc F$ is preinvex function with respect to η and by Theorem 3.1, F is K-preinvex with respect to η on U. ∎

Example 3.3. Let $X = \mathbb{R}$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $U = [-2, 2]$, and define

$$\begin{aligned} \eta(x,y) &= x - y && \text{if} \quad x \ge 0, y \ge 0; \\ &= x - y && \text{if} \quad x \le 0, y \le 0; \\ &= -2 - y && \text{if} \quad x > 0, y < 0; \\ &= 2 - y && \text{if} \quad x < 0, y > 0. \end{aligned}$$

Then η satisfies in assumption C on U, and if we define $F : X \longrightarrow 2^Y$ by $F(x) = [-|x| + 2, 2]$, this function satisfies in assumption A, and F, is a K-preinvex with respect to η on U.

Now, we obtain the following infinite dimensional version of Theorem 3.1 of [17].

Lemma 3.2. *Suppose that X is a normed linear space and $U \subset X$ is an invex set with respect to η such that η satisfies Assumption C. Assume that $f : U \longrightarrow \mathbb{R}$ is an upper semi continuous real valued function satisfying*

$$f(y+\eta(x,y)) \le f(x), \qquad \textit{for all } x, y \in U.$$

Then f is a preinvex function on U if and only if there exists an $\alpha \in (0, 1)$ such that

$$f(y+\alpha\eta(x,y)) \le \alpha f(x) + (1-\alpha)f(y), \qquad \text{for all } x, y \in U.$$

Proof. Similar to the proof of Theorem 3.1 of [17]. ■

Theorem 3.3. *Suppose that U is a nonempty invex subset of a normed linear space X, Y is a normed linear space partially ordered by a convex cone K and η satisfies Assumption C. Let $F : U \longrightarrow 2^Y$ be a lower hemi-continuous set-valued map satisfying Assumption A, and $F(x) + K$ be a closed convex set for all $x \in U$. Then F is a K-preinvex function on U if and only if there exists an $\alpha \in (0, 1)$ such that for every $x, y \in U$,*

$$\alpha F(x) + (1 - \alpha)F(y) \subset F(y + \alpha\eta(x, y)) + K.$$

Proof. The necessity follows directly from the definition of the K-preinvex function. Suppose that F is lower hemi-continuous, then for all $p \in Y^*$, we have $x \mapsto \sup_{y\in F(x)}\langle p, y\rangle$ is lower semi continuous. Hence if $x_0 \in U$,

$$\sup_{z\in F(x_0)} \langle p, z\rangle \leq \liminf_{x\to x_0} \sup_{z\in F(x)} \langle p, z\rangle, \quad \text{for all } p \in Y^*.$$

Now we consider $-\ell = p \in Y^*$ for every $\ell \in K^+$, then we have

$$-\inf_{z\in F(x_0)} \langle \ell, z\rangle \leq -\limsup_{x\to x_0} \inf_{z\in F(x)} \langle \ell, z\rangle, \quad \text{for all } \ell \in K^+.$$

Hence

$$\limsup_{x\to x_0} (\ell \circledcirc F)(x) \leq (\ell \circledcirc F)(x_0), \quad \text{for all } \ell \in K^+.$$

Therefore, $\ell \circledcirc F$ is upper semi continuous for every $\ell \in K^+$. By our assumptions, there exists an $\alpha \in (0, 1)$ such that for every $x, y \in U$

$$\alpha F(x) + (1 - \alpha)F(y) \subset F(y + \alpha\eta(x, y)) + K.$$

If $\ell \in K^+$, by the same argument as in the proof of Theorem 3.2 we deduce

$$(\ell \circledcirc F)(y + \alpha\eta(x, y)) \leq \alpha(\ell \circledcirc F)(x) + (1 - \alpha)(\ell \circledcirc F)(y).$$

and

$$(\ell \circledcirc F)(y + \eta(x, y)) \leq (\ell \circledcirc F)(x) \qquad \text{for all } x, y \in U.$$

Therefore, the function $\ell \circledcirc F : U \longrightarrow \mathbb{R}$ satisfies all of the conditions of Lemma 3.2 for all $\ell \in K^+$ and therefore $\ell \circledcirc F$ is preinvex function with respect to η. Now Theorem 3.1 implies that F is K-preinvex with respect to η. ■

4. Characterization of K-Prequasiinvex Set-valued Maps

In this section, we will establish a characterization of a K-prequasiinvex set valued map in terms of prequasiinvexity of a certain extended real function.

Lemma 4.1. *Suppose that U is a nonempty invex subset of a vector space X and Y is a Banach space partially ordered by a closed convex cone K which generates Y. Let $F : U \longrightarrow 2^Y$ be a K-prequasiinvex set-valued map with respect to η. Then, for every $\ell \in$ extdK^+, the set-valued map $\ell \circ F$ is $\mathbb{R}_+$-prequasiinvex with respect to η on U and indeed $\ell \circledcirc F$ is prequasiinvex with respect to η on U.*

Proof. Similar to the proof of Proposition 3.1 of [3]. ■

Theorem 4.1 *Suppose that U is a nonempty invex subset of a vector space X, Y is a Banach space partially ordered by a closed convex cone K which generates Y and K^+ is the weak-star closed convex hull of extdK^+. Let $F : U \longrightarrow 2^Y$ be a set-valued map such that a smallest element exist in each of its nonempty values. Then the following assertions are equivalent:*

(1) *F is K-prequasiinvex with respect to η.*

(2) *$\ell \circ F$ is $\mathbb{R}_+$-prequasiinvex with respect to η, for every $\ell \in$ extdK^+.*

(3) *$\ell \circledcirc F$ is prequasiinvex with respect to η, for every $\ell \in$extdK^+.*

Proof. One can prove by a similar proof as that of Theorem 3.1 of [3]. ■

The following remarks which are mentioned in [3] are also suitable for our cases.

Remark 4.1. If the interior of the closed convex cone K is nonempty, then K^+ has a bounded hence weak-star compact base and K generates Y [2]. As shown by Theorem 4.2 below, in this case the K-prequasiinvexity may be characterized in terms of scaler prequasiinvexity in a different manner, under less restrictive assumptions on the structure of the values of F.

Remark 4.2. Certain Banach spaces are partially ordered by a closed convex cone with empty interior, which however generates the space and for which the nonnegative polar cone is the weak-star closed convex hull of its extreme directions. For an example, consider the space $Y = l^p(1 \leq p < +\infty)$ and the cone $K = l^p_+ = \{(y_i)_{i\in\mathbb{N}} \in l^p : y_i \geq 0, \forall i \in \mathbb{N}\}$. In these case we have $K^+ = l^q_+$, where $Y^* = (l^p)^*$ is identified, as usual, with l^q $(1/p + 1/q = 1)$.

Remark 4.3. In the particular case where $f : U \longrightarrow Y$ is a vector-valued function, defined on some nonempty invex subset U of X and the set-valued map $F : U \longrightarrow 2^Y$ defined for all $x \in U$ by $F(x) = \{f(x)\}$, then $f(x)$ is actually the smallest element of $F(x)$, for each $x \in \mathrm{Dom}(F) = U$.

We can obtain a characterization of K-prequasiinvex set-valued map in terms of its marginal function.

Theorem 4.2. *Suppose that U is a nonempty invex subset of a vector space X, Y is a topological vector space partially ordered by a convex cone K with nonempty interior and fix some $e \in intK$. Let $F : U \longrightarrow 2^Y$ be a set-valued map such that $F(x)+K$ be closed for all $x \in U$. For every $a \in Y$, define the set-valued map $\phi_a : U \longrightarrow 2^{\mathbb{R}}$ by*

$$\phi_a(x) = \{\alpha \in \mathbb{R} : x \in F^{-1}(a + \alpha e - K)\} \quad \text{for all } x \in U,$$

and denote by $\varphi_a : U \longrightarrow \overline{\mathbb{R}}$ its marginal function, i.e.

$$\varphi_a(x) = \inf \phi_a(x) \quad \text{for all } x \in U.$$

Then F is K-prequasiinvex with respect to η if and only if φ_a is prequasiinvex with respect to η for all $a \in Y$.

Proof. It is similar to the proof of convex cases [see Theorem 3.2 of 3]. ■

We need the following infinite dimensional version of Theorem 2.3 of [18], for establishing our next result.

Lemma 4.2. *Suppose that X is a normed linear space, $U \subset X$ is a nonempty invex set with respect to η such that η satisfies Assumption C. Assume that $f : U \longrightarrow \mathbb{R}$ is a lower semi continuous real valued function satisfying*

$$f(y + \eta(x, y)) \leq f(x), \qquad \text{for all } x, y \in U.$$

Then f is a prequasiinvex function on U if and only if for all $x, y \in U$, there exists an $\alpha \in (0, 1)$ such that

$$f(y + \alpha\eta(x, y)) \leq \max\{f(x), f(y)\}.$$

Proof. Similar to the proof of Theorem 2.3 of [18]. ■

In the following theorem we give a characterization of upper hemicontinuous K-prequasiinvex set valued map.

Theorem 4.3. *Suppose that U is a nonempty invex subset of a normed linear space X, Y is a Banach space, partially ordered by a closed convex cone K which generates Y, K^+ is the weak-star closed convex hull of extdK^+ and η satisfies Assumption C. Let $F : U \longrightarrow 2^Y$ be an upper hemi-continuous set-valued map such that a smallest element exist in each of its nonempty values and satisfying Assumption A. Then F is a K-prequasiinvex function on U if and only if for every $x, y \in U$, there exists an $\alpha \in (0, 1)$ such that*

$$(F(x) + K) \cap (F(y) + K) \subset F(y + \alpha\eta(x, y)) + K.$$

Proof. The necessity follows directly from the definition of the K-prequasiinvex function. Suppose that F is upper hemi-continuous, then similar to the proof of Theorem 3.2 we deduce that for every $\ell \in K^+$, $\ell \circledcirc F$ is lower semi continuous. By our condition for every $x, y \in U$, there exists an $\alpha \in (0,1)$ such that

$$(F(x)+K) \cap (F(y)+K) \subset F(y+\alpha\eta(x,y)) + K.$$

If $\ell \in K^+$, we have

$$\inf_{z \in F(y+\alpha\eta(x,y))+K} \langle \ell, z\rangle \leq \inf_{z \in (F(x)+K)\cap(F(y)+K)} \langle \ell, z\rangle.$$

Hence

$$\inf_{z \in F(y+\alpha\eta(x,y))} \langle \ell, z\rangle + \inf_{k \in K}\langle \ell, k\rangle \leq$$
$$\max\{(\inf_{z_1 \in F(x)} \langle \ell, z_1\rangle + \inf_{k\in K}\langle \ell, k\rangle), (\inf_{z_2 \in F(y)} \langle \ell, z_2\rangle + \inf_{k\in K}\langle \ell, k\rangle)\}.$$

Since $\ell \in K^+$, we deduce

$$(\ell \circledcirc F)(y+\alpha\eta(x,y)) \leq \max\{(\ell \circledcirc F)(x), (\ell \circledcirc F)(y)\}.$$

On the other hand by Assumption A we have

$$F(x) \subset F(y+\eta(x,y)) + K, \qquad \text{for all } x, y \in U.$$

Then in similar way we deduce

$$(\ell \circledcirc F)(y+\eta(x,y)) \leq (\ell \circledcirc F)(x) \qquad \text{for all } x, y \in U.$$

Therefore, the function $\ell \circledcirc F : U \longrightarrow \mathbb{R}$ satisfies all of the conditions of Lemma 4.2 for all $\ell \in K^+$ and therefore $\ell \circledcirc F$ is prequasiinvex function with respect to η. Thus by Theorem 4.1, F is K-prequasiinvex with respect to η. ∎

We obtain the following infinite dimensional extension of Theorem 2.1 of [11].

Lemma 4.3. *Suppose that X is a normed linear space and $U \subset X$ is invex with respect to η such that η satisfies Assumption C. Assume that $f : U \longrightarrow \mathbb{R}$ is an upper semi continuous real valued function satisfying*

$$f(y+\eta(x,y)) \leq f(x), \qquad \textit{for all } x, y \in U.$$

Then f is a prequasiinvex function on U if and only if there exist an $\alpha \in (0,1)$ such that

$$f(y+\alpha\eta(x,y)) \leq \max\{f(x), f(y)\}, \qquad \text{for all } x, y \in U.$$

Proof. The proof of Theorem 2.1 of [11] can be easily applied to our case. ■

Theorem 4.4. *Suppose that U is a nonempty invex subset of a normed linear space X, Y is a Banach space partially ordered by a closed convex cone K which generates Y, K^+ is the weak-star closed convex hull of extdK^+ and η satisfies Assumption C. Let $F : U \longrightarrow 2^Y$ be a lower hemi-continuous set-valued map such that a smallest element exist in each of its nonempty values and satisfying Assumption A. Then F is a K-prequasiinvex function on U if and only if there exists an $\alpha \in (0,1)$ such that for every $x, y \in U$,*

$$(F(x)+K)\cap(F(y)+K)\subset F(y+\alpha\eta(x,y))+K.$$

Proof. The necessity follows directly from the definition of the K-prequasiinvex function. Suppose that F is lower hemi-continuous then similar to the proof of Theorem 3.3 we deduce $\ell \circledcirc F$ is upper semi continuous for every $\ell \in K^+$. By condition of theorem there exists an $\alpha \in (0,1)$ such that for every $x, y \in U$

$$(F(x)+K)\cap(F(y)+K)\subset F(y+\alpha\eta(x,y))+K.$$

If $\ell \in K^+$, by the same argument as that of the proof of Theorem 4.3 we deduce

$$(\ell \circledcirc F)(y+\alpha\eta(x,y)) \leq \max\{(\ell \circledcirc F)(x), (\ell \circledcirc F)(y)\}.$$

and

$$(\ell \circledcirc F)(y+\eta(x,y)) \leq (\ell \circledcirc F)(x) \qquad \text{for all } x, y \in U.$$

therefore the function $\ell \circledcirc F : U \longrightarrow \mathbb{R}$ satisfies all of the conditions of Lemma 4.3 for all $\ell \in K^+$ and therefore $\ell \circledcirc F$ is prequasiinvex function with respect to η and by Theorem 4.1, F is K-prequasiinvex with respect to η. ■

Theorem 4.5. *Suppose that U is a nonempty invex subset of a normed linear space X, Y is a Banach space partially ordered by a closed convex cone K with int$K \neq \emptyset$, and η satisfies Assumption C. Let $F : U \longrightarrow 2^Y$ be a set-valued map satisfying Assumption A. Then F is strictly K-prequasiinvex function on U if and only if F is K-prequasiinvex function on U and there exists an $\alpha \in (0,1)$ such that for every $x, y \in U$, $x \neq y$,*

$$(F(x)+K)\cap(F(y)+K)\subset F(y+\alpha\eta(x,y))+\textit{int}K,$$

and

$$(F(x)+K)\cap(F(y)+K)\subset F(y+(1-\alpha)\eta(x,y))+\textit{int}K.$$

Proof. The necessity follows directly from the definition of the strictly K-prequasiinvex function and Assumption A. Conversely suppose that $t \in (0,1)$ and $x, y \in U$, $x \neq y$. Let $z = y + t\eta(x,y)$, if $t \leq \alpha$ we consider

$$z_1 = y + \frac{t}{\alpha}\eta(x,y),$$

then, by Remark 3.1, we have

$$y + \alpha\eta(z_1, y) = y + \alpha\eta(y + \frac{t}{\alpha}\eta(x,y), y) = y + t\eta(x,y) = z.$$

Then by our assumptions, we deduce

$$(F(z_1) + K) \cap (F(y) + K) \subset F(y + \alpha\eta(z_1, y)) + \text{int}K = F(z) + \text{int}K.$$

On the other hand by K-prequasiinvexity of F, we have

$$(F(x) + K) \cap (F(y) + K) \subset F(z_1) + K.$$

Thus

$$(F(x) + K) \cap (F(y) + K) \subset (F(z_1) + K) \cap (F(y) + K) \subset F(z) + \text{int}K.$$

If $t \leq 1 - \alpha$, in a similar way for

$$z_1 = y + \frac{t}{1-\alpha}\eta(x,y),$$

we deduce

$$(F(x) + K) \cap (F(y) + K) \subset F(z) + \text{int}K.$$

If $t > \max\{\alpha, (1-\alpha)\}$, then $0 < \frac{t-1+\alpha}{\alpha} < 1$, let

$$z_2 = y + \frac{t-1+\alpha}{\alpha}\eta(x,y).$$

Then by Assumption C, we have

$$z_2 + (1-\alpha)\eta(x, z_2) = y + \frac{t-1+\alpha}{\alpha}\eta(x,y) + (1-\alpha)\eta(x, y + \frac{t-1+\alpha}{\alpha}\eta(x,y))$$

$$y + [\frac{t-1+\alpha}{\alpha} + (1-\alpha)(1 - \frac{t-1+\alpha}{\alpha})]\eta(x,y) = y + t\eta(x,y) = z,$$

and by our assumptions, we have

$$(F(x) + K) \cap (F(z_2) + K) \subset F(z) + \text{int}K.$$

On the other hand since F is K-prequasiinvex, we deduce

$$(F(x)+K)\cap(F(y)+K)\subset F(z_2)+K.$$

Thus

$$(F(x)+K)\cap(F(y)+K)\subset(F(x)+K)\cap(F(z_2)+K)\subset F(z)+\mathrm{int}K.$$

The proof is completed. ■

5. Almost η-Segmentary-valued Maps

Throughout this section we denote by X and Y two vector spaces, the last one being partially ordered by a convex cone K and U is a nonempty invex subset of X. Given a set-valued map $F:U\to 2^Y$, for every $(x,y)\in U^2$,

$$C_F(x,y)=\{t\in[0,1]: tF(x)+(1-t)F(y)\subset F(y+t\eta(x,y))+K\}, \tag{5.1}$$

$$Q_F(x,y)=\{t\in[0,1]: (F(x)+K)\cap(F(y)+K)\subset F(y+t\eta(x,y))+K\}. \tag{5.2}$$

By Definition 2.2, F is said to be

(a) K-preinvex with respect to η if $C_F(x,y)=[0,1]$ for all $(x,y)\in U^2$;

(b) K-prequasiinvex with respect to η if $Q_F(x,y)=[0,1]$ for all $(x,y)\in U^2$.

Then F will be called

(a') weakly K-preinvex with respect to η if $C_F(x,y)\cap\,]0,1[\,\neq\emptyset$ for all $(x,y)\in U^2$;

(b') weakly K-prequasiinvex with respect to η if $Q_F(x,y)\cap\,]0,1[\,\neq\emptyset$ for all $(x,y)\in U^2$.

The aim of this section is to give sufficient conditions for a weakly K-preinvex (respectively, weakly K-prequasiinvex) set-valued map to be K-preinvex (respectively, K-prequasiinvex). However, we also focus on vector-valued functions. A vector-valued function $f:U\longrightarrow Y$ defined on a nonempty invex subset U of X will be called weakly K-preinvex (respectively weakly K-prequasiinvex) if the set-valued map $F:U\longrightarrow 2^Y$ defined by $F(x)=\{f(x)\}$, for $x\in U$, is weakly K-preinvex (respectively weakly K-prequasiinvex).
For each pair $(x,y)\in U^2$, we define the function $\ell_{x,y}:[0,1]\longrightarrow U$ for all $t\in[0,1]$ by $\ell_{x,y}(t)=y+t\eta(x,y)$. For any points $x,y\in U$, we have
$[y,y+\eta(x,y)]=\ell_{x,y}([0,1])$, and if $\eta(x,y)\neq 0$ then $]y,y+\eta(x,y)[=\ell_{x,y}(]0,1[)$.
The following basic lemma will be often used in the sequel.

Lemma 5.1. *If η satisfies Assumption C, then for every points $x, y \in X$ and for any numbers $t, s, r \in [0,1]$, we have*

$$\ell_{u,v}(r) = \ell_{x,y}(rt + (1-r)s), \quad \text{where } u = \ell_{x,y}(t) \text{ and } v = \ell_{x,y}(s).$$

Proof. First let $t < s$, then by Assumption C, we have

$$\begin{aligned}
\ell_{u,v}(r) &= v + r\eta(u,v) = y + s\eta(x,y) + r\eta(y + t\eta(x,y), y + s\eta(x,y)) \\
&= y + s\eta(x,y) + r\eta(y + t\eta(x,y), y + t\eta(x,y) + (s-t)\eta(x,y)) \\
&= y + s\eta(x,y) + r\eta(y + t\eta(x,y), y + t\eta(x,y) + \frac{s-t}{1-t}\eta(x, y + t\eta(x,y))) \\
&= y + s\eta(x,y) + r\left(\frac{t-s}{1-t}\right)\eta(x, y + t\eta(x,y)) \\
&= y + s\eta(x,y) + r\left(\frac{t-s}{1-t}\right)(1-t)\eta(x,y)) \\
&= y + (rt + (1-r)s)\eta(x,y) = \ell_{x,y}(rt + (1-r)s).
\end{aligned}$$

Now suppose that $t > s$, then by Remark 3.1 and Assumption C, we have

$$\begin{aligned}
\ell_{u,v}(r) &= v + r\eta(u,v) = y + s\eta(x,y) + r\eta(y + t\eta(x,y), y + s\eta(x,y)) \\
&= y + s\eta(x,y) + r\eta(y + s\eta(x,y) + (t-s)\eta(x,y), y + s\eta(x,y)) \\
&= y + s\eta(x,y) + r\eta(y + s\eta(x,y) + \frac{t-s}{1-s}\eta(x, y + s\eta(x,y)), y + s\eta(x,y)) \\
&= y + s\eta(x,y) + r\left(\frac{t-s}{1-s}\right)\eta(x, y + s\eta(x,y)) \\
&= y + s\eta(x,y) + r\left(\frac{t-s}{1-s}\right)(1-s)\eta(x,y) \\
&= y + (rt + (1-r)s)\eta(x,y) = \ell_{x,y}(rt + (1-r)s).
\end{aligned}$$

and if $t = s$, then $u = v$ and by Assumption C, we have $\eta(u,v) = 0$. Therefore,

$$\ell_{u,v}(r) = v = y + r\eta(x,y) = y + (rt + (1-r)s)\eta(x,y) = \ell_{x,y}(rt + (1-r)s). \qquad \blacksquare$$

Definition 5.1. Let $S : U^2 \longrightarrow 2^U$ be a set-valued map, which assigns to each pair (x,y) of points of U a subset $S(x,y)$ of U. We say that S is almost η-segmentary-valued if it satisfies the following conditions:

(C1) $\{y, y + \eta(x,y)\} \subset S(x,y) \subset [y, y + \eta(x,y)]$ for all $x, y \in U$;

(C2) $S(x,y) \cap\,]y, y + \eta(x,y)[\, \neq \emptyset$ for all $x, y \in U$ with $\eta(x,y) \neq 0$;

(C3) $S(u,v) \subset S(x,y)$ for all $x, y \in U$ and $u, v \in S(x,y)$.

Example 5.1. Let T be one of the sets [0,1] or $\mathbb{Q} \cap [0,1]$. Then, it can be easily seen that the set-valued map $S : U^2 \longrightarrow 2^U$, defined for all $x, y \in U$ by $S(x,y) = \ell_{x,y}(T)$ is almost η-segmentary-valued.

Lemma 5.2. *If $S : U^2 \longrightarrow 2^U$ is almost η-segmentary-valued and η satisfies Assumption C, then for every $x, y \in U$, the following assertion are equivalent:*

(A1) $S(x,y) = [y, y + \eta(x,y)]$*;*

(A2) $\ell_{x,y}^{-1}(S(x,y))$ *is closed in* $\mathbb{R}$*.*

Proof. The proof is similar to the Lemma 2.3 of [4], hence it is omitted. ■

Note that the proof of Lemma 5.2 shows that (A2) implies the density of $\ell_{x,y}^{-1}(S(x,y))$ in [0,1]. The converse is not true, as shown by Example 5.1 when $T = \mathbb{Q} \cap [0,1]$.
In what follows we shall assume that the space Y is a topological vector space.

Definition 5.2. A set-valued map $F : U \longrightarrow 2^Y$ is called η-segmentary epi-closed for all $x, y \in \mathrm{Dom}(F)$, the epigraph

$$\mathrm{Epi}(F \circ \ell_{x,y}) = \{(t,z) \in [0,1] \times Y : z \in F(y + t\eta(x,y)) + K\}$$

of the composed set-valued map $F \circ \ell_{x,y} : [0,1] \longrightarrow 2^Y$ is closed in $\mathbb{R} \times Y$.

Note that, in the particular case where X is also a topological vector space, any set-valued map $F : X \longrightarrow 2^Y$ which has a closed epigraph is η-segmentary epi-closed.

Theorem 5.1. *Assume that $F : U \longrightarrow 2^Y$ is η-segmentary epi-closed and η and F satisfying Assumption C and A, respectively. Then F is K-preinvex with respect to η if and only if it is weakly K-preinvex with respect to η.*

Proof. The proof is similar to the proof of Theorem 3.2 of [4], and the only major difference is that in the place of property (4) of [4], we use our Lemma 5.1. ■

Theorem 5.2. *Assume that $F : U \longrightarrow 2^Y$ is η-segmentary epi-closed and η and F satisfying Assumption C and A, respectively. Then, F is K-prequasiinvex with respect to η if and only if it is weakly K-prequasiinvex with respect to η.*

Proof. Similar to the proof of Theorem 3.3 of [4]. ■

As a consequence of our Theorems 5.1 and 5.2, we show that the notion of K-preinvex (resp. K-prequasiinvex) is equivalent to the notion of weak K-preinvex (resp. weak K-prequasiinvex) for a vector-valued function under some additional conditions.

Corollary 5.1. *Let $f : U \longrightarrow Y$ be a function defined on a nonempty invex subset U of X. Assume that, for every $x, y \in U$, the epigraph of the composed function $f \circ \ell_{x,y} : [0,1] \longrightarrow Y$, i.e.,*

$$\mathit{Epi}(f \circ \ell_{x,y}) = \{(t,z) \in [0,1] \times Y : z \in f(y + t\eta(x,y)) + K\}$$

is closed in $\mathbb{R} \times Y$, η satisfies Assumption C and f satisfies

$$f(x) \in f(y + \eta(x,y)) + K \qquad \text{for all } x, y \in U.$$

Then, the following assertions hold:

(i) *f is K-preinvex with respect to η if and only if it is weakly K-preinvex with respect to η.*

(ii) *f is K-prequasiinvex with respect to η if and only if it is weakly K-prequasiinvex with respect to η.*

Proof. Consider the set-valued map $F : U \longrightarrow 2^Y$ defined by

$$F(x) = \{f(x)\} \qquad \text{if } x \in U,$$

Then $\mathrm{Dom}(F) = U$, F is η-segmentary epi-closed, since for all $x, y \in U$ the set $\mathrm{Epi}(F \circ \ell_{x,y}) = \mathrm{Epi}(f \circ \ell_{x,y})$ is closed by hypothesis and F satisfies Assumption A. Whence (i) and (ii) are direct consequences of Theorem 5.1 and 5.2, respectively. ■

6. Applications of K-Preinvex and K-Prequasiinvex Maps

In this section we give some conditions under which a local solution of set-valued scalar optimization is a global efficient solution for vector optimization problem. Let X be a normed linear space and Y and Z be two normed linear spaces with norm dual spaces Y^* and Z^*, respectively. Let $K \subset Y$ and $D \subset Z$ be pointed closed convex cones with $\mathrm{int}K \neq \emptyset$ and $\mathrm{int}D \neq \emptyset$. Consider the following vector optimization problem with set-valued maps:

$$\text{(VP)} \qquad \min F(x) \qquad \text{s. t. } x \in A = \{x \in X : G(x) \cap (-D) \neq \emptyset\},$$

where $F : X \longrightarrow 2^Y$ and $G : X \longrightarrow 2^Z$ are two set-valued maps with nonempty values. The set A is the set of all feasible solutions of (VP).

We associate the following set-valued scalar optimization with (VP). For an $\ell \in Y^* \setminus \{0\}$:

$$\text{(SP)}_\ell \qquad \min \ell(F(x)) \qquad \text{s. t. } x \in A = \{x \in X : G(x) \cap (-D) \neq \emptyset\}.$$

A convex subset B of K is a base of convex cone K if $0 \notin \overline{B}$ and $K = \cup_{\lambda \geq 0} \lambda B$. For a nonempty subset V of Y, the generated cone of V is given by

$$\text{con}(V) = \{\lambda v | \lambda \geq 0, v \in V\} = \cup_{\lambda \geq 0} \lambda V.$$

Definition 6.1. Let V be a nonempty subset in Y. The set of all efficient points of V with respect to the convex cone K is defined by

$$E(V, K) = \{y_0 \in V | V \cap (y_0 - K) = \{y_0\}\}$$

The set of all Borwein's supper efficient point of V with respect to convex cone K is defined by

$$SE(V, K) = \{y_0 \in V | \exists N > 0 \text{ such that } (B_Y - K) \cap \overline{\text{con}(V - y_0)} \subset NB_Y\},$$

where B_Y is the closed unit ball of Y.

It was proved by Borwein and Zhuang [5] that $SE(V, K) \subset E(V, K)$.

The following lemma was proved by Rong and Wu [13].

Lemma 6.1. *If the convex pointed ordering cone K has a closed bounded base B and if V is a nonempty subset in Y, then $SE(V, K) = SE(V + K, K)$.*

The next lemma is due to Borwein and Zhuang [5]

Lemma 6.2. *If the convex pointed ordering cone K has a closed bounded base B and if V is a convex subset in Y, then $y_0 \in SE(V, K)$ if and only if there exists $\ell \in$ intK^+ such that $\ell(y - y_0) \geq 0$ for all $y \in V$.*

Definition 6.2. (1) $\bar{x} \in A$ is said to be global super efficient solution of (VP), if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \in SE(F(A), K)$.

$\bar{x} \in A$ is said to be local super efficient solution of (VP), if there exist $\bar{y} \in F(\bar{x})$ and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that $\bar{y} \in SE(F(A \cap N(\bar{x})), K)$.

Definition 6.3. Let $\ell \in Y^* \setminus \{0_{Y^*}\}$. Consider the problem

$$(SP)_\ell \qquad \min_{x \in A} \ell(F(x)).$$

A point $\bar{x} \in A$, is said to be a global solution of $(SP)_\ell$ if there exists $\bar{y} \in F(\bar{x})$ such that

$$\ell(y) \geq \ell(\bar{y}), \qquad \forall x \in A, \forall y \in F(x).$$

A point $\bar{x} \in A$, is said to be a local solution of $(SP)_\ell$, if there exist a neighborhood $N(\bar{x})$ and $\bar{y} \in F(\bar{x})$ such that

$$\ell(y) \geq \ell(\bar{y}), \qquad \forall x \in A \cap N(\bar{x}), \forall y \in F(x).$$

A point $\bar{x} \in A$, is said to be a strict local solution of $(SP)_\ell$ if there exist a neighborhood $N(\bar{x})$ and $\bar{y} \in F(\bar{x})$ such that

$$\ell(y) > \ell(\bar{y}), \qquad \forall x \in A \cap N(\bar{x}) \setminus \{\bar{x}\}, \ \forall y \in F(x).$$

Theorem 6.1. *Suppose that the convex pointed ordering cone K has a closed bounded base B, F is K-preinvex and G is D-prequasiinvex functions with respect to η. If there exists $\ell \in$ intK^+ such that $\bar{x}$ is a local solution for $(SP)_\ell$, then $\bar{x}$ is a global super efficient solution for (VP).*

Proof. Let there exists $\ell \in \text{int}K^+$ such that $\bar{x}$ is a local solution for $(SP)_\ell$. Then by definition, there exist $\bar{y} \in F(\bar{x})$ and a neighborhood $N(\bar{x})$ such that

$$\ell(y) \geq \ell(\bar{y}), \qquad \forall x \in A \cap N(\bar{x}), \forall y \in F(x).$$

If $\hat{x} \in A$ and $\hat{y} \in F(\hat{x})$, then by K-preinvexity of F for $t \in (0,1)$, we have

$$t\hat{y} + (1-t)\bar{y} \in tF(\hat{x}) + (1-t)F(\bar{x}) \subset F(\bar{x} + t\eta(\hat{x}, \bar{x})) + K.$$

Hence, there exist $y_t \in F(\bar{x}+t\eta(\hat{x},\bar{x}))$ and $k_t \in K$ such that $t\hat{y}+(1-t)\bar{y} = y_t+k_t$. On the other hand by Lemma 2.1, A is invex with respect to η. Thus for any $t \in (0,1)$ small enough, $\bar{x} + t\eta(\hat{x}, \bar{x}) \in N(\bar{x}) \cap A$. Then for such $t > 0$ we deduce

$$t\ell(\hat{y}) + (1-t)\ell(\bar{y}) = \ell(y_t) + \ell(k_t) \geq \ell(\bar{y}),$$

therefore,

$$\ell(\hat{y}) - \ell(\bar{y}) \geq 0.$$

Since $\hat{x} \in A$ is arbitrary $\bar{x}$ is a global solution for $\min_{x \in A} \ell(F(x))$. Let $z \in F(A) + K$, then there exist $y \in F(A)$ and $k \in K$ such that $z = y + k$. Hence

$$\ell(z) = \ell(y) + \ell(k) \geq \ell(\bar{y}),$$

and as $F(A) + K$ is convex subset of Y, by Lemma 6.2, $\bar{y} \in SE(F(A) + K, K)$ and by Lemma 6.1, $\bar{y} \in SE(F(A), K)$. ∎

Example 6.1. Let in the above theorem, $X = Y = Z = \mathbb{R}$, $K = D = \mathbb{R}_+$, $F : X \longrightarrow 2^Y$, defined by

$$F(x) = [-|x| + 1, 1] \quad \text{if} \quad -2 \leq x \leq 2;$$
$$= [-1, 1] \quad \text{if} \quad \text{otherwise},$$

and define $G : X \longrightarrow 2^Z$, by

$$G(x) = [-|x|, 0] \quad \text{if} \quad -2 \le x \le 2;$$
$$= [x-2, x-1] \quad \text{if} \quad -2 \le x \le 2;$$
$$= [-x-2, -x-1] \quad \text{if} \quad -2 \le x \le 2,$$

and $\eta : X \times X \longrightarrow X$ defined by

$$\eta(x,y) = x - y \quad \text{if} \quad x \ge 0, y \ge 0;$$
$$= x - y \quad \text{if} \quad x \le 0, y \le 0;$$
$$= -2 - y \quad \text{if} \quad x > 0, y \le 0;$$
$$= 2 - y \quad \text{if} \quad x \le 0, y > 0.$$

Then function F is K-preinvex, and $G$is not D-preinvex, but that is D-prequasiinvex with respect to η on X,

$$A = \{x \in X : G(x) \cap (-D) \ne \emptyset\} = [-2, 2],$$

we consider $\bar{x} = -2$, and $-1 = \bar{y} \in F(\bar{x})$, $N(\bar{x}) = (-3, -1)$. If $\ell \in \text{int}K^+ = (0, +\infty)$ then for each $x \in A \cap N(\bar{x}) = [-2, -1)$, and each $y \in F(x) \subset [-1, 1]$, we deduce, $-\ell = \ell\bar{y} \le \ell y$ therefore $\bar{x}$ is a local solution for $(SP)_\ell$. On the other hand

$$SE(F(A), K)$$
$$= \{y_0 \in [-1, 1] : \exists N > 0 \text{ such that } (-\infty, 1) \cap [-1 - y_0, 1 - y_0] \subset [-N, N]\},$$

therefore $\bar{y} \in SE(F(A), K)$ and $\bar{x}$ is a global super efficient solution of (VP).

Theorem 6.2. *Suppose that the convex pointed ordering cone K has a closed bounded base B, F is K-preinvex and G is D-prequasiinvex with respect to η on X. If $\bar{x} \in A$ is a local super efficient solutions of (VP), then there exists $\ell \in$ intK^+ such that $\bar{x}$ is a global solution of $(SP)_\ell$.*

Proof. Let $\bar{x}$ be a local supper efficient solution for (VP). Then there exist $\bar{y} \in F(\bar{x})$ and a neighborhood $N(\bar{x})$ of $\bar{x}$ such that $\bar{y} \in SE(F(A \cap N(\bar{x})), K)$ and from Lemma 6.1 we derive $SE(F(A \cap N(\bar{x})), K) = SE(F(A \cap N(\bar{x})) + K, K)$. Then by Definition 6.1 there exists $N > 0$ such that $(B_Y - K) \cap \overline{\text{con}(F(A \cap N(\bar{x})) + K - \bar{y})} \subset NB_Y$. We show that

$$\overline{\text{con}(F(A) - \bar{y})} \subset \overline{\text{con}(F(A \cap N(\bar{x})) + K - \bar{y})}.$$

Let $x \in A$, $y \in F(x)$ for $0 < t < 1$ small enough, we have $\bar{x}+t\eta(x,\bar{x}) \in A\cap N(\bar{x})$, hence

$$\begin{aligned} t(y-\bar{y}) &= ty + (1-t)\bar{y} - \bar{y} \in tF(x) + (1-t)F(\bar{x}) - \bar{y} \\ &\subset F(\bar{x}+t\eta(x,\bar{x})) + K - \bar{y} \subset F(A\cap N(\bar{x})) + K - \bar{y}. \end{aligned}$$

Then

$$y - \bar{y} \in \overline{\text{con}(F(A\cap N(\bar{x})) + K - \bar{y})}.$$

Hence, we deduce

$$\overline{\text{con}(F(A) - \bar{y})} \subset \overline{\text{con}(F(A\cap N(\bar{x})) + K - \bar{y})}.$$

Therefore $(B_Y - K) \cap \overline{\text{con}(F(A) - \bar{y})} \subset NB_Y$ that means $\bar{y} \in SE(F(A), K)$, hence $\bar{x}$ is a global super efficient solution of (VP). By Lemma 6.1, $\bar{y} \in SE(F(A)+K, K)$, and Lemma 6.2 implies that there exists $\ell \in \text{int}K^+$ such that $\ell(y+k-\bar{y}) \geq 0$ for all $y \in F(A)$ and $k \in K$. Then $\ell(y) \geq \ell(\bar{y})$ for all $y \in F(A)$. Hence $\bar{x}$ is a global solution of $(SP)_\ell$. ■

Remark 6.1. Suppose that the convex pointed ordering cone K has a closed bounded base B, F is K-preinvex and G is D-prequasiinvex with respect to η on X. Then the following are equivalent.

(a) $\bar{x} \in A$ is a local super efficient solutions of (VP).

(b) There exists $\ell \in \text{int}K^+$ such that $\bar{x}$ is a global solution of $(SP)_\ell$.

(c) $\bar{x}$ is a global super efficient solution for (VP).

(d) There exists $\ell \in \text{int}K^+$ such that $\bar{x}$ is a local solution for $(SP)_\ell$.

Definition 6.4. A point $\bar{x} \in A$ is called a global efficient solution of problem (VP) if there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(A\backslash\{\bar{x}\}) - \bar{y}) \cap (-K) = \emptyset.$$

A point $\bar{x} \in A$ is called a local efficient solution of problem (VP) if there are a neighborhood $N(\bar{x})$ of $\bar{x}$ and $\bar{y} \in F(\bar{x})$ such that

$$(F(A\cap N(\bar{x})\backslash\{\bar{x}\}) - \bar{y}) \cap (-K) = \emptyset.$$

A point $\bar{x} \in A$ is called a global weakly efficient solution of problem (VP) if there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(A) - \bar{y}) \cap (-\text{int}K) = \emptyset.$$

A point $\bar{x} \in A$ is called a local weakly efficient solution of problem (VP) if there are a neighborhood $N(\bar{x})$ of $\bar{x}$ and $\bar{y} \in F(\bar{x})$ such that

$$(F(A\cap N(\bar{x})) - \bar{y}) \cap (-\text{int}K) = \emptyset.$$

Theorem 6.3. *Let F be K-prequasiinvex, G be D-prequasiinvex maps with respect to η, and $\eta(x,y) \neq 0$ for $x \neq y$. Then, any local efficient solution of (VP) is a global efficient solution of (VP).*

Proof. Let $\bar{x}$ be a local efficient solution of (VP), then there exist a neighborhood $N(\bar{x})$ of $\bar{x}$ and $\bar{y} \in F(\bar{x})$ such that

$$(F(A \cap N(\bar{x}) \backslash \{\bar{x}\}) - \bar{y}) \cap (-K) = \emptyset. \tag{6.1}$$

If $\bar{x}$ is not a global efficient solution of (VP), then there exist a $u \in A$, $u \neq \bar{x}$ and $y_0 \in F(u)$ such that $y_0 - \bar{y} \in -K$. Therefore, by K-prequasiinvexity of F for each $t \in [0,1]$ we have

$$\bar{y} \in (y_0 + K) \cap (\bar{y} + K) \subset F(\bar{x} + t\eta(u,\bar{x})) + K.$$

This means that for each $t \in [0,1]$, there exists $y_t \in F(\bar{x} + t\eta(u,\bar{x}))$ such that $y_t - \bar{y} \in -K$. On the other hand for $t \in (0,1)$ small enough, $\bar{x} + t\eta(u,\bar{x}) \in N(\bar{x})$ and by D-prequasiinvexity of G and Lemma 2.1, A is invex with respect to η. Then for $t \in (0,1)$ small enough, $\bar{x} + t\eta(u,\bar{x}) \in A \cap N(\bar{x}) \backslash \{\bar{x}\}$. Thus we deduce $y_t - \bar{y} \in (F(A \cap N(\bar{x}) \backslash \{\bar{x}\}) - \bar{y}) \cap (-K)$ which contradicts (6.1). ∎

Theorem 6.4. *Let F be strictly K-prequasiinvex and G be D-prequasiinvex maps with respect to η. Then, any local weakly efficient solution of (VP) is a global weakly efficient solution of (VP).*

Proof. Let $\bar{x}$ be a local weakly efficient solution of (VP), then there exists a neighborhood $N(\bar{x})$ of $\bar{x}$ and $\bar{y} \in F(\bar{x})$ such that

$$(F(A \cap N(\bar{x})) - \bar{y}) \cap (-\text{int}K) = \emptyset. \tag{6.2}$$

If $\bar{x}$ is not a global weakly efficient solution of (VP). Then there exist a $u \in A$ and $y_0 \in F(u)$ such that $y_0 - \bar{y} \in -\text{int}K$. Therefore, by strict K-prequasiinvexity of F for each $t \in (0,1)$ we have

$$\bar{y} \in (y_0 + K) \cap (\bar{y} + K) \subset F(\bar{x} + t\eta(u,\bar{x})) + \text{int}K.$$

This means that for each $t \in (0,1)$ there exists $y_t \in F(\bar{x} + t\eta(u,\bar{x}))$ such that $y_t - \bar{y} \in -\text{int}K$. On the other hand for $t \in (0,\delta)$ with $\delta > 0$ small enough, $\bar{x} + t\eta(u,\bar{x}) \in N(\bar{x})$ and by D-prequasiinvexity of G and Lemma 2.1, A is an invex set with respect to η. Then, $\bar{x} + t\eta(u,\bar{x}) \in A \cap N(\bar{x})$ for $t \in (0,1)$ small enough, then we deduce $y_t - \bar{y} \in (F(A \cap N(\bar{x})) - \bar{y}) \cap (-\text{int}K)$, which contradicts (6.2). ∎

Theorem 6.5. *Let F be strictly K-prequasiinvex and G be D-prequasiinvex maps with respect to η. If there exists $\ell \in K^+ \setminus \{0_{Y^*}\}$ such that $\bar{x} \in A$ is a local solution of $(SP)_\ell$, then $\bar{x}$ is a global weakly efficient solution of (VP).*

Proof. Since $\bar{x}$ is a local solution of $(SP)_\ell$, then there exists a neighborhood $N(\bar{x})$ and $\bar{y} \in F(\bar{x})$ such that

$$\ell(y) \geq \ell(\bar{y}), \qquad \forall x \in A \cap N(\bar{x}), \forall y \in F(x).$$

If $\bar{x}$ is not a global weakly efficient solution of (VP), then

$$(F(A) - \bar{y}) \cap (-\text{int}K) \neq \emptyset.$$

Hence there exist $x_0 \in A$, and $y_0 \in F(x_0)$, such that $\bar{y} - y_0 \in \text{int}K$. Thus by strict K-prequasiinvexity of F for each $t \in (0,1)$, we have

$$\bar{y} \in (F(\bar{x}) + K) \cap (F(x_0) + K) \subset F(\bar{x} + t\eta(x_0, \bar{x})) + \text{int}K.$$

Then for each $t \in (0,1)$, there exists

$$y_t \in F(\bar{x} + t\eta(x_0, \bar{x})), \text{ such that } \bar{y} \in y_t + \text{int}K.$$

Therefore,

$$\ell(\bar{y}) > \ell(y_t).$$

On the other hand for $t > 0$ small enough, $\bar{x} + t\eta(x_0, \bar{x}) \in A \cap N(\bar{x})$, which is a contradiction. ■

Theorem 6.6. *Let F be K-prequasiinvex, G be D-prequasiinvex maps with respect to η and $\eta(x, y) \neq 0$ for $x \neq y$. If there exists $\ell \in K^+ \setminus \{0_{Y^*}\}$ such that $\bar{x} \in A$ is an strict local solution of $(SP)_\ell$, then $\bar{x}$ is a global efficient solution of (VP).*

Proof. Since $\bar{x}$ is an strict local solution of $(SP)_\ell$, then there exist a neighborhood $N(\bar{x})$ and $\bar{y} \in F(\bar{x})$ such that

$$\ell(y) > \ell(\bar{y}), \qquad \forall x \in A \cap N(\bar{x}) \setminus \{\bar{x}\}, \forall y \in F(x).$$

If $\bar{x}$ is not a global efficient solution of (VP), then

$$(F(A \setminus \{\bar{x}\}) - \bar{y}) \cap (-K) \neq \emptyset.$$

Therefore, there exist $x_0 \in A \setminus \{\bar{x}\}$, and $y_0 \in F(x_0)$, such that $\bar{y} - y_0 \in K$. Thus by K-prequasiinvexity of F for each $t \in (0,1)$ we have

$$\bar{y} \in (F(\bar{x}) + K) \cap (F(x_0) + K) \subset F(\bar{x} + t\eta(x_0, \bar{x})) + K.$$

Then for each $t \in (0,1)$, there exists

$$y_t \in F(\bar{x} + t\eta(x_0, \bar{x})), \text{ such that } \bar{y} \in y_t + K.$$

Therefore,

$$\ell(\bar{y}) \geq \ell(y_t).$$

On the other hand for $t > 0$ small enough, $\bar{x} + t\eta(x_0, \bar{x}) \in A \cap N(\bar{x}) \setminus \{\bar{x}\}$, which is a contradiction. ■

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