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ON DISCRETE QUASICONVEXITY CONCEPTS FOR SINGLE VARIABLE SCALAR FUNCTIONS

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Abstract. The aim of this paper is to propose quasiconvexity concepts for discrete single variable functions and state some related optimality conditions. Four classes of discrete quasiconvex single variable functions are introduced, compared and characterized. Two different algorithm procedures for determining a minimum are provided.

1. INTRODUCTION

Generalized convexity properties have been widely used in Mathematics and in Economics due to their usefulness in optimization problems (e.g., both critical points and local minima are global optimum points). As it is well known, these concepts regard to functions defined over convex sets. Unfortunately, many applicative problems arising in Operations Research and in Management Science belong to integer programming. As a consequence, some efforts have been done in the literature in order to determine convexity concepts suitable for discrete problems (see for all [3, 5, 9, 10, 11, 12, 13]). On the other hand, very few results have been presented regarding quasiconvexity notions [11, 12], expecially from an applicative point of view.

The aim of this paper is to study discrete generalized convexity concepts following the lines recently proposed by Murota and Shioura in [11] and by Cambini, Riccardi and Yüceer in [3]. Such an approach resulted to be suitable for concrete applicative problems (see for example [3, 4, 13]) and for solution algorithms. In particular, these concepts will allow to generalize some of the results stated by Murota and Shioura in [11] and to provide optimality conditions and efficient algorithms for determining the minimum points. It is worth noticing that the optimality properties of the proposed discrete generalized convex functions do not coincide with the

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known ones related to scalar functions defined over convex sets; just as an example, it will be shown that the minimum of discrete strictly quasiconvex functions is not necessarily unique.

In Section 2 the notion of discrete convexity is recalled and the one of discrete strict convexity is proposed. These concepts are also characterized generalizing the results in [3]. In Section 3 we introduce the classes of discrete quasiconvex, discrete strictly quasiconvex, discrete semistriclty quasiconvex, discrete semi quasiconvex functions, providing various examples and comparing them with the discrete convex functions and with the discrete strictly convex ones. It will be also pointed out that the proposed discrete quasiconvexity concepts do not coincide with the quasiconvexity of the corresponding linear piecewise extensions. In Section 4 the introduced classes of discrete quasiconvex functions are used in order to generalize some of the results stated by Murota and Shioura in [11]. In Section 5 some optimality results concerning discrete quasiconvex functions are given and two different procedures for determining a minimum are provided.

2. Discrete Convexity Concepts

In this section we first recall the concept of discrete convexity introduced and studied by Cambini-Riccardi-Yüceer in [3]. Then, some of the results stated in [3] will be generalized. With this aim, let us preliminarily provide the following notations, where $x, y \in Z$:

$$[x, y]_Z = \{z \in Z : \min\{x, y\} \leq z \leq \max\{x, y\}\}$$
$$]x, y[_Z = \{z \in Z : \min\{x, y\} < z < \max\{x, y\}\}$$

Definition 2.1. A set $X \subseteq Z$ is said to be a *discrete reticulum* if

$$[x, y]_Z \subseteq X \quad \forall x, y \in X.$$

Definition 2.2. Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. Function f is said to be a *discrete convex function* if for all $x \in X$ such that $x + 1 \in X$ and $x - 1 \in X$, it is:

$$f(x + 1) + f(x - 1) \geq 2f(x) \tag{1}$$

Function f is said to be a *discrete strictly convex function* if for all $x \in X$ such that $x + 1 \in X$ and $x - 1 \in X$, it is:

$$f(x + 1) + f(x - 1) > 2f(x) \tag{2}$$

Notice that, by means of the definitions, a discrete strictly convex function is also discrete convex. The previous definition tries to implement, in the discrete case, the very well known property of convex functions related to the nonnegativeness of their second order derivatives. Finally, notice that condition (1) has been also used in [8] with the aim of studying convexity properties on Abelian groups.

The following theorems provide characterizations of the discrete convexity concepts generalizing the results given in [3] with respect to single variable discrete convex functions.

Theorem 2.1. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. The following conditions are equivalent:*

(i) *function f is discrete convex;*

(ii) *the following inequality holds for all $h, k \geq 1$ such that $x+h, x-k \in X$:*

$$\frac{f(x+h)-f(x)}{h} \geq \frac{f(x)-f(x-k)}{k} \tag{3}$$

(iii) *the following inequality holds for all $x, y \in X$ such that $x-1 \in X$:*

$$f(y) \geq [f(x)-f(x-1)]\,(y-x)+f(x) \tag{4}$$

Proof. $(i) \Rightarrow$ *(ii)* Let us first prove, as a preliminary result, that the discrete convexity of f implies:

$$f(x+h)-f(x+h-1) \geq f(x+1)-f(x) \quad \forall h \geq 1 \tag{5}$$

This property is trivial in the case $h=1$, while for $h>1$ condition (1) implies:

$$\begin{aligned} &(f(x+h)-f(x+h-1))-(f(x+1)-f(x)) \\ =&\sum_{j=1}^{h-1}((f(x+j+1)-f(x+j))-(f(x+j)-f(x+j-1))) \\ =&\sum_{j=1}^{h-1}(f(x+j+1)+f(x+j-1)-2f(x+j)) \geq 0 \end{aligned}$$

Notice also that from (5) it yields:

$$f(x)-f(x-1) \geq f(x-k+1)-f(x-k) \quad \forall k \geq 1 \tag{6}$$

Conditions (5) and (6) allow us to prove that:

$$f(x+h)-f(x)=\sum_{j=1}^{h}(f(x+j)-f(x+j-1)) \geq h(f(x+1)-f(x))$$

$$f(x) - f(x-k) = \sum_{j=1}^{k} (f(x-j+1) - f(x-j)) \leq k(f(x) - f(x-1))$$

As a conclusion, the discrete convexity of f implies:

$$\frac{f(x+h) - f(x)}{h} \geq f(x+1) - f(x) \geq f(x) - f(x-1) \geq \frac{f(x) - f(x-k)}{k}$$

so that the result is proved.

(ii) $\Rightarrow$ *(iii)* Follows just setting $h = y - x$ and $k = 1$.

(iii) $\Rightarrow$ *(i)* Follows just setting $y = x + 1$. ∎

Theorem 2.2. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. The following conditions are equivalent:*

(i) *function f is discrete strictly convex;*

(ii) *the following inequality holds for all $h, k \geq 1$ such that $x + h, x - k \in X$:*

$$\frac{f(x+h) - f(x)}{h} > \frac{f(x) - f(x-k)}{k} \tag{7}$$

(iii) *the following inequality holds for all $x, y \in X$ such that $x - 1 \in X$:*

$$f(y) > [f(x) - f(x-1)]\,(y - x) + f(x) \tag{8}$$

Proof. The result follows analogously to Theorem 2.1 noticing that a discrete strictly convex function is also discrete convex. ∎

Notice that in the previous Theorems 2.1 and 2.2 condition *ii)* represents, in the discrete case, the very well known property of convex functions given by the nondecreaseness of the marginal increments, while condition *iii)* implements in the discrete case the relationship existing between the graph of convex functions and their tangent lines.

3. Discrete Quasiconvexity Concepts

The aim of this section is to introduce some discrete quasiconvexity concepts suitable for obtaining optimality conditions and for generalizing some of the results in [11]. The following concepts of discrete quasiconvex functions are introduced following the lines of Murota and Shioura in [11] and of Cambini-Riccardi-Yüceer in [3].

Definition 3.1. Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. Function f is said to be:

(i) *discrete quasiconvex* if for all $x, y \in X$, $x \neq y$, it holds:

$$f(y) \leq f(x) \quad \Rightarrow \quad f(c) \leq f(x) \ \forall c \in]x, y[_Z$$

(ii) *discrete strictly quasiconvex* if for all $x, y \in X$, $x \neq y$, it holds:

$$f(y) \leq f(x) \quad \Rightarrow \quad f(c) < f(x) \ \forall c \in]x, y[_Z$$

(iii) *discrete semistrictly quasiconvex* if for all $x, y \in X$, $x \neq y$, it holds:

$$f(y) < f(x) \quad \Rightarrow \quad f(c) < f(x) \ \forall c \in]x, y[_Z$$

(iv) *discrete semi quasiconvex* if for all $x, y \in X$, $x \neq y$, it holds:

$$f(y) < f(x) \quad \Rightarrow \quad f(c) \leq f(x) \ \forall c \in]x, y[_Z$$

Clearly, a discrete quasiconvex function is also discrete semi quasiconvex, a discrete strictly quasiconvex function is also discrete quasiconvex and discrete semistrictly quasiconvex, a discrete semistrictly quasiconvex function is also discrete semi quasiconvex. It is worth also focusing on the relationships existing between discrete convexity and discrete quasiconvexity. With this aim, the following result is provided.

Theorem 3.1. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. The following properties hold:*

(i) *if f is a discrete convex function then it is also discrete semistrictly quasiconvex;*

(ii) *if f is a discrete convex function then it is also discrete quasiconvex;*

(iii) *if f is a discrete strictly convex function then it is also discrete strictly quasiconvex.*

Proof. (i) Suppose by contradiction that f is not discrete semistrictly quasiconvex, that is to say that there exist $x, y, a \in X$, with $x \neq y$ and $a \in]x, y[_Z$, such that $f(y) < f(x) \leq f(a)$. Let us assume, without loss of generality, that $y < x$; from Theorem 2.1 we get:

$$\frac{f(x) - f(a)}{x - a} \geq \frac{f(a) - f(y)}{a - y}$$

Table 1: Inclusion relationships among the classes

$$
\begin{array}{ccccc}
\boxed{\text{cx}} & \subset & \boxed{\text{qcx}} & \subset & \boxed{\text{sm.qcx}} \\
 & & \cup & & \cup \\
 & & \boxed{\text{s.qcx}} & \subset & \boxed{\text{ss.qcx}} \\
 & & \cup & & \cup \\
 & & \boxed{\text{s.cx}} & \subset & \boxed{\text{cx}}
\end{array}
$$

Since $x - a > 0$, $a - y > 0$ and $f(x) - f(a) \leq 0$, it yields that $f(a) - f(y) \leq 0$ which contradicts the discrete convexity of function f.

(ii), (iii) The proofs are analogous to the one of (i). ■

The inclusion relationships between the classes of functions defined so far are represented in Table 1.

In Examples 3.1 it is pointed out that these various classes of functions do not coincide.

Example 3.1. Let us present now some counterexamples showing that the classes of functions defined so far do not coincide.

(i) The following function $f : Z \to \Re$ is both discrete semistrictly quasiconvex and discrete semi quasiconvex but neither discrete quasiconvex nor discrete convex:

$$f(x) = \begin{cases} 0 & \text{if } x \in Z, x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

(ii) The following function $f : Z \to \Re$ is discrete quasiconvex but not discrete semistrictly quasiconvex:

$$f(x) = \begin{cases} 0 & \text{if } x \in Z, x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$

(iii) The function $f : Z \to \Re$ given by $f(x) = |x| - x$ is both discrete quasiconvex and discrete semistrictly quasiconvex but not discrete strictly quasiconvex.

(iv) The function $f : Z_{++} \to \Re$ given by $f(x) = \log(x)$ is discrete strictly quasiconvex but not discrete convex.

(v) The function $f : Z \to \Re$ given by $f(x) = x$ is discrete convex but not discrete strictly convex.

It is worth pointing out that the previously introduced discrete quasiconvexity concepts do not guarantee that their continuous extensions verify the corresponding quasiconvex property. With this aim, given a discrete function $f : Z \to \Re$, we will say that $\tilde{f} : \Re \to \Re$ is:

- an *extension* of f if $f(Z) = \tilde{f}(Z)$,
- a *linear piecewise extension* of f if $f(Z) = \tilde{f}(Z)$ and for all $x \in Z$, for all $y \in (x, x+1)$, it is $\tilde{f}(y) = f(x) + [f(x+1) - f(x)](y - x)$.

First of all consider the discrete function $f : Z \to \Re$ given by $f(x) = \left|x - \frac{1}{2}\right| - \frac{1}{2}$ which is discrete strictly quasiconvex while its corresponding linear piecewise extension is not strictly quasiconvex. Then, consider the discrete function $f : Z \to \Re$ in *i)* of Examples 3.1. Such a function is both discrete semistrictly quasiconvex and discrete semi quasiconvex while its corresponding linear piecewise extension is not generalized convex.

Nevertheless, the following properties hold for a discrete function $f : Z \to \Re$ (see [2] for real scalar semi quasiconvex functions):

- f is discrete quasiconvex if and only if its linear piecewise extension is quasiconvex;
- f is discrete semi quasiconvex [discrete semistrictly quasiconvex, discrete strictly quasiconvex] if and only if there exists at least an extension $\tilde{f} : \Re \to \Re$ of f which is semi quasiconvex [semistrictly quasiconvex, strictly quasiconvex].

Notice that a discrete semistrictly quasiconvex function could not allow continuous generalized convex extensions (see the function in *i)* of Examples 3.1). The existence of generalized convex extensions is a theoretical tool which results to be not so useful in stating optimality conditions, since it does not guarantee integer optima, while it can be used for example in order to show that the classical result by Fenchel [6] holds also for discrete quasiconvex functions.

Theorem 3.2. *Let $f : X \to Z$, where $X \subset Z$ is a discrete reticulum, let $g : f(X) \to \Re$ and consider the composite function $g(f(x))$. The following properties hold:*

(i) *if f is discrete quasiconvex and g is nondecreasing then $g(f(x))$ is also discrete quasiconvex;*

(ii) *if f is discrete semi quasiconvex, discrete semistrictly quasiconvex, discrete strictly quasiconvex, and g is increasing then $g(f(x))$ is also discrete semi quasiconvex, discrete semistrictly quasiconvex, discrete strictly quasiconvex, respectively.*

4. Characterizations of Quasiconvexity Concepts

The aim of this section is to generalize some of the results stated by Murota and Shioura in [11]. These results are related to characterizations for the various classes of generalized discrete quasiconvex functions. Notice that the following results are related to discrete functions and cannot be used in order to characterize real functions.

Theorem 4.1. *Let* $f : X \to \Re$*, where* $X \subset Z$ *is a discrete reticulum. The following conditions are equivalent:*

(i) *function* f *is discrete quasiconvex;*

(ii) *for all* $x, y \in X$*,* $x < y$*, it holds* $f(c) \leq \max\{f(x), f(y)\}$ $\forall c \in]x, y[_Z$*;*

(iii) *the following inequality holds for all* $x, y \in X$*,* $x < y$*:*

$$\min\{f(x+1), f(y-1)\} \leq \max\{f(x), f(y)\}$$

(iv) $\nexists x, y \in X$*,* $x < y$*, such that* $f(x) < f(x+1)$ *and* $f(y) < f(y-1)$*;*

(v) *for all* $x, y \in X$*,* $x < y$*, the following implication holds:*

$$f(x) < f(x+1) \;\Rightarrow\; f(y) \geq f(y-1)$$

Proof. $(i) \Leftrightarrow (ii)$ The equivalence follows straightforward from the definition.

$(ii) \Rightarrow (iii)$ The result follows being $\{x+1, y-1\} \subset]x, y[_Z$.

$(iii) \Rightarrow (iv)$ Suppose by contradiction that $\exists x, y \in X$, $x < y$, such that $f(x) < f(x+1)$ and $f(y) < f(y-1)$; define also $M = \max_{z \in [x,y]_Z}\{f(z)\}$. Then, there necessarily exists $m_1, m_2 \in [x, y]_Z$, $m_1 < m_2$, such that $M = f(m_1+1) = f(m_2-1)$, $f(m_1) < f(m_1+1)$ and $f(m_2) < f(m_2-1)$ (notice that m_1+1 and m_2-1 may coincide); as a consequence it is $M = \min\{f(m_1+1), f(m_2-1)\} > \max\{f(m_1), f(m_2)\}$ which contradicts the hypothesis for the values m_1 and m_2.

$(iv) \Rightarrow (v)$ The result is trivial.

$(v) \Rightarrow (ii)$ Suppose by contradiction that $\exists x, y \in X$, $x < y$, $\exists c \in]x, y[_Z$ such that $f(c) > \max\{f(x), f(y)\}$; define also $M = \max_{z \in [x,y]_Z}\{f(z)\} \geq f(c) > \max\{f(x), f(y)\}$. Then, there necessarily exists $m_1, m_2 \in [x, y]_Z$, $m_1 < m_2$, such that $M = f(m_1+1) = f(m_2-1)$, $f(m_1) < f(m_1+1)$ and $f(m_2) < f(m_2-1)$ (notice that m_1+1 and m_2-1 may coincide), and this contradicts the hypothesis for the values m_1 and m_2. ■

Notice that condition *ii)* of the previous theorem has been used by Murota and Shioura in [11] as the definition of discrete quasiconvexity and that in the same paper also condition *iii)* has been given. Notice also that condition *v)* implicitly handles a sort of monotonicity property of discrete generalized convex functions. Finally, it is worth pointing out that condition *v)* is the most useful in order to concretely verify the discrete quasiconvexity of a function. Analogous results can be proved similarly also for the other classes of generalized discrete convex functions.

Theorem 4.2. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. The following conditions are equivalent:*

(i) *function f is discrete strictly quasiconvex;*

(ii) *for all $x, y \in X$, $x < y$, it holds $f(c) < \max\{f(x), f(y)\}\ \forall c \in]x, y[_Z$;*

(iii) *the following inequality holds for all $x, y \in X$, $x < y$:*

$$\min\{f(x+1), f(y-1)\} < \max\{f(x), f(y)\}$$

(iv) *$\nexists x, y \in X$, $x < y$, such that $f(x) \leq f(x+1)$ and $f(y) \leq f(y-1)$;*

(v) *for all $x, y \in X$, $x < y$, the following implication holds:*

$$f(x) \leq f(x+1) \ \Rightarrow \ f(y) > f(y-1)$$

Theorem 4.3. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. The following conditions are equivalent:*

(i) *function f is discrete semistrictly quasiconvex;*

(ii) *for all $x, y \in X$, $x < y$, $f(x) \neq f(y)$, it holds:*

$$f(c) < \max\{f(x), f(y)\}\ \forall c \in]x, y[_Z$$

(iii) *the following inequality holds for all $x, y \in X$, $x < y$, $f(x) \neq f(y)$:*

$$\min\{f(x+1), f(y-1)\} < \max\{f(x), f(y)\}$$

(iv) *$\nexists x, y \in X$, $x < y$, $f(x) \neq f(y)$, such that $f(x) \leq f(x+1)$ and $f(y) \leq f(y-1)$;*

(v) *for all $x, y \in X$, $x < y$, $f(x) \neq f(y)$, the following implication holds:*

$$f(x) \leq f(x+1) \ \Rightarrow \ f(y) > f(y-1)$$

Theorem 4.4. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. The following conditions are equivalent:*

(i) *function f is discrete semi quasiconvex;*

(ii) *for all $x, y \in X$, $x < y$, $f(x) \neq f(y)$, it holds:*

$$f(c) \leq \max\{f(x), f(y)\}\ \forall c \in]x, y[_Z$$

(iii) *the following inequality holds for all $x, y \in X$, $x < y$, $f(x) \neq f(y)$:*

$$\min\{f(x+1), f(y-1)\} \leq \max\{f(x), f(y)\}$$

(iv) *$\nexists x, y \in X$, $x < y$, $f(x) \neq f(y)$, such that $f(x) < f(x+1)$ and $f(y) < f(y-1)$;*

(v) *for all $x, y \in X$, $x < y$, $f(x) \neq f(y)$, the following implication holds:*

$$f(x) < f(x+1) \;\Rightarrow\; f(y) \geq f(y-1)$$

Notice that the definition proposed in this paper for the discrete semistrictly quasiconvex functions is weaker than the one proposed by Murota and Shioura in [11], where a discrete semistrictly quasiconvex function is requested to be also discrete quasiconvex. This is pointed out in the following result.

Theorem 4.5. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. The following conditions are equivalent:*

(i) *function f is both discrete quasiconvex and discrete semistrictly quasiconvex*

(ii) *for all $x, y \in X$, $x < y$, both the two following implications hold:*

$$f(x) = f(y) \quad \Rightarrow \quad f(c) \leq f(x) \;\forall c \in]x, y[_Z$$

$$f(x) \neq f(y) \quad \Rightarrow \quad f(c) < \max\{f(x), f(y)\} \;\forall c \in]x, y[_Z$$

(iii) *for all $x, y \in X$, $x < y$, both the two following implications hold:*

$$f(x) = f(y) \quad \Rightarrow \quad \min\{f(x+1), f(y-1)\} \leq f(x)$$

$$f(x) \neq f(y) \quad \Rightarrow \quad \min\{f(x+1), f(y-1)\} < \max\{f(x), f(y)\}$$

(iv) *$\nexists x, y \in X$, $x < y$, such that either $f(x) \leq f(x+1)$ and $f(y) < f(y-1)$ or $f(x) < f(x+1)$ and $f(y) \leq f(y-1)$*

(v) *for all $x, y \in X$, $x < y$, both the two following implications hold:*

$$f(x) \leq f(x+1) \quad \Rightarrow \quad f(y) \geq f(y-1)$$

$$f(x) < f(x+1) \quad \Rightarrow \quad f(y) > f(y-1)$$

Proof. $(i) \Leftrightarrow (ii)$ The equivalence follows straightforward from the definitions.

$(ii) \Rightarrow (iii)$ The result follows being $\{x+1, y-1\} \subset]x, y[_Z$.

$(iii) \Rightarrow (iv)$ Suppose by contradiction that $\exists x, y \in X$, $x < y$, such that either $f(x) \leq f(x+1)$ and $f(y) < f(y-1)$ or $f(y) \leq f(y-1)$ and $f(x) < f(x+1)$; define also $M = \max_{z \in [x,y]_Z}\{f(z)\}$.

In the case $f(x) = f(y)$, there exists $m_1, m_2 \in [x, y]_Z$, $m_1 < m_2$, such that $M = f(m_1 + 1) = f(m_2 - 1) > f(x) = f(y)$, $f(m_1) < f(m_1 + 1)$ and $f(m_2) < f(m_2 - 1)$ (notice that $m_1 + 1$ and $m_2 - 1$ may coincide), as a consequence $M = \min\{f(m_1 + 1), f(m_2 - 1)\} > \max\{f(m_1), f(m_2)\}$, and this contradicts the first assumption for the values m_1 and m_2.

In the case $f(x) \neq f(y)$, there exists $m_1, m_2 \in [x, y]_Z$, $m_1 < m_2$, such that $M = f(m_1 + 1) = f(m_2 - 1)$ and such that either $f(m_1) < f(m_1 + 1)$ and $f(m_2) \leq f(m_2 - 1)$ or $f(m_1) \leq f(m_1 + 1)$ and $f(m_2) < f(m_2 - 1)$ (the two cases depends on the point x or y which provides the maximum value between $f(x)$ and $f(y)$); if $f(m_1) = f(m_2)$ it results $M = \min\{f(m_1 + 1), f(m_2 - 1)\} > \max\{f(m_1), f(m_2)\}$ which contradicts the first assumption for the values m_1 and m_2. Otherwise, if $f(m_1) \neq f(m_2)$ then we have $M = \min\{f(m_1 + 1), f(m_2 - 1)\} \geq \max\{f(m_1), f(m_2)\}$ which contradicts the second assumption for the values m_1 and m_2.

$(iv) \Rightarrow (v)$ The result is trivial.

$(v) \Rightarrow (ii)$ Suppose by contradiction that $\exists x, y, c \in X$, $x < c < y$, such that either $f(c) > f(x) = f(y)$ or $f(x) \neq f(y)$ and $f(c) \geq \max\{f(x), f(y)\}$; define also $M = \max_{z \in [x,y]_Z}\{f(z)\}$.

In the case $f(c) > f(x) = f(y)$, there exists $m_1, m_2 \in [x, y]_Z$, $m_1 < m_2$, such that $M = f(m_1 + 1) = f(m_2 - 1)$, $f(m_1) < f(m_1 + 1)$ and $f(m_2) < f(m_2 - 1)$ (notice that $m_1 + 1$ and $m_2 - 1$ may coincide) and this contradicts the assumptions.

In the case $f(x) \neq f(y)$ with $f(c) \geq \max\{f(x), f(y)\}$, there exists $m_1, m_2 \in [x, y]_Z$, $m_1 < m_2$, such that $M = f(m_1 + 1) = f(m_2 - 1)$ and such that either $f(m_1) < f(m_1 + 1)$ and $f(m_2) \leq f(m_2 - 1)$ or $f(m_1) \leq f(m_1 + 1)$ and $f(m_2) < f(m_2 - 1)$ (the two cases depends on the point x or y which provides the maximum value between $f(x)$ and $f(y)$); in both the cases the assumptions are contradicted. ■

Notice that there exist functions which are both discrete quasiconvex and discrete semistrictly quasiconvex but which are not discrete strictly quasiconvex, as it is pointed out in *iii)* of Example 3.1.

Finally, it is worth providing the following further pointwise characterizations of discrete quasiconvex and of discrete strictly quasiconvex functions.

Theorem 4.6. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum. The following properties hold:*

(i) *function f is discrete quasiconvex if and only if the following logical implication holds for all $x, y \in X$, $x \neq y$:*

$$f(y) \leq f(x) \quad \Rightarrow \quad f\left(x + \frac{y - x}{|y - x|}\right) \leq f(x) \tag{9}$$

(ii) *function f is discrete strictly quasiconvex if and only if the following logical implication holds for all $x, y \in X$, $x \neq y$:*

$$f(y) \leq f(x) \quad \Rightarrow \quad f\left(x + \frac{y-x}{|y-x|}\right) < f(x) \tag{10}$$

Proof.

(i) If function f is discrete quasiconvex then (9) holds trivially since $\left(x + \frac{y-x}{|y-x|}\right) \in]x, y[_Z$. Assume now that (9) holds and suppose by contradiction that f is not discrete quasiconvex, that is to say that there exist $x, y, c \in X$, $x \neq y$ and $c \in]x, y[_Z$, such that $f(y) \leq f(x) < f(c)$. Let M be the maximum value of function f over the finite set $[x, y]_Z$ and notice that it results $M > f(x) \geq f(y)$. As a consequence, it is possible to determine $m_1, m_2 \in [x, y]_Z$, with $m_1 < m_2$, such that $f(m_1) < f(m_1+1) = M$ and $f(m_2) < f(m_2-1) = M$ (notice that $m_1 + 1$ and $m_2 - 1$ may coincide). It yields that (9) is not verified for the couple of points m_1 and m_2, and this contradicts the assumptions.

(ii) The proof is analogous to the previous one. ∎

Remark 4.1. It is worth noticing that in conditions (9) and (10) it is fundamental to compare points x and y such that $f(y) = f(x)$. In other words, conditions of the kind

$$f(y) < f(x) \quad \Rightarrow \quad f\left(x + \frac{y-x}{|y-x|}\right) < f(x) \tag{11}$$

do not guarantee function f to be discrete semi quasiconvex, as it is shown by the following function $f : [-2, 2]_Z \to \Re$:

$$f(x) = \begin{cases} |x| & \text{if } x \in [-2, 2]_Z, \ x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$$

This function verifies (11) for all $x, y \in [-2, 2]_Z$, $x \neq y$, but it is not discrete semi quasiconvex.

5. Optimality Properties

The aim of this section is to point out the usefulness in optimization of the quasiconvexity concepts introduced in Section 3.

In this light, it is worth pointing out that the optimality properties of discrete functions do not coincide with the ones verified by functions defined over convex sets. For example, consider the discrete function $f : Z \to \Re$ given by $f(x) = \left|x - \frac{1}{2}\right| - \frac{1}{2}$ which is discrete strictly quasiconvex. This function admits two different global minima, that are $x_1 = 0$ and $x_2 = 1$. In other words, discrete strictly quasiconvex functions allows more than one global minimum while it is very well known that for real strictly quasiconvex functions the minimum, if it exists, is unique. This behavior is described in the following result which provides a sort of "convexity property" for the set of global minima.

Theorem 5.1. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum and let $S \subseteq X$, $S \neq \emptyset$, be the set of global minima for f over X. The following properties hold:*

(i) *if f is discrete quasiconvex then S is a discrete reticulum;*

(ii) *if f is discrete strictly quasiconvex then S is a discrete reticulum having no more than two elements.*

Proof. *(i)* It follows directly from the definition.

(ii) From *(i)* S results to be a discrete reticulum. Suppose by contradiction that S has at least three elements and let $x = \min\{S\}$ and $y = \max\{S\}$, then the discrete strict quasiconvexity of f implies that $f(c) < f(x)\ \forall c \in]x, y[_Z$, which contradicts the global minimality of x and y. ■

Some results useful for determining minimum points are stated in the following theorem.

Theorem 5.2. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum, and let $x_0 \in X$ such that $x_0 - 1 \in X$ and $x_0 + 1 \in X$. The following properties hold:*

(i) *if f is discrete semi quasiconvex then:*

$$f(x_0) < f(x_0+1) \quad \Rightarrow \quad f(x_0) \leq f(x) \ \ \forall x \in X,\ x > x_0+1;$$
$$f(x_0) < f(x_0-1) \quad \Rightarrow \quad f(x_0) \leq f(x) \ \ \forall x \in X,\ x < x_0-1;$$

(ii) *if f is discrete quasiconvex then:*

$$f(x_0) < f(x_0+1) \quad \Rightarrow \quad f(x_0) < f(x) \ \ \forall x \in X,\ x > x_0+1;$$
$$f(x_0) < f(x_0-1) \quad \Rightarrow \quad f(x_0) < f(x) \ \ \forall x \in X,\ x < x_0-1;$$

(iii) *if f is discrete strictly quasiconvex then:*

$$f(x_0) \leq f(x_0+1) \quad \Rightarrow \quad f(x_0) < f(x) \ \ \forall x \in X,\ x > x_0+1;$$
$$f(x_0) \leq f(x_0-1) \quad \Rightarrow \quad f(x_0) < f(x) \ \ \forall x \in X,\ x < x_0-1;$$
$$f(x_0) = f(x_0+1) \quad \Rightarrow \quad f(x_0) < f(x) \ \ \forall x \in X \setminus \{x_0, x_0+1\};$$

(iv) *if f is discrete semistrictly quasiconvex then:*

$$f(x_0) \le f(x_0+1) \quad \Rightarrow \quad f(x_0) \le f(x) \ \ \forall x \in X,\ x > x_0+1;$$

$$f(x_0) \le f(x_0-1) \quad \Rightarrow \quad f(x_0) \le f(x) \ \ \forall x \in X,\ x < x_0-1;$$

$$f(x_0) = f(x_0+1) \quad \Rightarrow \quad f(x_0) \le f(x) \ \ \forall x \in X.$$

Proof.

(i) Assume by contradiction that there exists $y \in X$, $y \ge x_0$, such that $f(y) < f(x_0) < f(x_0+1)$; hence $y > x_0+1$, so that for the discrete semi quasiconvexity of f it is $f(c) \le f(x_0)$ $\forall c \in]x_0, y[_Z$ and this is a contradiction since $x_0+1 \in]x_0, y[_Z$ and $f(x_0+1) > f(x_0)$. The proof of the second implication is analogous.

(ii) The proofs for (ii), (iii)) and (iv) are analogous. ∎

Notice that these results imply the global optimality of local optima.

Corollary 5.1. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum, and let $x_0 \in X$. If one of the following conditions hold:*

(i) *function f is discrete semistrictly quasiconvex and*
$f(x_0) \le f(x)$ $\forall x \in \{x_0-1, x_0+1\} \cap X$*;*

(ii) *function f is discrete semi quasiconvex and*
$f(x_0) < f(x)$ $\forall x \in \{x_0-1, x_0+1\} \cap X$*;*
then, x_0 is a global minimum for f over X. Furthermore, if the following condition holds:

(iii) *function f is discrete quasiconvex and*
$f(x_0) < f(x)$ $\forall x \in \{x_0-1, x_0+1\} \cap X$*;*
then, x_0 is the unique global minimum for f over X.

Corollary 5.2. *Let $f : X \to \Re$, where $X \subset Z$ is a discrete reticulum, and let $x_0 \in X$ such that $x_0+1 \in X$ and $f(x_0) = f(x_0+1)$. The following properties hold:*

(i) *if function f is discrete semistrictly quasiconvex then x_0 and x_0+1 are global minima for f over X;*

(ii) *if function f is discrete strictly quasiconvex then x_0 and x_0+1 are the only global minima for f over X.*

Theorem 5.2 allows to propose the following algorithm for determining a global minimum of a discrete semistrictly quasiconvex function over a bounded discrete reticulum $[m, M]_Z$.

***Procedure* MinDiscrConv**(*inputs*: f, m, M ; *output*: ris)

Let $a := m$ and $b := M$;

while $a < b$ *do*

let $c := \left\lfloor \frac{a+b}{2} \right\rfloor$;

if $f(c+1) < f(c)$ *then* $a := c+1$

elseif $f(c+1) > f(c)$ *then* $b := c$

else $a := c$ and $b := c$

end if;

end while;

$ris := a$;

end proc.

It worth noticing that the proposed algorithm has a logarithmic complexity since in every iteration the current interval is divided into two equally long subintervals. In other words, it can be easily seen that after n iterations we have $(b-a) \approx \left(\frac{1}{2}\right)^n (M-m)$, the solution is then found when $(b-a) < 1$ and this happens for:

$$n \simeq \left\lceil \frac{\log(M-m)}{\log(2)} \right\rceil$$

Notice that in every iteration function f has to be evaluated twice, that is in c and $c+1$.

In order to reduce the total number of evaluations of function f we propose the following further algorithm based on the Golden Section method.

***Procedure* MinDiscrGolden**(*inputs*: f, m, M ; *output*: ris)

Let $a := m$ and $b := M$;

if $b - a > 2$ *then*

let $R := \frac{\sqrt{5}-1}{2}$, $\delta := \lceil R(b-a) \rceil$;

let $\alpha := b - \delta$, $\beta := \max\{a+\delta, \alpha+1\}$, $f_\alpha := f(\alpha)$, $f_\beta := f(\beta)$;

while $b - a > 2$ *do*

if $f_\alpha < f_\beta$ *then*

$b := \beta$, $\beta := \alpha$, $f_\beta := f_\alpha$;

$\delta := \lceil R(b-a) \rceil$, $\alpha := \min\{b-\delta, \beta-1\}$, $f_\alpha := f(\alpha)$;

else-if $f_\alpha > f_\beta$ *then*

$a := \alpha$, $\alpha := \beta$, $f_\alpha := f_\beta$;

$\delta := \lceil R(b-a) \rceil$, $\beta := \max\{a+\delta, \alpha+1\}$, $f_\beta := f(\beta)$;

else

$a := \alpha$, $b := \beta$, $\delta := \lceil R(b-a) \rceil$;

$\alpha := b - \delta$, $\beta := \max\{a + \delta, \alpha + 1\}$, $f_\alpha := f(\alpha)$, $f_\beta := f(\beta)$;

end if;

end do;

end if;

$ris := \arg \min_{x \in [a,b] \cap Z} \{f(x)\}$;

end proc.

It can be easily seen that after n iterations it is $(b-a) \approx \left(\frac{\sqrt{5}-1}{2}\right)^n (M-m)$; the algorithm leaves the while cycle when $(b-a) \leq 2$ and this happens for:

$$n \simeq \left\lceil \frac{\log(M-m)}{\log\left(\frac{\sqrt{5}+1}{2}\right)} \right\rceil$$

Notice that in every iteration function f has to be evaluated just once, that is in either α or β; as a consequence, the total number of evaluated points is smaller than the ones used in the bisection method even if the total number of iterations is greater (notice that $\frac{1}{2}\log(2) < \log\left(\frac{\sqrt{5}+1}{2}\right) < \log(2)$).

6. Conclusions

In this paper discrete convexity and discrete quasiconvexity concepts for single variable discrete functions have been proposed and studied in an unified framework. Their usefulness in optimization has been pointed out from both a theoretical and an algorithmic point of view. Some of the results in [3, 11] have been generalized. The applicative use of these concepts in Operations Research and Management Science suggests to deep on this research topic for both single variable and multi variables functions.

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