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ASYMPTOTICS OF THE LANDAU CONSTANTS AND THEIR RELATIONSHIP WITH HYPERGEOMETRIC FUNCTIONS

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Abstract. We examine the Landau constants defined by

$$G_n := \sum_{m=0}^{n} \frac{1}{2^{4m}} \binom{2m}{m}^2 \qquad (n = 0, 1, 2, \cdots)$$

by making use of the celebrated Ramanujan formula expressing G_n in terms of the Clausenian ${}_3F_2$ hypergeometric series. It is shown that it could be used to deduce other, mostly new, Ramanujan type formulas for the Landau constants involving the terminating and non-terminating hypergeometric series. In addition, by this approach we derive once again, in a simple and unified manner, almost all of the known results and also establish several new results for G_n. These new results include (for example) the generating function and asymptotic expansions and estimates for G_n.

1. Landau Constants

The *normalized* central binomial coefficients:

$$\mu_m \qquad (m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, 3, \cdots\})$$

can be variously defined in terms of the binomial coefficients and factorials as follows:

$$\mu_m = \frac{1}{2^{2m}} \binom{2m}{m} = (-1)^m \binom{-\frac{1}{2}}{m} = \frac{(2m)!}{(2^m \cdot m!)^2} \qquad (m \in \mathbb{N}_0) \tag{1}$$

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or, equivalently, as follows:

$$\mu_0 = 1 \qquad \text{and} \qquad \mu_m = \frac{(2m-1)!!}{(2m)!!} \qquad (m \in \mathbb{N}) \tag{2}$$

in terms of double factorials, which are given by

$$0!! := 1, \quad (2n)!! = 2{\cdot}4{\cdot}6\cdots(2n), \quad \text{and} \quad (2n-1)!! = 1{\cdot}3{\cdot}5\cdots(2n-1) \qquad (n \in \mathbb{N})$$

or, alternatively, as follows:

$$\mu_m = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} = \frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(m+1)} = \frac{\left(\frac{1}{2}\right)_m}{(1)_m} \qquad (m \in \mathbb{N}_0) \tag{3}$$

in terms of the familiar Gamma function $\Gamma(z)$ and the Pochhammer symbol $(\lambda)_n$ defined (for $\lambda \in \mathbb{C}$) by [17, p. 2]

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases} \tag{4}$$

The numbers G_0, G_1, G_2, $\cdots$ given by (see, for instance, [1, p. 216], [4, p. 159], [8, p. 252], [20, p. 310] and [21])

$$G_n = \sum_{m=0}^{n} \mu_m^2 = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 + \cdots + \left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\right)^2 \tag{5}$$

are known as the Landau constants (see also a recent investigation by Alzer *et al.* [2] involving various other important mathematical constants). The values of the first few of these constants are given by

$$G_0 = 1, \qquad G_1 = \frac{5}{4} = 1.25, \qquad G_2 = \frac{89}{64} = 1.3906250,$$

$$G_3 = \frac{381}{256} = 1.488281250, \qquad G_4 = \frac{25609}{16384} = 1.563049316406250,$$

$$G_5 = \frac{106405}{65536} = 1.62361145019531250,$$

and

$$G_6 = \frac{1755841}{1048576} = 1.67450046539306640625 0.$$

Now we choose to recall here the following two classical results concerning the Landau constants.

Theorem A. (Landau, 1913, [1, p. 215] and [10]). *Let*

$$f(z) = \sum_{m=0}^{\infty} a_m z^m$$

be a complex function of complex argument which is analytic inside the unit disk, *continuous on its boundary and satisfies the following inequality*:

$$|f(z)| < 1 \qquad whenever \qquad |z| < 1.$$

Then

$$\left| \sum_{m=0}^{n} a_m \right| \leqq G_n,$$

G_n *being the constants defined as in* (5).

Moreover, *if* $T_n(f)$ *is a polynomial operator associated with* $f(z)$, *then its norm is given by*

$$\|T_n\| = G_n.$$

Theorem B. (Ramanujan, 1913, [15, p. 351] and [19, p. 82]). *Let* ${}_3F_2$ *denote the generalized hypergeometric function defined as in Section* 2 *below. If* $n \in \mathbb{N}_0$, *then*

$$\begin{aligned} \left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right)^2 \cdot G_n &= \frac{1}{n+1}\, {}_3F_2 \left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, n+1; \\ 1, n+2; \end{matrix} \; 1 \right] \\ &= \frac{1}{n+1} + \left(\frac{1}{2}\right)^2 \frac{1}{n+2} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{1}{n+3} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{1}{n+4} + \cdots . \end{aligned} \tag{6}$$

In his first letter to Hardy, dated 16 January 1913, Ramanujan stated the formula (6) without proof. When letters of Ramanujan to Hardy were reproduced in full with comments in Ramanujan's collected papers published in 1927 and this result became publicly known, the formula (6) as well as its various generalizations attracted a great deal of attention.

It is interesting to remark here that the Ramanujan formula (6) is rarely cited in the newer literature on the Landau constants G_n. Here, we make use of this formula and exploit to the full its connection with the theory of the generalized hypergeometric functions and deduce other, mostly new, Ramanujan type formulas for the Landau constants involving the terminating and non-terminating hypergeometric series. Moreover, by this approach we again derive, in a simple and unified manner, almost all of the known results for G_n and also establish various new results for G_n such as (for example) the generating function (see, for details, [18]) as well as asymptotic expansions and estimates for the Landau constants G_n.

2. Landau Constants and Hypergeometric Functions

The generalized (Gauss) hypergeometric ${}_pF_q$ is, as usual, defined by the following hypergeometric series [17, p. 52]:

$$ {}_pF_q \left[\begin{array}{c} \alpha_1, \alpha_2, \cdots, \alpha_p; \\ \\ \beta_1, \beta_2, \cdots, \beta_q; \end{array} z \right] = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_m \cdots (\alpha_p)_m}{(\beta_1)_m (\beta_2)_m \cdots (\beta_q)_m} \frac{z^m}{m!} \tag{7} $$

whenever the series converges and (by analytic continuation) elsewhere. Here $(\cdot)_m$ denotes the Pochhammer symbol defined by (4), p and q are nonnegative integers, the complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_p$ and $\beta_1, \beta_2, \cdots, \beta_q$ are called, respectively, the numerator and denominator parameters, and z is called the variable. The denominator parameters are not allowed to be zero or negative integers and the ${}_pF_q$ function is symmetric in its numerator parameters, and likewise in its denominator parameters. We are concerned here only with the cases when $p = q + 1$. In these instances, the series defining ${}_pF_q$ converges when $|z| < 1$ for all choices of the parameters involved. If $z = 1$, the series converges for

$$ \Re(\beta_1 + \beta_2 + \cdots + \beta_q - \alpha_1 - \alpha_2 - \cdots - \alpha_p) > 0. $$

We begin by noting that (presumably) it has not been noticed so far that the Landau constants G_n admit a generating function. For this purpose, we need to recall the following definition [17, p. 49, Eq. (33)]:

$$ K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \qquad (k^2 < 1) $$

of the *complete elliptic integral* $K(k)$ *of the first kind* with the (elliptic) modulus k.

Proposition 1. *Consider the Landau constants defined by* (5) *and let* $K(k)$ *be the above-defined complete elliptic integral. Then*

$$ g(x) := \frac{2}{\pi} \frac{K(x)}{1 - x} = \sum_{n=0}^{\infty} G_n x^n. \tag{8} $$

Demonstration. Indeed, it is not difficult to show that $g(x)$ is the generating function for G_n by making use of the Leibniz product rule and the following well-known expansion of $K(k)$ in the Gauss hypergeometric series [17, p. 49, Eq. (33)]:

$$ \frac{2}{\pi} K(k) = {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; k^2 \right) = 1 + \sum_{m=1}^{\infty} \left(\frac{(2m-1)!!}{(2m)!!} \right)^2 k^{2m} \qquad (k^2 < 1). $$

We shall next record another interesting connection between G_n and $K(k)$ (*cf.* Watson [20, p. 314, Sect. 3] and Dutka [5, p. 472, Eq. (2.4)]).

Proposition 2. *If* $n \in \mathbb{N}_0$, *then*

$$G_n = \frac{4}{\pi}\left(\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right)^2 \int_0^1 k^{2n+1} K(k)\, dk. \tag{9}$$

Demonstration. Clearly, in view of the above series expansion of $K(k)$, the Ramanujan result (6) could be recast as follows:

$$\begin{aligned} &G_n \cdot \left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right)^2 \\ &= \frac{1}{n+1} + \left(\frac{1}{2}\right)^2 \frac{1}{n+2} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{1}{n+3} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{1}{n+4} + \cdots \\ &= 2\int_0^1 \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right] k^{2n+1}\, dk \\ &= \frac{4}{\pi}\int_0^1 k^{2n+1}\, K(k)\, dk, \end{aligned}$$

showing that the assertion (9) holds true.

We shall now establish four Ramanujan type formulas for the Landau constants of which only one can be found in the literature. Proposition 6 was recently proved by Eisinberg *et al.* [6, p. 59, Eq. (7)] by using some rather involved argument and without appealing to the theory of generalized hypergeometric functions.

Proposition 3. *The following explicit hypergeometric representation holds true*:

$$G_n = \frac{4}{\pi}\left(\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right)^2 {}_3F_2\left[\begin{matrix} -n, 1, 1; \\ \frac{3}{2}, \frac{3}{2}; \end{matrix}\ 1\right] \qquad (n \in \mathbb{N}_0). \tag{10}$$

Demonstration. The above-proposed identity follows at once from (6) upon making use of the following known transformation [14, p. 533, Entry 7.4.4.2]:

$${}_3F_2\left[\begin{matrix} a, b, c; \\ d, e; \end{matrix}\ 1\right] = \frac{\Gamma(d)\,\Gamma(e)\,\Gamma(s)}{\Gamma(a)\,\Gamma(b+s)\,\Gamma(c+s)}\, {}_3F_2\left[\begin{matrix} d-a, e-a, s; \\ b+s, c+s; \end{matrix}\ 1\right] \tag{11}$$

$$\Big(s := d+e-a-b-c;\ \ \min\{\Re(a), \Re(s)\} > 0\Big).$$

Proposition 4. *If* $n \in \mathbb{N}_0$, *then*

$$G_n = \frac{(2n+1)!!}{(2n)!!}\, {}_3F_2\left[\begin{matrix} -n, \frac{1}{2}, \frac{1}{2}; \\ \frac{3}{2}, 1; \end{matrix}\; 1\right]. \tag{12}$$

Demonstration. In order to verify this formula, it suffices to use Proposition 3 and the following transformation [14, p. 533, Entry 7.4.4.1]:

$${}_3F_2\left[\begin{matrix} a, b, c; \\ d, e; \end{matrix}\; 1\right] = \frac{\Gamma(d)\,\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b)\,\Gamma(d-c)}\; {}_3F_2\left[\begin{matrix} e-a, e-b, c; \\ d+e-a-b, e; \end{matrix}\; 1\right]$$

$$\big(\min\{\Re(d+e-a-b-c), \Re(d-c)\} > 0\big).$$

Further, in a similar fashion, we obtain the next two results. Indeed, upon combining (12) and (11), we arrive at Proposition 5, while Proposition 6 is obtained by making use of (12) and the following transformation [14, p. 546, Entry 7.4.5.2]:

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a, b, c; \\ 1+a-b, 1+a-c; \end{matrix}\; -1\right] \\ &= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)}{\Gamma(1+a)\,\Gamma(1+a-b-c)}\; {}_3F_2\left[\begin{matrix} \frac{1}{2}, b, c; \\ \frac{a}{2}+1, \frac{a+1}{2}; \end{matrix}\; 1\right]. \end{aligned}$$

Proposition 5. *If* $n \in \mathbb{N}_0$, *then*

$$G_n = \frac{1}{2(n+1)} \left(\frac{(2n+1)!!}{(2n)!!}\right)^2 {}_3F_2\left[\begin{matrix} \frac{1}{2}, 1, n+\frac{3}{2}; \\ \frac{3}{2}, n+2; \end{matrix}\; 1\right]. \tag{13}$$

Proposition 6. *If* $n \in \mathbb{N}_0$, *then*

$$G_n = \frac{4}{\pi(n+1)} \left(\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right)^2 {}_3F_2\left[\begin{matrix} -n, 1, \frac{1}{2}; \\ \frac{3}{2}, 2+n; \end{matrix}\; -1\right]. \tag{14}$$

Remark. It is well-known that, in general, the ${}_pF_q$ series terminates and, therefore, is a polynomial if a numerator parameter is a negative integer or zero, provided that no denominator parameter is a negative integer or zero.

The series in the original Ramanujan formula (6) is nonterminating, while the ${}_3F_2$ series given by (10), (12) and (14) are terminating, and they can be rewritten by making use of the following finite sum:

$$ {}_3F_2\left[\begin{matrix} -n, \alpha, \beta; \\ \gamma, \delta; \end{matrix}\; x\right] = \sum_{m=0}^{n} \frac{(-n)_m\,(\alpha)_m\,(\beta)_m}{(\gamma)_m\,(\delta)_m\,(1)_m}\; x^m. $$

Thus, in light of this and the following identities (see, for instance, [14, p. 758]):

$$ (-n)_m = (-1)^m m! \binom{n}{m}, \qquad (n)_m = \frac{(n+m-1)!}{(n-1)!}, $$

$$ \left(\frac{1}{2}\right)_m = \frac{1}{2^{2m}} \frac{(2m)!}{m!}, \qquad \text{and} \qquad \left(\frac{3}{2}\right)_m = \frac{1}{2^{2m}} \frac{(2m+1)!}{m!}, $$

starting from (10), (12) and (14), after some simple algebra, we can write down a number of formulas for G_n. We list a few examples in Proposition 7: Part (a), Parts (b) and (c), and Parts (d), (e) and (f) follow from (10), (12), and (14), respectively.

Proposition 7. *If* $n \in \mathbb{N}_0$, *then*

$$ \text{(a)} \qquad G_n = \left(\frac{(2n+1)!!}{(2n)!!}\right)^2 \sum_{m=0}^{n} \frac{(-1)^m\, 2^{4m}}{(2m+1)^2} \binom{n}{m} \binom{2m}{m}^{-2}; $$

$$ \text{(b)} \qquad G_n = \frac{2n+1}{2^{2n}} \binom{2n}{n} \sum_{m=0}^{n} \frac{(-1)^m}{(2m+1)\, 2^{2m}} \binom{n}{m} \binom{2m}{m}; $$

$$ \text{(c)} \qquad G_n = \frac{2n+1}{2^{2n}} \binom{2n}{n} \sum_{m=0}^{n} \frac{1}{(2m+1)} \binom{n}{m} \binom{-\frac{1}{2}}{m}; $$

$$ \text{(d)} \qquad G_n = 2(-1)^{n+1} \sum_{m=0}^{n} \binom{n+\frac{1}{2}}{m} \binom{n-\frac{1}{2}-m}{2n+1-m}; $$

$$ \text{(e)} \qquad G_n = \frac{(2n+1)^2}{n+1} \binom{-\frac{1}{2}}{n}^2 \sum_{m=0}^{n} \frac{1}{2m+1} \binom{n}{m} \binom{n+m+1}{m}^{-1}; $$

$$ \text{(f)} \qquad G_n = \frac{(2n+1)!}{(2^{2n} n!)^2} \sum_{m=0}^{n} \binom{2n+1}{m} \frac{1}{2(n-m)+1}. $$

It should be noted that the formula (f) in Proposition 7 has been known for a long time (see Schönhage [16, p. 64] and Mills and Smith [12, pp. 115 and 116]) and that the formulas (d) and (e) were recently deduced by Eisinberg *at al.* [6, p. 61, Eqs. (17) and (18)], while the formulas (a), (b) and (c) are (presumably) new.

It is known that, in a ${}_3F_2$ form, if a numerator parameter exceeds any denominator parameter by a positive integer, say n, the ${}_3F_2$ series may be expressed as the sum of $n+1$ hypergeometric ${}_2F_1$ series (see, for instance, [14, p. 439, Entry 7.2.3.15]). If the latter can be summed, then so can the ${}_3F_2$ be. We note that the ${}_3F_2$ series given by (13) exhibits this property and we here present a very simple and straightforward derivation of the identity given by Proposition 8 in sharp contrast to two rather lengthy and complicated proofs given by Montaldi and Zucchelli [13, pp. 562 and 563].

Proposition 8. *The following identity holds true*:

$$G_n = \frac{(\frac{3}{2})_n}{n!} \sum_{m=0}^{n} \frac{(\frac{1}{2})_m}{m!} \frac{1}{2(n-m)+1}.$$

Demonstration. Indeed, by making use of the transformation [14, p. 439, Entry 7.2.3.15]) in conjunction with the Gauss summation theorem [14, p. 489, Entry 7.3.5.2]:

$${}_2F_1 \left[\begin{matrix} a, b; \\ \\ c; \end{matrix} \; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \big(\Re(c-a-b) > 0\big)$$

in Proposition 5, we have

$$\begin{aligned} G_n &= \frac{1}{2(n+1)} \left(\frac{\left(\frac{3}{2}\right)_n}{n!} \right)^2 \sum_{m=0}^{n} \binom{n}{m} \frac{\left(\frac{1}{2}\right)_m (1)_m}{\left(\frac{3}{2}\right)_m (n+2)_m} \; {}_2F_1 \left[\begin{matrix} \frac{1}{2}+m, 1+m; \\ \\ n+2+m; \end{matrix} \; 1 \right] \\ &= \frac{1}{2} \frac{\left(\frac{3}{2}\right)_n}{n!} \sum_{m=0}^{n} \frac{1}{2m+1} \frac{\left(\frac{3}{2}\right)_n}{(n-m)!} \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+\frac{3}{2})} \\ &= \frac{1}{2} \frac{\left(\frac{3}{2}\right)_n}{n!} \sum_{m=0}^{n} \frac{1}{2(n-m)+1} \frac{\left(\frac{3}{2}\right)_n}{m!} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(n+\frac{3}{2})} \\ &= \frac{(\frac{3}{2})_n}{n!} \sum_{m=0}^{n} \frac{(\frac{1}{2})_m}{m!} \frac{1}{2(n-m)+1}. \end{aligned}$$

A particularly short proof of the next proposition can be obtained if we consider G_n as partial sums of the (divergent) hypergeometric series as follows [*cf.* Eqs. (3) and (5)]:

$$G_n = \sum_{m=0}^{n} \frac{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_m}{(1)_m} \frac{1}{m!}$$

and then apply the following known result [11, p. 109, Eq. (34)]:

$$ {}_2F_1\left[\begin{matrix} a, b; \\ \\ a+b; \end{matrix}\; 1\right]_n := \sum_{m=0}^{n} \frac{(a)_m (b)_m}{(a+b)_m} \frac{1}{m!} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} $$
$$ \cdot \left(\psi(b+n+1) - \psi(a) - \psi(b) + \psi(1) - \frac{b(1-a)}{b+n+1}\, {}_4F_3\left[\begin{matrix} b+1, 2-a, 1, 1; \\ \\ b+n+2, 2, 2; \end{matrix}\; 1\right]\right), $$

where $\psi(x)$ is the Psi (or Digamma) function defined by [17, p. 13, Eq. (1)]

$$ \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \frac{d}{dz}\{\log \Gamma(z)\}. $$

We note that Eq. (15) below is proven by different arguments in several places (see Dutka [5, p. 474, Eq. (3.4)], Mills and Smith [12, p. 114, Theorem 2] and Cvijović and Klinowski [4, p. 161, Theorem 1]).

Proposition 9. *Let*

$$ \gamma = -\psi(1) = 0.5772156649015328606065 12 \cdots $$

denote the Euler-Mascheroni constant and let $\psi(x)$ be the Psi function. Then

$$ G_n = \frac{1}{\pi}\, \psi\left(n + \frac{3}{2}\right) + \frac{1}{\pi}\,(\gamma + 4\,\log 2) - \delta_n, \tag{15} $$

where

$$ \delta_n = \frac{1}{\pi}\, \frac{1}{4\left(n+\frac{3}{2}\right)}\, {}_4F_3\left[\begin{matrix} \frac{3}{2}, \frac{3}{2}, 1, 1; \\ \\ n+\frac{5}{2}, 2, 2; \end{matrix}\; 1\right]. $$

In conclusion of this section, we shall obtain a new formula for G_n given by (16) below. It follows from (15), since, by making use of the following well-known identity of the Psi function [14, p. 760]:

$$ \psi(n+z) = \psi(z) + \sum_{m=0}^{n-1} \frac{1}{m+z}, $$

we have

$$ \psi\left(n + \frac{5}{4}\right) - \psi\left(\frac{5}{4}\right) = 4 \sum_{m=0}^{n-1} \frac{1}{4m+5} $$

and

$$\psi\left(n+\frac{3}{2}\right)-\psi\left(\frac{3}{2}\right)=2\sum_{m=0}^{n-1}\frac{1}{2m+3},$$

so that

$$\psi\left(n+\frac{5}{4}\right)=\psi\left(n+\frac{3}{2}\right)+2-\log 2-\frac{\pi}{2}+2\sum_{m=0}^{n-1}\frac{1}{(2m+3)(4m+5)},$$

since (see [14, p. 761])

$$\psi\left(\frac{3}{2}\right)=-\gamma-2\log 2+2 \qquad \text{and} \qquad \psi\left(\frac{5}{4}\right)=-\gamma-\frac{\pi}{2}-3\log 2+4.$$

This completes the proof of Proposition 10 below.

Proposition 10. *Let γ and $\psi(x)$ have the same meaning as above. Then*

$$G_n=\frac{1}{\pi}\,\psi\left(n+\frac{5}{4}\right)+\frac{1}{\pi}\,(\gamma+4\,\log 2)+\Delta_n, \tag{16}$$

where

$$-\pi\,\Delta_n=2-\log 2-\frac{\pi}{2}+\sum_{m=0}^{n-1}\frac{2}{(2m+3)(4m+5)}$$
$$+\frac{1}{4\left(n+\frac{3}{2}\right)}\,{}_4F_3\left[\begin{matrix}\frac{3}{2},\frac{3}{2},1,1;\\ \\ n+\frac{5}{2},2,2;\end{matrix}\;1\right].$$

3. Estimates and Asymptotics for the Landau Constants

Let

$$A=\frac{1}{\pi}\,(\gamma+4\log 2)=1.06627\cdots. \tag{17}$$

We now use the closed-form expressions for the Landau constants given by (15) and (16) to establish, in a simple and unified manner, several upper and lower bounds for G_n in terms of the Psi and logarithm functions.

Proposition 11. *The following two-sided inequality holds true*:

$$A-\delta_0+\frac{1}{\pi}\psi\left(n+\frac{3}{2}\right)<G_n<A+\frac{1}{\pi}\psi\left(n+\frac{3}{2}\right) \qquad (n\in\mathbb{N}_0), \tag{18}$$

where

$$\delta_0 = \frac{2}{\pi}(\log 2 + 1) - 1 \quad \textit{and} \quad A - \delta_0 = 1 - \frac{1}{\pi}(\gamma + 2\log 2 - 2) \approx 0.988385.$$

Proposition 12. *The following two-sided inequality holds true*:

$$A + \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) < G_n < A + \Delta_0 + \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) \qquad (n \in \mathbb{N}_0), \tag{19}$$

where

$$\Delta_0 = \frac{3}{2} - \frac{1}{\pi}(4 + \log 2) \quad \textit{and} \quad A + \Delta_0 = \frac{3}{2} - \frac{1}{\pi}(4 - \gamma - 3\log 2) \approx 1.07240.$$

Proposition 13. *The following inequality holds true*:

$$1 + \frac{1}{\pi}\log(n+1) \leqq G_n < A + \frac{1}{\pi}\log(n+1) \qquad (n \in \mathbb{N}_0). \tag{20}$$

Proposition 14. *The following inequality holds true*:

$$A + \frac{1}{\pi}\log\left(n + \frac{3}{4}\right) < G_n \leqq A + \Delta_0^* + \frac{1}{\pi}\log\left(n + \frac{3}{4}\right) \qquad (n \in \mathbb{N}_0), \tag{21}$$

where

$$A + \Delta_0^* = 1 - \frac{1}{\pi}\log\left(\frac{3}{4}\right) \approx 1.09157.$$

Demonstrations. In order to prove Propositions 11 and 12, it is sufficient to show that the sequences $\{\delta_n\}_{n=0}^{\infty}$ and $\{\Delta_n\}_{n=0}^{\infty}$ in (15) and (16) are strictly decreasing. To do so, in the case of $\{\Delta_n\}_{n=0}^{\infty}$ in (16), we need only to verify that

$$\begin{aligned}\Delta_n - \Delta_{n-1} &= G_n - G_{n-1} - \frac{1}{\pi}\frac{4}{4n+1} \\ &= \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 - \frac{1}{\pi}\,\frac{4}{4n+1} < 0 \quad (n \in \mathbb{N})\end{aligned}$$

leads to

$$\pi < \frac{4}{4n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \quad (n \in \mathbb{N}),$$

and then to appeal to the following Gurland inequalities [9]:

$$\frac{4n+3}{(2n+1)^2}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 < \pi < \frac{4}{4n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \quad (n \in \mathbb{N}). \tag{22}$$

Similarly, we show that the sequence $\{\delta_n\}_{n=0}^{\infty}$ in (15) is strictly decreasing by making use of the well-known Wallis formula:

$$\frac{2}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 < \pi < \frac{1}{n}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \quad (n \in \mathbb{N}),$$

and, thus, demonstrate that the proposed bounds in (18) are valid.

In order to prove Propositions 13 and 14, we consider the following sequences:

$$\delta_n^* = \frac{1}{\pi}\log(n+1) + A - G_n \quad (n \in \mathbb{N}_0) \tag{23}$$

and

$$\Delta_n^* = G_n - \frac{1}{\pi}\log\left(n+\frac{3}{4}\right) - A \quad (n \in \mathbb{N}_0), \tag{24}$$

which, respectively, follow from (15) and (16) in conjunction with

$$\psi\left(n+\frac{3}{2}\right) \sim \log(n+1) \quad (n \to \infty)$$

and

$$\psi\left(n+\frac{5}{4}\right) \sim \log\left(n+\frac{3}{4}\right) \quad (n \to \infty),$$

which are obtained by means of the following asymptotic formula:

$$\psi\left(z+\frac{1}{2}\right) = \log z + O\left(\frac{1}{z^2}\right) \quad (z \to \infty),$$

which, in turn, is readily derivable from the following asymptotic expansion for the psi function [11, p. 33, Eq. (9)]:

$$\psi(z+a) = \log z - \sum_{m=0}^{n-1}(-1)^{m+1}\frac{B_{m+1}(a)}{m+1}\,z^{-m-1} + O\left(\frac{1}{z^{n+1}}\right) \quad (z \to \infty)$$

$$\Big(|\arg(z)| \leqq \pi - \epsilon \qquad (\epsilon > 0)\Big),$$

$B_n(x)$ being the classical Bernoulli polynomials (see, for example, [17, p. 59 *et seq.*]).

Now, in order to prove the inequalities (20) and (21) asserted by Propositions 13 and 14, it suffices to show, in a similar fashion as above, that the sequences $\{\delta_n^*\}_{n=0}^{\infty}$ and $\{\Delta_n^*\}_{n=0}^{\infty}$ in (23) and (24) are strictly decreasing by using the inequalities in (22).

We note that the estimates in (18) and (19) were found by Cvijović and Klinowski [4], while Falaleev [7] and Brutman [3] obtained (20) and (21), respectively.

It can be shown, by a simple numerical computation, that (for all sufficiently large n) the lower bound in (20) improves the one given in (21), whereas the upper bound given in (21) is better than the one in (20). We thus have arrived at the new estimates contained in Proposition 15 below.

Proposition 15. *The following inequality holds true*:

$$1+\frac{1}{\pi}\log(n+1)\leqq G_n\leqq 1-\frac{1}{\pi}\log\left(\frac{3}{4}\right)+\Delta_0^*+\frac{1}{\pi}\log\left(n+\frac{3}{4}\right)\quad (n\in\mathbb{N}_0). \tag{25}$$

We now state (*without proof*) a recent result of Alzer [1, p. 218, Theorem 1]. Indeed, as noted already by Alzer [1], the upper and lower bounds for G_n given by (25) improve the bounds presented in (18) to (21).

Proposition 16. *Let* $\psi^{-1}(z)$ *denote the inverse function of* $\psi(z)$*. Then*

$$A+\frac{1}{\pi}\psi(n+\alpha)<G_n<A+\frac{1}{\pi}\psi(n+\beta)\qquad (n\in\mathbb{N}_0), \tag{26}$$

with the following best possible constants:

$$\alpha=\frac{5}{4}\qquad \text{and}\qquad \beta=\psi^{-1}\big(\pi(1-A)\big)\approx 1.26621\cdots.$$

In conclusion, we remark that Landau [10] first studied the asymptotic behavior of G_n and showed that

$$G_n\sim\frac{1}{\pi}\log n\qquad (n\to\infty),$$

while Watson [20, p. 315, Eq. (8)] proved the following asymptotic expansion:

$$G_n=\frac{1}{\pi}\log(n+1)+A-\frac{1}{4\pi(n+1)}+O\left(\frac{1}{n^2}\right)\qquad (n\to\infty).$$

In this connection, we find it to be worthwhile to restate the results asserted by Propositions 11 to 14 as follows.

Proposition 17. *Assume that* $n\in\mathbb{N}_0$ *and let* A *be a constant given by* (17)*. Then*

(a) $$G_n\sim\tfrac{1}{\pi}\psi\left(n+\tfrac{3}{2}\right)+A\qquad (n\to\infty); \tag{27}$$

(b) $$G_n\sim\tfrac{1}{\pi}\psi\left(n+\tfrac{5}{4}\right)+A\qquad (n\to\infty); \tag{28}$$

(c) $$G_n\sim\tfrac{1}{\pi}\log(n+1)+A\qquad (n\to\infty); \tag{29}$$

$$\text{(d)} \qquad G_n \sim \tfrac{1}{\pi} \log\left(n + \tfrac{3}{2}\right) + A \qquad (n \to \infty). \tag{30}$$

Finally, it is not difficult to show that δ_n in (15) is of the following form:

$$\delta_n = \left(\frac{1}{2}\right) \frac{1}{1 \cdot (2n+3)} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right) \frac{1 \cdot 3}{2 \cdot (2n+3)(2n+5)} + \cdots ,$$

so that, by making use of (15) and (16), we can deduce the (presumably) new asymptotic expansions asserted by Proposition 18 below.

Proposition 18. *Let A be a constant given by* (17). *Then*

$$\text{(a)} \quad G_n = \frac{1}{\pi}\psi\left(n + \frac{3}{2}\right) + A - \frac{1}{2\pi(2n+3)} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty); \tag{31}$$

$$\text{(b)} \quad G_n = \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) + A - \frac{1}{2\pi(2n+3)} - \frac{B}{\pi} + O\left(\frac{1}{n^2}\right) \quad (n \to \infty), \tag{32}$$

where

$$B = 2 - \log 2 - \frac{\pi}{2} + \sum_{m=0}^{n-1} \frac{2}{(2m+3)(4m+5)}.$$

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