TAIWANESE JOURNAL OF MATHEMATICS
Vol. 13, No. 2B, pp. 753-755, April 2009
This paper is available online at http://www.tjm.nsysu.edu.tw/

WHAT IS INVEXITY WITH RESPECT TO THE SAME η?

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Abstract. Many papers on both scalar and multiobjective optimization problems use the assumption that the objective and constraint functions are invex with respect to the same function η. In this note we characterize the finite families of functions for which this condition holds.

1. Introduction

One of the most frequently used generalized convexity notions is the concept of invexity:

Definition 1. [6]. A differentiable function f defined on an open subset X of $\mathbb{R}^n$ is invex if there exists a vector function $\eta : X \times X \to \mathbb{R}^n$ such that

$$f(y) \geq f(x) + \langle \nabla f(x), \eta(y,x) \rangle \qquad (x, y \in X).$$

This notion was introduced in order to provide a sufficient condition for Kuhn-Tucker points of nonlinear programming problems to be optimal. Some time later the following simple characterization of invexity clarified the essence of this notion:

Theorem 2. [3]. *A differentiable function f defined on an open subset X of $\mathbb{R}^n$ is invex if and only if every stationary point is a global minimum.*

In both scalar and vector constrained programming problems, it is usually required that all functions involved are invex with respect to the same function η (see, e.g., [8, 7, 1, 4, 2]). However, the problem of finding a characterization of those finite families of functions that are invex with respect to a common function η has apparently received no attention. This note provides such a characterization, which in fact follows from Gale's theorem of the alternative for linear inequalities in a rather straightforward way.

Received December 15, 2008.
2000 *Mathematics Subject Classification*: 26B25, 90C26.
Key words and phrases: Invex function, Gale's theorem of the alternative.
This research has been supported by the Ministerio de Ciencia y Tecnología, Project MTM2008-06695-C03-03/MTM, the Barcelona GSE Research Network and the Generalitat de Catalunya.

Theorem 3. [5]. *(Gale's theorem of the alternative for linear inequalities). For a given $m \times n$ matrix A and a given coumn vector $b \in \mathbb{R}^m$, either*

$$\text{the system } Ax \leq b \text{ has a solution } x \in \mathbb{R}^n$$

or

$$\text{the system } A^T\lambda = 0,\ \ \langle b, \lambda \rangle = -1 \text{ has a solution } \lambda \geq 0,$$

but never both.

Thus, according to Gale's theorem of the alternative, if a linear inequality system

$$\langle a_i, x \rangle \leq b_i \qquad (i = 1, , , m)$$

(with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$) has no solution x then there exist $\lambda_i \geq 0$ $(i = 1, ..., m)$ such that $\sum_{i=1}^{m} \lambda_i a_i = 0$ and $\sum_{i=1}^{m} \lambda_i b_i = -1$.

The next theorem characterizes invexity with respect to a common function η.

Theorem 4. *Let $f_1, ..., f_p$ be differentiable functions defined on an open subset X of $\mathbb{R}^n$. The following statements are equivalent:*

- (i) *The functions $f_1, ..., f_p$ are invex with respect to the same η.*
- (ii) *The functions $\sum_{i=1}^{p} \lambda_i f_i$ $\left((\lambda_1, ..., \lambda_p) \in \mathbb{R}_+^p\right)$ are invex with respect to the same η.*
- (iii) *The functions $\sum_{i=1}^{p} \lambda_i f_i$ $\left((\lambda_1, ..., \lambda_p) \in \mathbb{R}_+^p\right)$ are invex.*
- (iv) *For every $(\lambda_1, ..., \lambda_p) \in \mathbb{R}_+^p$, every stationary point of $\sum_{i=1}^{p} \lambda_i f_i$ is a global minimum.*

Proof. Implications *(i)* $\Longrightarrow$ *(ii)* $\Longrightarrow$ *(iii)* $\Longrightarrow$ *(iv)* are obvious, so we only have to prove implication *(iv)* $\Longrightarrow$ *(i)*. To this aim, assume, by contradiction, that there is no function $\eta : X \times X \to \mathbb{R}^n$ such that

$$f_i(y) \geq f_i(x) + \langle \nabla f_i(x), \eta(y, x) \rangle \qquad (x, y \in X;\ i = 1, , , p).$$

In other words, there exist $x, y \in X$ such that the linear inequality system

$$\langle \nabla f_i(x), \eta(y, x) \rangle \leq f_i(y) - f_i(x) \qquad (i = 1, , , p)$$

in the unknown vector $\eta(y, x)$ has no solution. Hence, by Thm. 3, there is $(\lambda_1, ..., \lambda_p) \in \mathbb{R}_+^p$ such that $\sum_{i=1}^{p} \lambda_i \nabla f_i(x) = 0$ and $\sum_{i=1}^{p} \lambda_i (f_i(y) - f_i(x)) = -1$. Therefore $\sum_{i=1}^{p} \lambda_i f_i$ has a stationary point x which is not a global minimum, since $\sum_{i=1}^{p} \lambda_i f_i(y) = \sum_{i=1}^{p} \lambda_i f_i(x) - 1 < \sum_{i=1}^{p} \lambda_i f_i(x)$. This contradicts (iv). ∎

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