

**CONVERGENCE OF THE FAMILY OF EULER-HALLEY TYPE
METHODS ON RIEMANNIAN MANIFOLDS
UNDER THE γ -CONDITION**

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Abstract. A convergence criterion of the family of Euler-Halley type methods for the vector fields on Riemannian manifolds whose covariant derivatives satisfy the γ -condition is established. The corresponding results due to [12] are extended. An application to analytic vector fields is provided.

1. INTRODUCTION

Numerical problems posed on manifolds arise in many natural contexts. Classical examples are given by eigenvalue problems, symmetric eigenvalue problems, invariant subspace computations, optimization problems with equality constraints, etc, see for example [8, 10, 21, 22, 23]. For such problems, one often has to compute solutions of a system of equations or to find zeros of a vector field on a Riemannian manifold. One of the most famous methods to approximately solve these problems is Newton's method. An analogue of the well known Kantorovich theorem [15, 16] was given in [9] for Newton's method on Riemannian manifolds while the extensions of the famous Smale's α -theory and γ -theory in [20] to analytic vector fields on Riemannian manifolds were done in [6]. To extend and improve the Smale's γ -theory and α -theory of Newton's method for operators in Banach spaces, Wang proposed in [25, 26] the notion of the γ -condition, which is weaker than the Smale's assumption in [20] for analytic operators. In the recent paper [19], we extended the notion of the γ -condition to vector fields on Riemannian manifolds and

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then established the γ -theory and α -theory of Newton's method for the vector fields on Riemannian manifolds satisfying the γ -condition, which consequently extend the results in [6]. The radii of uniqueness balls of zeroes of vector fields satisfying the γ -conditions were studied in [24]. Other extensions about local behavior of Newton's method on Riemannian manifolds have been studied in paper [18], where we estimated the radii of convergence balls of Newton's method and uniqueness balls of zeroes of vector fields on Riemannian manifolds under the assumption that the covariant derivatives of the vector fields satisfy some kind of general Lipschitz condition.

As is well known, there are several kinds of cubic generalizations for Newton's method. The most important two are the Euler method and the Halley method, see e.g. [1, 4, 5, 13, 14, 28]. Another more general family of the cubic extensions is the family of Euler-Halley type methods in Banach spaces, which includes the Euler method and the Halley method as its special cases and has been studied extensively in [11, 12, 27]. In particular, Han established in [12] the cubic convergence of this family for operators satisfy the γ -condition. The purpose of the present paper is to extend the family of Euler-Halley type methods to vector fields on Riemannian manifolds and study the cubic convergence of this family for vector fields whose covariant derivatives satisfying the γ -condition. The convergence criterion of the family of Euler-Halley type methods for the vector fields on Riemannian manifolds whose covariant derivatives satisfy the γ -condition is established in Section 3, and an application to analytic vector fields is provided in Section 4.

2. NOTIONS AND PRELIMINARIES

Throughout this paper, M denotes a real complete connected m -dimensional Riemannian manifold. Let $p \in M$ and let $T_p M$ denote the tangent space at p to M . Let $\langle \cdot, \cdot \rangle_p$ be the scalar product on $T_p M$ with the associated norm $\| \cdot \|_p$. The subscript p is usually deleted whenever there is no possibility of confusion. For any two distinct elements $q, p \in M$, let $c : [0, 1] \rightarrow M$ be a piecewise smooth curve connecting q and p . Then the arc length of c is defined by $l(c) := \int_0^1 \| c'(t) \| dt$, and the Riemannian distance from q to p by $d(q, p) := \inf_c l(c)$, where the infimum is taken over all piecewise smooth curves $c : [0, 1] \rightarrow M$ connecting q and p . Thus (M, d) is a complete metric space by the Hopf-Rinow Theorem (cf. [3, 7, 17]).

For a finitely dimensional space or Riemannian manifold Z , let $\mathbf{B}_Z(p, r)$ and $\overline{\mathbf{B}}_Z(p, r)$ denote respectively the open metric ball and the closed metric ball at p with radius r , that is,

$$\begin{aligned} \mathbf{B}_Z(p, r) &= \{q \in Z : d(p, q) < r\}, \\ \overline{\mathbf{B}}_Z(p, r) &= \{q \in Z : d(p, q) \leq r\}. \end{aligned}$$

In particular, we write respectively $\mathbf{B}(p, r)$ and $\overline{\mathbf{B}(p, r)}$ for $\mathbf{B}_M(p, r)$ and $\overline{\mathbf{B}_M(p, r)}$ in the case when M is a Riemannian manifold.

Noting that M is complete, the exponential map at p , $\exp_p : T_pM \rightarrow M$ is well-defined on T_pM . Recall that a geodesic in M connecting q and p is called a minimizing geodesic if its arc length equals its Riemannian distance between q and p . Note that there is at least one minimizing geodesic connecting q and p . In particular, the curve $c : [0, 1] \rightarrow M$ connecting q and p is a minimizing geodesic if and only if there exists a vector $v \in T_qM$ such that $\|v\| = d(q, p)$ and $c(t) = \exp_q(tv)$ for each $t \in [0, 1]$.

Let ∇ denote the Levi-Civita connection on M . For any two vector fields X and Y on M , the covariant derivative of X with respect to Y is denoted by $\nabla_Y X$. Define the linear map $DX(p) : T_pM \rightarrow T_pM$ by

$$DX(p)(u) = \nabla_Y X(p), \quad \forall u \in T_pM, \quad (2.1)$$

where Y is a vector field satisfying $Y(p) = u$. Then the value $DX(p)(u)$ of $DX(p)$ at u depends only on the tangent vector $u = Y(p) \in T_pM$ since ∇ is tensorial in Y . Let $c : \mathbb{R} \rightarrow M$ be a C^∞ curve and let $P_{c, \cdot, \cdot}$ denote the parallel transport along c , which is defined by

$$P_{c, c(b), c(a)}(v) = V(c(b)), \quad \forall a, b \in \mathbb{R} \text{ and } v \in T_{c(a)}M,$$

where V is the unique C^∞ vector field satisfying $\nabla_{c'(t)}V = 0$ and $V(c(a)) = v$. Then, for any $a, b \in \mathbb{R}$, $P_{c, c(b), c(a)}$ is an isometry from $T_{c(a)}M$ to $T_{c(b)}M$. Note that, for any $a, b, b_1, b_2 \in \mathbb{R}$,

$$P_{c, c(b_2), c(b_1)} \circ P_{c, c(b_1), c(a)} = P_{c, c(b_2), c(a)} \quad \text{and} \quad P_{c, c(b), c(a)}^{-1} = P_{c, c(a), c(b)}.$$

In particular, we write $P_{q, p}$ for $P_{c, q, p}$ in the case when c is a minimizing geodesic connecting p and q . Moreover, for a positive integer i , $P_{p, q}^i$ stands for the map from $(T_qM)^i$ to $(T_pM)^i$ defined by

$$P_{p, q}^i(v_1 \cdots v_i) = P_{p, q}v_1 \cdots P_{p, q}v_i, \quad \forall (v_1, \cdots, v_i) \in (T_qM)^i.$$

Let $\kappa \in \mathbb{N} \cup \{\infty, \omega\}$. We use $C^\kappa(TM)$ to denote the set of all the C^κ -vector fields of M . In the particular cases when $\kappa = \infty$, or ω , a C^κ -vector field X is called a smooth vector field or an analytic vector field, respectively.

Let j be a positive integer and let X be a C^κ -vector field. We now define inductively the covariant derivative of order j for X (cf. [7, P.102]). Recall that ∇ is the Levi-Civita connection on M . Define the map $DX : C^\kappa(TM) \rightarrow C^{\kappa-1}(TM)$ by

$$DX(Y) = \nabla_Y X \quad \text{for each } Y \in C^\kappa(TM),$$

and define the map $D^j X : (C^\kappa(TM))^j \rightarrow C^{\kappa-j}(TM)$ by

$$D^j X(Y_1, \dots, Y_{j-1}, Y) = \nabla_Y(D^{j-1}X(Y_1, \dots, Y_{j-1})) - \sum_{i=1}^{j-1} D^{j-1}X(Y_1, \dots, \nabla_Y Y_i, \dots, Y_{j-1}) \quad (2.2)$$

for each $Y_1, \dots, Y_{j-1}, Y \in C^\kappa(TM)$. Then, one can use mathematical induction to prove easily that $D^j X(Y_1, \dots, Y_{j-1}, Y)$ is tensorial with respect to each component, that is, j multi-linear map from $(C^\kappa(TM))^j$ to $C^{\kappa-j}(TM)$, where the linearity refers to the structure of $C^j(M)$ -module with $C^j(M)$ the set of all C^j -mappings from M to \mathbb{R} . This implies that the value of $D^j X(Y_1, \dots, Y_{j-1}, Y)$ at $p \in M$ only depends on the j -tuple of tangent vectors $(v_1, \dots, v_j) = (Y_1(p), \dots, Y_{j-1}(p), Y(p)) \in (T_p M)^j$. Consequently, for a given $p \in M$, the map $D^j X(p) : (T_p M)^j \rightarrow T_p M$, defined by

$$D^j X(p)v_1 \cdots v_j := D^j X(Y_1, \dots, Y_j)(p) \quad \text{for any } (v_1, \dots, v_j) \in (T_p M)^j,$$

is well-defined, where $Y_i \in C^\kappa(TM)$ satisfy $Y_i(p) = v_i$ for each $i = 1, \dots, j$. Let $p_0 \in M$ be such that $DX(p_0)^{-1}$ exists. Thus, for any piecewise geodesic curve c connecting p_0 and p , $DX(p_0)^{-1}P_{c,p_0,p}D^j X(p)$ is a j -multilinear map from $(T_p M)^j$ to $T_{p_0} M$. We define the norm of $DX(p_0)^{-1}P_{c,p_0,p}D^j X(p)$ by

$$\|DX(p_0)^{-1}P_{c,p_0,p}D^j X(p)\| = \sup \|DX(p_0)^{-1}P_{c,p_0,p}D^j X(p)v_1 v_2 \cdots v_j\|_{p_0},$$

where the supremum is taken over all j -tuple of vectors $(v_1, \dots, v_j) \in (T_p M)^j$ with each $\|v_i\|_p = 1$. Furthermore, for any geodesic $c : \mathbb{R} \rightarrow M$ on M , since $\nabla_{c'(s)} c'(s) = 0$, it follows from (2.2) that

$$D^k X(c(s))(c'(s))^k = D_{c'(s)}(D^{k-1}X(c(s))(c'(s))^{k-1}) \quad \text{for each } s \in \mathbb{R}.$$

Let X be a C^2 vector field on M and let $p_0 \in M$. The family of Euler-Halley iterations with parameter $\lambda \in [0, 2]$ for solving $X(p) = 0$ with initial point p_0 is defined by

$$p_{n+1} = T_{X,\lambda}(p_n) = \exp_{p_n}(u_X(p_n) + v_{X,\lambda}(p_n)), \quad n = 0, 1, 2, \dots, \quad (2.3)$$

where

$$\begin{aligned} u_X(p) &= -DX(p)^{-1}X(p), \\ v_{X,\lambda}(p) &= -\frac{1}{2}DX(p)^{-1}D^2X(p)u_X(p)Q_{X,\lambda}(p)u_X(p), \\ Q_{X,\lambda}(p) &= \{\mathbf{I}_{T_p M} + \frac{\lambda}{2}DX(p)^{-1}D^2X(p)u_X(p)\}^{-1}, \end{aligned}$$

and $\mathbf{I}_{T_p M}$ is the identity on $T_p M$.

The γ -condition for operators in Banach spaces was first presented by Wang [25, 26] for the study of Smale's point estimate theory and extended to vector fields on Riemannian manifolds in [19]. The following definition gives an analogue of the γ -conditions of order 1 and 2 to the case of vector fields on Riemannian manifolds M . Throughout the whole paper, we always assume that X is a C^3 vector field on M .

Definition 2.3. Let $r > 0$ and $\gamma > 0$. Let $p_0 \in M$ be such that $DX(p_0)^{-1}$ exists. Then X is said to satisfy

- (i) the 2-piece γ -condition of order 1 at p_0 in $\mathbf{B}(p_0, r)$, if for any two points $p, q \in \mathbf{B}(p_0, r)$, any geodesic c_2 connecting p, q and minimizing geodesic c_1 connecting p_0, p with $l(c_1) + l(c_2) < r$,

$$\|DX(p_0)^{-1}P_{c_1,p_0,p} \circ P_{c_2,p,q}D^2X(q)\| \leq \frac{2\gamma}{(1 - \gamma(l(c_1) + l(c_2)))^3}; \tag{2.4}$$

- (ii) the 2-piece γ -condition of order 2 at p_0 in $\mathbf{B}(p_0, r)$, if for any two points $p, q \in \mathbf{B}(p_0, r)$, any geodesic c_2 connecting p, q and minimizing geodesic c_1 connecting p_0, p with $l(c_1) + l(c_2) < r$,

$$\|DX(p_0)^{-1}D^2X(p_0)\| \leq 2\gamma. \tag{2.5}$$

and

$$\|DX(p_0)^{-1}P_{c_1,p_0,p} \circ P_{c_2,p,q}D^3X(q)\| \leq \frac{6\gamma^2}{(1 - \gamma(l(c_1) + l(c_2)))^4}. \tag{2.6}$$

Note that the 2-piece γ -condition of order 1 is also called the 2-piece γ -condition in [19]. The following lemma will play a key role.

Lemma 2.1. Let $c : \mathbb{R} \rightarrow M$ be a geodesic and Y a C^k vector field on M such that $\nabla_{c'(s)}Y(c(s)) = 0$. Then, for each $k = 0, 1, 2$,

$$P_{c,c(0),c(t)}D^kX(c(t))Y(c(t))^k = D^kX(c(0))Y(c(0))^k + \int_0^t P_{c,c(0),c(s)}(D^{k+1}X(c(s))(c(s))^k c'(s))ds. \tag{2.7}$$

In particular,

$$P_{c,c(0),c(t)}D^kX(c(t))c'(t)^k = D^kX(c(0))c'(0)^k + \int_0^t P_{c,c(0),c(s)}(D^{k+1}X(c(s))(c'(s))^{k+1})ds. \tag{2.8}$$

Proof. The case when $k = 0$ results from [9, p.308]. Below, we will show that the case when $k = 1$ is true, that is,

$$\begin{aligned} P_{c,c(0),c(t)}DX(c(t))Y(c(t)) &= DX(c(0))Y(c(0)) \\ &+ \int_0^t P_{c,c(0),c(s)}(D^2X(c(s))Y(c(s))c'(s))ds, \end{aligned} \quad (2.9)$$

while the proof for the case when $k = 2$ is similar and so is omitted here. To this end, let $\xi = DX(Y)$. Since (2.7) is true for $k = 0$, it follows that

$$P_{c,c(0),c(t)}\xi(c(t)) = \xi(c(0)) + \int_0^t P_{c,c(0),c(s)}(D\xi(c(s))c'(s))ds. \quad (2.10)$$

By (2.2) (with $j = 2$), one has

$$\begin{aligned} (D^2X(c(s))Y(c(s))c'(s)) &= \nabla_{c'(s)}((DX(c(s))Y(c(s))) - DX(c(s))(\nabla_{c'(s)}Y(c(s)))) \\ &= \nabla_{c'(s)}((DX(c(s))Y(c(s)))) \\ &= \nabla_{c'(s)}\xi(c(s)) \\ &= D\xi(c(s))c'(s) \end{aligned}$$

thanks to the assumption that $\nabla_{c'(s)}Y(c(s)) = 0$. This combining with (2.10) yields (2.9). The proof is complete. \blacksquare

The following proposition shows that the 2-piece γ -condition of order 2 implies the 2-piece γ -condition of order 1.

Proposition 2.2. *Let $r > 0$ and $\gamma > 0$. Let $p_0 \in M$ be such that $DX(p_0)^{-1}$ exists. Suppose that X satisfies the 2-piece γ -condition of order 2 at p_0 in $\mathbf{B}(p_0, r)$. Then X satisfies the 2-piece γ -condition of order 1 at p_0 in $\mathbf{B}(p_0, r)$.*

Proof. For any $p, q \in \mathbf{B}(p_0, r)$, let c_1 be a minimizing geodesic connecting p_0, p and c_2 a geodesic connecting p, q such that $l(c_1) + l(c_2) < r$. To complete the proof, it suffices to show that

$$\|DX(p_0)^{-1}P_{c_1,p_0,p} \circ P_{c_2,p,q}D^2X(q)\| \leq \frac{2\gamma}{(1 - \gamma(l(c_1) + l(c_2)))^3}. \quad (2.11)$$

By the assumption that X satisfies the 2-piece γ -condition of order 2 at p_0 in $\mathbf{B}(p_0, r)$, we have that

$$\|DX(p_0)^{-1}D^2X(p_0)\| \leq 2\gamma \quad (2.12)$$

and

$$\|DX(p_0)^{-1}P_{c_1,p_0,p} \circ P_{c_2,p,q}D^3X(q)\| \leq \frac{6\gamma^2}{(1 - \gamma(l(c_1) + l(c_2)))^4}. \tag{2.13}$$

Below, we claim that

$$\begin{aligned} & \|DX(p_0)^{-1}P_{c_1,p_0,p}P_{c_2,p,q}D^2X(q)P_{c_2,q,p}^2P_{c_1,p,p_0}^2 - DX(p_0)^{-1}D^2X(p_0)\| \\ & \leq \frac{2\gamma}{(1 - \gamma(l(c_1) + l(c_2)))^3} - 2\gamma. \end{aligned} \tag{2.14}$$

Granting this, by (2.12), (2.11) is seen to hold because $P_{c_2,q,p}$ and P_{c_1,p,p_0} are isometries.

To verify (2.14), let $v \in T_{p_0}M$. Let $v_1 \in T_{p_0}M$ and $v_2 \in T_pM$ be such that $c_1(t) := \exp_{p_0}(tv_1), t \in [0, 1]$, and $c_2(t) := \exp_p(tv_2), t \in [0, 1]$. Note that there exist vector fields Y_1 and Y_2 such that $Y_1(c_1(0)) = v, D_{c_1'(t)}Y_1(c_1(t)) = 0, Y_2(c_2(0)) = P_{c_1,p,p_0}v$ and $D_{c_2'(t)}Y_2(c_2(t)) = 0$. Then we apply Lemma 2.1 (with $k = 2$) to conclude that

$$\begin{aligned} & (DX(p_0)^{-1}P_{c_1,p_0,p} \circ P_{c_2,p,q}D^2X(q)P_{c_2,q,p}^2 \circ P_{p,p_0}^2 - DX(p_0)^{-1}D^2X(p_0))v^2 \\ & = DX(p_0)^{-1}P_{c_1,p_0,p}[P_{c_2,p,q}D^2X(q)Y_2(c_2(1))^2 - D^2X(p)Y_2(c_2(0))^2] \\ & \quad + DX(p_0)^{-1}[P_{c_1,p_0,p}D^2X(p)Y_1(c_1(1))^2 - D^2X(p_0)Y_1(c_1(0))^2] \\ & = DX(p_0)^{-1}P_{c_1,p_0,p} \int_0^1 P_{c_2,p,c_2(s)}D^3X(c_2(s))Y_2(c_2(s))^2c_2'(s)ds \\ & \quad + DX(p_0)^{-1} \int_0^1 P_{c_1,p_0,c_1(s)}D^3X(c_1(s))Y_1(c_1(s))^2c_1'(s)ds. \end{aligned}$$

Hence, it follows from (2.13) that

$$\begin{aligned} & \|(DX(p_0)^{-1}P_{c_1,p_0,p} \circ P_{c_2,p,q}D^2X(q)P_{c_2,q,p}^2 \circ P_{p,p_0}^2 \\ & \quad - DX(p_0)^{-1}D^2X(p_0))v^2\| \\ & \leq \int_0^1 \|DX(p_0)^{-1}P_{c_1,p_0,p}P_{c_2,p,c_2(s)}D^3X(c_2(s))\| \|Y_2(c_2(s))^2\| \|c_2'(s)\| ds \\ & \quad + \int_0^1 \|DX(p_0)^{-1}P_{c_1,p_0,c_1(s)}D^3X(c_1(s))\| \|Y_1(c_1(s))^2\| \|c_1'(s)\| ds \\ & \leq \int_0^1 \frac{6\gamma^2}{(1 - \gamma(\|v_1\| + s\|v_2\|))^4} \|v\|^2 \|v_2\| ds \\ & \quad + \int_0^1 \frac{6\gamma^2}{(1 - \gamma s\|v_1\|)^4} \|v\|^2 \|v_1\| ds \\ & = \left(\frac{2\gamma}{(1 - \gamma(l(c_1) + l(c_2)))^3} - 2\gamma \right) \|v\|^2, \end{aligned} \tag{2.15}$$

where the last equation is because $\|v_1\| = l(c_1)$ and $\|v_2\| = l(c_2)$. As $v \in T_{p_0}M$ is arbitrary, (2.14) is seen to hold. \blacksquare

Lemma 2.2. *Let $\gamma > 0$ and $0 < r \leq \frac{2-\sqrt{2}}{2\gamma}$. Let $p_0 \in M$ be such that $DX(p_0)^{-1}$ exists. Suppose that X satisfies the 2-piece γ -condition of order 2 at p_0 in $\mathbf{B}(p_0, r)$. Then,*

- (i) for each point $q \in \mathbf{B}(p_0, r)$, $DX(q)^{-1}$ exists and, for any two points $q, p \in \mathbf{B}(p_0, r)$, any geodesic c_2 connecting p, q and minimizing geodesic c_1 connecting p_0, p with $l(c_1) + l(c_2) < r$,

$$\begin{aligned} & \|DX(q)^{-1}P_{c_2,q,p} \circ P_{c_1,p,p_0}DX(p_0)\| \\ & \leq \frac{(1 - \gamma(l(c_1) + l(c_2)))^2}{1 - 4\gamma(l(c_1) + l(c_2)) + 2\gamma^2(l(c_1) + l(c_2))^2}; \end{aligned} \tag{2.16}$$

- (ii) for any two points $p, q \in \mathbf{B}(p_0, r)$, any geodesic c_2 connecting p, q and minimizing geodesic c_1 connecting p_0, p with $l(c_1) + l(c_2) < r$,

$$\begin{aligned} & \left\| DX(p_0)^{-1}P_{c_1,p_0,p}(P_{c_2,p,q}D^2X(q)P_{c_2,q,p}^2 - D^2X(p)) \right\| \\ & \leq \frac{2\gamma}{(1 - \gamma(l(c_1) + l(c_2)))^3} - \frac{2\gamma}{(1 - \gamma l(c_1))^3}. \end{aligned} \tag{2.17}$$

Proof. (i) This result follows from Proposition 2.2 and [19, Lemma 2.3].

(ii) To verify (2.17), let $v \in T_pM$. For any $p, q \in \mathbf{B}(p_0, r)$, let c_1 be a minimizing geodesic connecting p_0, p and c_2 a geodesic connecting p, q such that $l(c_1) + l(c_2) < r$. Let $v_1 \in T_pM$ be such that $c_2(t) := \exp_p(tv_1)$, $t \in [0, 1]$. Note that there exists a vector field Y such that $Y(c_2(0)) = v$ and $D_{c_2(t)}Y(c_2(t)) = 0$. Using Lemma 2.1 with $k = 2$, one gets that

$$\begin{aligned} & (DX(p_0)^{-1}P_{c_1,p_0,p}(P_{p,q}D^2X(q)P_{q,p}^2 - D^2X(p)))v^2 \\ & = DX(p_0)^{-1}P_{c_1,p_0,p}(P_{p,q}D^2X(q)Y(c_2(1))^2 - D^2X(p)Y(c_2(0))^2) \\ & = DX(p_0)^{-1}P_{c_1,p_0,p} \int_0^1 P_{c_2,p,c_2(s)}D^3X(c_2(s))Y(c_2(s))^2c_2'(s)ds. \end{aligned} \tag{2.18}$$

Thus, it follows from (2.18) and (2.6) that

$$\begin{aligned} & \|(DX(p_0)^{-1}P_{c_1,p_0,p}(P_{p,q}D^2X(q)P_{q,p}^2 - D^2X(p)))v^2\| \\ & \leq \int_0^1 \|DX(p_0)^{-1}P_{c_1,p_0,p}P_{c_2,p,c_2(s)}D^3X(c_2(s))\| \|v\|^2 \|v_1\| ds \\ & \leq \int_0^1 \frac{6\gamma^2}{(1 - \gamma(l(c_1) + s\|v_1\|))^4} \|v\|^2 \|v_1\| ds \\ & = \left(\frac{2\gamma}{(1 - \gamma(l(c_1) + l(c_2)))^3} - \frac{2\gamma}{(1 - \gamma l(c_1))^3} \right) \|v\|^2. \end{aligned} \tag{2.19}$$

As $v \in T_{p_0}M$ is arbitrary, (2.17) is seen to hold. \blacksquare

Finally we introduce the majoring function h used by Wang [25, 26] and some related properties. Let $\beta > 0$ and $\gamma > 0$. Define

$$h(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t} \quad \text{for each } 0 \leq t < \frac{1}{\gamma}. \quad (2.20)$$

Then we have the following lemma, see [25, 26].

Lemma 2.3. *Assume that $\alpha = \gamma\beta \leq 3 - 2\sqrt{2}$. Then the zeros of h are*

$$r_1 = \frac{1 + \alpha - \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma}, \quad r_2 = \frac{1 + \alpha + \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma} \quad (2.21)$$

and satisfy

$$\beta \leq r_1 \leq \left(1 + \frac{1}{\sqrt{2}}\right)\beta \leq \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma} \leq r_2 \leq \frac{1}{2\gamma}. \quad (2.22)$$

Let $\{t_n\}$ denote the sequence generated by the Euler-Halley iteration (with parameter $\lambda \in [0, 2]$) for $h(t)$ with initial point $t_0 = 0$, that is,

$$t_{n+1} = T_{h,\lambda}(t_n) = t_n + u_h(t_n) + v_{h,\lambda}(t_n), \quad n = 0, 1, 2, \dots,$$

where

$$\begin{aligned} u_h(t) &= -h'(t)^{-1}h(t) \\ v_{h,\lambda}(t) &= -\frac{1}{2}h'(t)^{-1}h''(t)u_h(t)Q_{h,\lambda}(t)u_h(t) \\ Q_{h,\lambda}(t) &= \left(1 + \frac{\lambda}{2}h'(t)^{-1}h''(t)u_h(t)\right)^{-1}. \end{aligned}$$

Then the following lemma holds from [27].

Lemma 2.4. *Suppose that $\alpha = \gamma\beta \leq 3 - 2\sqrt{2}$. Then, for each $t \in [0, r_1]$,*

- (i) $0 < H_h(t) = h'(t)^{-2}h''(t)h(t) < 1$;
- (ii) $T_{h,\lambda}(t) \in [0, r_1]$ and $T_{h,\lambda}(t)$ is monotonically increasing on $[0, r_1]$ for each $\lambda \in [0, 2]$;
- (iii) $t \leq T_{h,\lambda}(t)$.
- (iv) $\{t_n\}$ is increasing monotonically and convergent to r_1 .

3. CONVERGENCE CRITERION

Recall that h is defined by (2.20) and $t_{n+1} = T_{h,\lambda}(t_n) = t_n + u_h(t_n) + v_{h,\lambda}(t_n)$

for each $n = 0, 1, 2, \dots$, with $t_0 = 0$. The following lemma is taken from [11], see also [27].

Lemma 3.1. *For any $n = 0, 1, 2, \dots$,*

$$\begin{aligned} h(t_{n+1}) &= \frac{1}{2}h''(t_n)\{(2-\lambda)u_h(t_n) + v_{h,\lambda}(t_n)\}v_{h,\lambda}(t_n) \\ &\quad + \int_0^1 \int_0^\tau \{h''(t_n + s(t_{n+1} - t_n)) - h''(t_n)\}dsd\tau(t_{n+1} - t_n)^2. \end{aligned}$$

The similar expression for vector fields is described in the following lemma. Recall that

$$p_{n+1} = \exp_{p_n}(u_X(p_n) + v_{X,\lambda}(p_n)), \quad n = 0, 1, 2, \dots$$

Lemma 3.2. *Let n be a nonnegative integer and write*

$$w_n = u_X(p_n) + v_{X,\lambda}(p_n). \quad (3.1)$$

Let c_n be the curve defined by $c_n(t) := \exp_{p_n}(tw_n)$ for each $t \in [0, 1]$. Then

$$\begin{aligned} &P_{c_n, p_n, p_{n+1}}X(p_{n+1}) \\ &= \frac{1}{2}D^2X(p_n)\{(2-\lambda)u_X(p_n) + v_{X,\lambda}(p_n)\}v_{X,\lambda}(p_n) \\ &\quad + \int_0^1 \int_0^\tau (P_{c_n, p_n, c_n(s)}D^2X(c_n(s))P_{c_n, c_n(\tau), p_n}^2 - D^2X(p_n))w_n^2dsd\tau. \end{aligned} \quad (3.2)$$

Proof. By Lemma 2.1 with $k = 0$, we have

$$P_{c_n, p_n, p_{n+1}}X(p_{n+1}) - X(p_n) = \int_0^1 P_{c_n, p_n, c_n(\tau)}DX(c_n(\tau))c_n'(\tau)d\tau. \quad (3.3)$$

Since $c_n'(0) = w_n$, one has by Lemma 2.1 (with $k = 0, 1$) that

$$\begin{aligned} &P_{c_n, p_n, p_{n+1}}X(p_{n+1}) - X(p_n) - DX(p_n)w_n \\ &= \int_0^1 (P_{c_n, p_n, c_n(\tau)}DX(c_n(\tau))c_n'(\tau) - DX(p_n)c_n'(0))d\tau \\ &= \int_0^1 \int_0^\tau P_{c_n, p_n, c_n(s)}D^2X(c_n(s))c_n'(s)^2dsd\tau \\ &= \int_0^1 \int_0^\tau P_{c_n, p_n, c_n(s)}D^2X(c_n(s))P_{c_n, c_n(\tau), p_n}^2w_n^2dsd\tau \end{aligned} \quad (3.4)$$

Consequently,

$$\begin{aligned} & P_{c_n, p_n, p_{n+1}} X(p_{n+1}) \\ &= X(p_n) + DX(p_n)w_n + \frac{1}{2}D^2X(p_n)w_n^2 \\ & \quad + \int_0^1 \int_0^\tau (P_{c_n, p_n, c_n(s)} D^2X(c_n(s)) P_{c_n, c_n(\tau), p_n}^2 - D^2X(p_n)) w_n^2 ds d\tau. \end{aligned} \quad (3.5)$$

Below, we will show that

$$\begin{aligned} & X(p_n) + DX(p_n)w_n + \frac{1}{2}D^2X(p_n)w_n^2 \\ &= \frac{1}{2}D^2X(p_n)\{(2-\lambda)u_X(p_n) + v_{X,\lambda}(p_n)\}v_{X,\lambda}(p_n). \end{aligned} \quad (3.6)$$

Granting this, (3.2) results from (3.5) and (3.6). Since $w_n = u_X(p_n) + v_{X,\lambda}(p_n)$ and $u_X(p_n) = -DX(p_n)^{-1}X(p_n)$, to prove (3.6), it's sufficient to verify that

$$DX(p_n)v_{X,\lambda}(p_n) + \frac{1}{2}D^2X(p_n)u_X(p_n)^2 = -\frac{\lambda}{2}D^2X(p_n)u_X(p_n)v_{X,\lambda}(p_n). \quad (3.7)$$

Recalling that $v_{X,\lambda}(p_n) = -\frac{1}{2}DX(p_n)^{-1}D^2X(p_n)u_X(p_n)Q_{X,\lambda}(p_n)u_X(p_n)$ and $Q_{X,\lambda}(p_n) = \{\mathbf{I}_{T_{p_n}M} + \frac{\lambda}{2}DX(p_n)^{-1}D^2X(p_n)u_X(p_n)\}^{-1}$, one has

$$\begin{aligned} & DX(p_n)v_{X,\lambda}(p_n) + \frac{1}{2}D^2X(p_n)u_X(p_n)^2 \\ &= DX(p_n)\left(-\frac{1}{2}DX(p_n)^{-1}D^2X(p_n)u_X(p_n)Q_{X,\lambda}(p_n)u_X(p_n)\right) \\ & \quad + \frac{1}{2}D^2X(p_n)u_X(p_n)^2 \\ &= \frac{1}{2}D^2X(p_n)u_X(p_n)(\mathbf{I}_{T_{p_n}M} - Q_{X,\lambda}(p_n))u_X(p_n) \\ &= \frac{1}{2}D^2X(p_n)u_X(p_n)\frac{\lambda}{2}DX(p_n)^{-1}D^2X(p_n)u_X(p_n)Q_{X,\lambda}(p_n)u_X(p_n) \\ &= -\frac{\lambda}{2}D^2X(p_n)u_X(p_n)v_{X,\lambda}(p_n). \end{aligned}$$

Hence, (3.7) is seen to hold and the proof is completed. \blacksquare

In the remainder of this paper, we always assume that X is a C^3 vector field and that $p_0 \in M$ such that $DX(p_0)^{-1}$ exists. Furthermore, we define

$$\beta = \|DX(p_0)^{-1}X(p_0)\|, \quad \alpha = \gamma\beta.$$

Then the main theorem of the present paper is stated as follows.

Theorem 3.1. *Suppose that*

$$\alpha = \beta\gamma \leq 3 - 2\sqrt{2}$$

and X satisfies the 2-piece γ -condition of order 2 at p_0 in $\mathbf{B}(p_0, r_1)$, where r_1 is given by (2.21). Then the sequence $\{p_n\}$ generated by (2.3) with initial point p_0 is well-defined for all $\lambda \in [0, 2]$ and converges to a singular point p^* of X in $\overline{\mathbf{B}(p_0, r_1)}$. Moreover,

$$d(p^*, p_n) \leq r_1 - t_n.$$

Proof. It's sufficient to show that the sequence $\{p_n\}$ generated by (2.3) with initial point p_0 is well-defined for all $\lambda \in [0, 2]$ and satisfies

$$d(p_n, p_{n+1}) \leq \|u_X(p_n)\| + \|v_{X,\lambda}(p_n)\| \leq t_{n+1} - t_n$$

for each $n = 0, 1, \dots$. To do this, we will use mathematical induction to prove that the generated sequence $\{p_n\}$ is well-defined and the following statements hold for each $n = 0, 1, \dots$:

- (a) $\|u_X(p_n)\| \leq u_h(t_n)$;
- (b) $Q_{X,\lambda}(p_n)$ exists and $\|Q_{X,\lambda}(p_n)\| \leq Q_{h,\lambda}(t_n)$;
- (c) $\|v_{X,\lambda}(p_n)\| \leq v_{h,\lambda}(t_n)$;
- (d) $d(p_n, p_{n+1}) \leq \|w_n\| \leq \|u_X(p_n)\| + \|v_{X,\lambda}(p_n)\| \leq t_{n+1} - t_n$, where w_n is defined by (3.1).

Indeed, in the case when $n = 0$, (a) results from

$$\|u_X(p_0)\| = \|DX(p_0)^{-1}X(p_0)\| = \beta = u_h(t_0). \quad (3.8)$$

By (2.5), (3.8) and Lemma 2.4 (i), we have

$$\left\| -\frac{\lambda}{2}DX(p_0)^{-1}D^2X(p_0)u_X(p_0) \right\| \leq -\frac{\lambda}{2}h'(t_0)^{-1}h''(t_0)u_h(t_0) < 1.$$

Then, using the Banach Lemma, $Q_{X,\lambda}(p_0)$ exists and

$$\|Q_{X,\lambda}(p_0)\| \leq \frac{1}{1 + \frac{\lambda}{2}h'(t_0)^{-1}h''(t_0)u_h(t_0)} = Q_{h,\lambda}(t_0). \quad (3.9)$$

Thus, (b) and (c) follow. As $\|w_0\| \leq \|u_X(p_0)\| + \|v_{X,\lambda}(p_0)\| \leq t_1 - t_0 \leq \beta$ and $p_1 = \exp_{p_0}(w_0)$,

$$d(p_0, p_1) \leq \|w_0\| \leq \|u_X(p_0)\| + \|v_{X,\lambda}(p_0)\| \leq t_1 - t_0. \quad (3.10)$$

Therefore, (d) holds for $n = 0$. Now assume that p_1, \dots, p_{k+1} are well-defined and that (a)-(d) are true for $n = 0, 1, \dots, k$. Then,

$$d(p_k, p_{k+1}) \leq \|w_k\| \leq \|u_X(p_k)\| + \|v_{X,\lambda}(p_k)\| \leq t_{k+1} - t_k \tag{3.11}$$

and

$$d(p_0, p_{k+1}) \leq t_{k+1} < r_1. \tag{3.12}$$

Below, we will show that (a)-(d) are true for $n = k + 1$. Let c be a minimizing geodesic connecting p_0 and p_k . Define the curve c_k by $c_k(t) := \exp_{p_k}(tw_k)$, $t \in [0, 1]$. By (3.12) and Lemma 2.2 (i), $DX(p_{k+1})^{-1}$ exists and

$$\|DX(p_{k+1})^{-1}P_{c_k,p_{k+1},p_k} \circ P_{c,p_k,p_0}DX(p_0)\| \leq -h'(t_{k+1})^{-1}. \tag{3.13}$$

By Lemma 3.2, one has

$$\begin{aligned} & \|DX(p_0)^{-1}P_{c,p_0,p_k} \circ P_{c_k,p_k,p_{k+1}}X(p_{k+1})\| \\ & \leq \frac{1}{2}\|DX(p_0)^{-1}P_{c,p_0,p_k}D^2X(p_k)\|\{(2 - \lambda)\|u_X(p_k)\| \\ & \quad + \|v_{X,\lambda}(p_k)\|\}\|v_{X,\lambda}(p_k)\| \\ & \quad + \int_0^1 \int_0^\tau DX(p_0)^{-1}P_{c,p_0,p_k}(P_{c_k,p_k,c_k(s)}D^2X(c_k(s))P_{c_k,c_k(\tau),p_k}^2 \\ & \quad - D^2X(p_k))w_k^2 ds d\tau. \end{aligned} \tag{3.14}$$

By Propositions 2.2 and (2.4), one has

$$\begin{aligned} & \|DX(p_0)^{-1}P_{c,p_0,p_k} \circ P_{c_k,p_k,p_{k+1}}D^2X(p_{k+1})\| \\ & \leq \frac{2\gamma}{(1 - \gamma(l(c) + l(c_k)))^3} \leq h''(t_{k+1}) \end{aligned} \tag{3.15}$$

and

$$\|DX(p_0)^{-1}P_{c,p_0,p_k}D^2X(p_k)\| \leq \frac{2\gamma}{(1 - \gamma l(c))^3} \leq h''(t_k) \tag{3.16}$$

By induction assumptions, we have

$$\|u_X(p_k)\| \leq u_h(t_k), \quad \|v_{X,\lambda}(p_k)\| \leq v_{h,\lambda}(t_k) \quad \text{and} \quad \|w_k\| \leq t_{k+1} - t_k. \tag{3.17}$$

Moreover, it follows from Lemma 2.2 (ii) that

$$\begin{aligned} & \|DX(p_0)^{-1}P_{c,p_0,p_k}(P_{c_k,p_k,c_k(\tau)}D^2X(c_k(\tau))P_{c_k,c_k(\tau),p_k}^2 - D^2X(p_k))\| \\ & \leq h''(t_k + \tau(t_{k+1} - t_k)) - h''(t_k). \end{aligned} \tag{3.18}$$

Thus, combining (3.14), (3.15-3.17) and Lemma 3.1 yields that

$$\|DX(p_0)^{-1}P_{c,p_0,p_k} \circ P_{c_k,p_k,p_{k+1}}X(p_{k+1})\| \leq h(t_{k+1}). \tag{3.19}$$

Since

$$\begin{aligned}
\|u_X(p_{k+1})\| &= \|-DX(p_{k+1})^{-1}X(p_{k+1})\| \\
&\leq \|DX(p_{k+1})^{-1}P_{c_k,p_{k+1},p_k} \circ P_{c,p_k,p_0}DX(p_0)\| \\
&\quad \cdot \|DX(p_0)^{-1}P_{c,p_0,p_k} \circ P_{c_k,p_k,p_{k+1}}X(p_{k+1})\| \\
&\leq u_h(t_{k+1})
\end{aligned} \tag{3.20}$$

thanks to (3.13) and (3.19), (a) is true for $k + 1$, that is,

$$\|u_X(p_{k+1})\| \leq u_h(t_{k+1}). \tag{3.21}$$

Note that

$$\begin{aligned}
&\|-\frac{\lambda}{2}DX(p_{k+1})^{-1}D^2X(p_{k+1})u_X(p_{k+1})\| \\
&\leq \frac{\lambda}{2}\|DX(p_{k+1})^{-1}P_{c_k,p_{k+1},p_k} \circ P_{c,p_k,p_0}DX(p_0)\| \\
&\quad \cdot \|DX(p_0)^{-1}P_{c,p_0,p_k} \circ P_{c_k,p_k,p_{k+1}}D^2X(p_{k+1})\| \cdot \|u_X(p_{k+1})\|.
\end{aligned} \tag{3.22}$$

It follows from (3.13), (3.15) and (3.21) that

$$\begin{aligned}
&\|-\frac{\lambda}{2}DX(p_{k+1})^{-1}D^2X(p_{k+1})u_X(p_{k+1})\| \\
&\leq -\frac{\lambda}{2}h'(t_{k+1})^{-1}h''(t_{k+1})u_h(t_{k+1}) < 1,
\end{aligned} \tag{3.23}$$

where the last inequality is because of Lemma 2.4 (i). Thus, by the Banach Lemma, (3.23) implies that $Q_{X,\lambda}(p_{k+1})$ exists and

$$\begin{aligned}
Q_{X,\lambda}(p_{k+1}) &= \|(\mathbf{I}_{T_{p_k}M} + \frac{\lambda}{2}DX(p_{k+1})^{-1}D^2X(p_{k+1})u_X(p_{k+1}))^{-1}\| \\
&\leq \frac{1}{1 + \frac{\lambda}{2}h'(t_{k+1})^{-1}h''(t_{k+1})u_h(t_{k+1})} \\
&= Q_{h,\lambda}(t_{k+1}).
\end{aligned} \tag{3.24}$$

Hence, p_{k+2} is well-defined and (b) is true for $n = k + 1$. Since

$$\begin{aligned}
&\|v_{X,\lambda}(p_{k+1})\| \\
&= \|\frac{1}{2}DX(p_{k+1})^{-1}D^2X(p_{k+1})u_X(p_{k+1})Q_{X,\lambda}(p_{k+1})u_X(p_{k+1})\| \\
&\leq \frac{1}{2}\|DX(p_{k+1})^{-1}D^2X(p_{k+1})u_X(p_{k+1})\| \|Q_{X,\lambda}(p_{k+1})\| \|u_X(p_{k+1})\|,
\end{aligned}$$

it follows from (3.23), (3.24) and (3.21) that (c) holds for $k + 1$. Consequently,

$$\|w_{k+1}\| \leq \|u_X(p_{k+1})\| + \|v_{X,\lambda}(p_{k+1})\| \leq \|u_h(t_{k+1})\| + \|v_{h,\lambda}(t_{k+1})\| = t_{k+2} - t_{k+1}.$$

This implies that (d) is true for $n = k + 1$. The proof is complete. ■

4. APPLICATION TO ANALYTIC VECTOR FIELDS

Throughout this section, we always assume that M is an analytic complete m -dimensional Riemannian Manifold and X is analytic on M . Let $p_0 \in M$ be such that $DX(p_0)^{-1}$ exists. Following [6], we define

$$\gamma(X, p_0) = \sup_{k \geq 2} \left\| DX(p_0)^{-1} \frac{D^k X(p_0)}{k!} \right\|_{p_0}^{\frac{1}{k-1}}. \tag{4.1}$$

Also we adopt the convention that $\gamma(X, p_0) = \infty$ if $DX(p_0)$ is not invertible. Note that this definition is justified and in the case when $DX(p_0)$ is invertible, by analyticity, $\gamma(X, p_0)$ is finite. The following Taylor formula for vector fields will play a key role in the remainder of the present paper.

Lemma 4.1. *Let $r = \frac{1}{\gamma(X, p_0)}$. Let $p \in \mathbf{B}(p_0, r)$ and $v \in T_{p_0}M$ be such that $p = \exp_{p_0}(v)$ and $\|v\| < r$. Then, for each $j = 0, 1, 2, \dots$,*

$$D^j X(p) = P_{c,p,p_0} \left(\sum_{k=0}^{\infty} \frac{1}{k!} D^{k+j} X(p_0) v^k \right) P_{c,p_0,p}^j, \tag{4.2}$$

where c is the geodesic defined by $c(t) := \exp_{p_0}(tv)$ for each $t \in [0, 1]$ and $c(1) = p$.

Proof. We first verify (4.2) for the case when $j = 0$:

$$X(p) = P_{c,p,p_0} \left(\sum_{k=0}^{\infty} \frac{1}{k!} D^k X(p_0) v^k \right). \tag{4.3}$$

Let $\{e_i\}$ be a basis of $T_{p_0}M$ such that $\{DX(p_0)^{-1}e_i\}$ is an orthonormal basis of $T_{p_0}M$. Since X is an analytic vector field, there exist m analytic functions $X^i(t)$, $i = 1, 2, \dots, m$, such that

$$X(c(t)) = \sum_{i=1}^m X^i(t) P_{c,c(t),p_0} e_i \tag{4.4}$$

(cf [17, p.19]). Then by (2.2) and mathematical induction, one has

$$D^k X(c(t))(c'(t))^k = \sum_{i=1}^m \frac{d^k X^i(t)}{dt^k} P_{c,c(t),p_0} e_i \quad \text{for each } k = 0, 1, \dots \quad (4.5)$$

In particular, as $c'(0) = v$, we get

$$D^k X(p_0)v^k = \sum_{i=1}^m \frac{d^k X^i(t)}{dt^k} \Big|_{t=0} e_i \quad \text{for each } k = 0, 1, \dots \quad (4.6)$$

Let $j = 1, 2, \dots, m$. It follows that

$$\begin{aligned} & \langle DX(p_0)^{-1} D^k X(p_0)v^k, DX(p_0)^{-1} e_j \rangle \\ &= \frac{d^k X^j(t)}{dt^k} \Big|_{t=0} \quad \text{for each } k = 0, 1, \dots, \end{aligned} \quad (4.7)$$

because $\{DX(p_0)^{-1} e_i\}$ is an orthonormal basis of $T_{p_0}M$. Note that

$$\begin{aligned} & \overline{\lim}_{k \rightarrow \infty} \left(\frac{\|DX(p_0)^{-1} D^k X(p_0)\|}{k!} \right)^{\frac{1}{k}} \\ & \leq \sup_{k \geq 2} \left(\frac{\|DX(p_0)^{-1} \mathcal{D}^k X(p_0)\|}{k!} \right)^{\frac{1}{k-1}} = \gamma(X, p_0). \end{aligned}$$

This together with (4.7) yields that

$$\begin{aligned} & \overline{\lim}_{k \rightarrow \infty} \left(\frac{1}{k!} \left| \frac{d^k X^j(t)}{dt^k} \Big|_{t=0} \right| \right)^{\frac{1}{k}} \\ & \leq \overline{\lim}_{k \rightarrow \infty} \left(\frac{1}{k!} \|DX(p_0)^{-1} \mathcal{D}^k X(p_0)\| \right)^{\frac{1}{k}} \|v\| < 1 \end{aligned} \quad (4.8)$$

since $\|v\| < r = \frac{1}{\gamma(X, p_0)}$. Hence

$$X^j(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k X^j(t)}{dt^k} \Big|_{t=0} t^k \quad \text{for each } t \in [0, 1]. \quad (4.9)$$

Combining this with (4.4) and the fact that $p = c(1)$ gives that

$$X(p) = \sum_{i=1}^m X^i(1) P_{c,p,p_0} e_i = \sum_{i=1}^m \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k X^i(t)}{dt^k} \Big|_{t=0} P_{c,p,p_0} e_i. \quad (4.10)$$

Noting that P_{c,p,p_0} is a linear isomorphism from $T_{p_0}M$ to T_pM , one has that

$$X(p) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=1}^m \frac{d^k X^i(t)}{dt^k} \Big|_{t=0} P_{c,p,p_0} e_i = \mathcal{P}_{c,p,p_0} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=1}^m \frac{d^k X^i(t)}{dt^k} \Big|_{t=0} e_i \right).$$

This, together with (4.6), implies (4.3).

Below we will show that (4.2) holds. To do this, let $j = 1, 2, \dots$ and $v_1, \dots, v_j \in T_pM$. It's sufficient to prove that

$$\begin{aligned} & D^j X(p)(v_1, \dots, v_j) \\ &= P_{c,p,p_0} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \mathcal{D}^{k+j} X(p_0)(P_{c,p_0,p} v_1, \dots, P_{c,p_0,p} v_j, \underbrace{v, \dots, v}_k) \right) \end{aligned} \quad (4.11)$$

because, by the analyticity of X ,

$$\begin{aligned} & \mathcal{D}^{k+j} X(p_0)(\underbrace{v, \dots, v}_k, P_{c,p_0,p} v_1, \dots, P_{c,p_0,p} v_j) \\ &= \mathcal{D}^{k+j} X(p_0)(P_{c,p_0,p} v_1, \dots, P_{c,p_0,p} v_j, \underbrace{v, \dots, v}_k) \end{aligned}$$

holds for each k .

To show (4.11), for each $i = 1, \dots, j$, let Y_i be the vector field such that $Y_i(p) = v_i$, $\nabla_{c'(s)} Y_i = 0$ and $Y_i(p_0) = P_{c,p_0,p} v_i$. Let $\eta = D^j X(Y_1, \dots, Y_j)$. Then η is a vector field. Thus, applying (4.3) with X replaced by η , we have

$$\eta(p) = P_{c,p,p_0} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \mathcal{D}^k \eta(p_0) v^k \right). \quad (4.12)$$

In view of the definitions of η and $D^k \eta$, one can use mathematical induction to verify that

$$D^k \eta(\underbrace{c'(s), \dots, c'(s)}_k) = D^{k+j} X(Y_1, \dots, Y_j, \underbrace{c'(s), \dots, c'(s)}_k) \text{ for each } k=0, 1, \dots.$$

Since $\mathcal{D}^j X(p)(v_1, \dots, v_j) = \eta(p)$ and $Y_i(p_0) = P_{c,p_0,p} v_i$ for each $i = 1, \dots, j$, it follows that

$$D^k \eta(p_0)(\underbrace{v, \dots, v}_k) = D^{k+j} X(p_0)(Y_1(p_0), \dots, Y_j(p_0), \underbrace{v, \dots, v}_k) \text{ for each } k=0, 1, \dots.$$

Combining this with (4.12), (4.11) is seen to hold and the proof is complete. \blacksquare

The following two lemmas will be used. The first one was given in [2] while the proof for the second one is almost the same as that of [6, Lemma 4.3]. Let $p_0 \in M$ be such that $DX(p_0)^{-1}$ exists. For convenience, we use the function ψ defined by

$$\psi(u) := 1 - 4u + 2u^2, \quad u \in [0, 1 - \frac{\sqrt{2}}{2}].$$

Note that ψ is strictly monotonically decreasing on $[0, 1 - \frac{\sqrt{2}}{2}]$.

Lemma 4.2. *Let $|r| < 1$ and let k be a positive integer. Then*

$$\sum_{j=0}^{\infty} \frac{(k+j)!}{k!j!} r^j = \frac{1}{(1-r)^{k+1}}.$$

Lemma 4.3. *Let $p \in M$ and let c be a geodesic connecting p_0 and p such that $u := \gamma l(c) < 1 - \frac{\sqrt{2}}{2}$, where $l(c)$ is the arc length of c . Then $DX(p)^{-1}$ exists and*

$$\gamma(X, p) \leq \frac{\gamma(X, p_0)}{(1-u)\psi(u)}. \quad (4.13)$$

Recall that $p_0 \in M$ such that $DX(p_0)^{-1}$ exists. The following lemma shows that an analytic vector field satisfies the 2-piece γ -condition of order 2 at p_0 in $\mathbf{B}(p_0, \frac{2-\sqrt{2}}{2\gamma(X, p_0)})$.

Lemma 4.4. *Let $\gamma = \gamma(X, p_0)$ and $0 < r \leq \frac{2-\sqrt{2}}{2\gamma}$. Then X satisfies the 2-piece γ -condition of order 2 at p_0 in $\mathbf{B}(p_0, r)$.*

Proof. Note that

$$\gamma = \gamma(X, p_0) = \sup_{k \geq 2} \left\| DX(p_0)^{-1} \frac{D^k X(p_0)}{k!} \right\|_{\frac{1}{k-1}}. \quad (4.14)$$

Then

$$\|DX(p_0)^{-1} D^2 X(p_0)\| \leq 2\gamma. \quad (4.15)$$

For any $p, q \in \mathbf{B}(p_0, r)$, let c_1 be a minimizing geodesic connecting p_0, p and c_2 a geodesic connecting p, q such that $l(c_1) + l(c_2) < r$. To complete the proof, it remains to verify that

$$\|DX(p_0)^{-1} P_{c_1, p_0, p} \circ P_{c_2, p, q} D^3 X(q)\| \leq \frac{6\gamma^2}{(1 - \gamma(l(c_1) + l(c_2)))^4}. \quad (4.16)$$

Since

$$u = \gamma(X, p_0)l(c_1) < r \leq \frac{2 - \sqrt{2}}{2\gamma}, \quad (4.17)$$

Lemma 4.3 is applicable. It follows that

$$\gamma(X, p) \leq \frac{\gamma(X, p_0)}{(1-u)\psi(u)}. \quad (4.18)$$

Since $(1-u)\psi(u) \geq 1 - \frac{\sqrt{2}}{2} - u$, we obtain that

$$\frac{(1-u)\psi(u)}{\gamma(X, p_0)} \geq \frac{2 - \sqrt{2}}{2\gamma(X, p_0)} - l(c_1).$$

This, together with (4.18), implies that

$$l(c_2) \leq \frac{2 - \sqrt{2}}{2\gamma(X, p_0)} - l(c_1) \leq \frac{(1-u)\psi(u)}{\gamma(X, p_0)} \leq \frac{1}{\gamma(X, p)}, \quad (4.19)$$

thanks to the fact that $l(c_1) + l(c_2) < r \leq \frac{2 - \sqrt{2}}{2\gamma(X, p_0)}$. Let $v_0 \in T_{p_0}M$ and $v_1 \in T_pM$ such that $c_1(t) = \exp_{p_0}(tv_0)$ for each $t \in [0, 1]$ and $c_2(t) = \exp_p(tv_1)$ for each $t \in [0, 1]$. As $l(c_1) = \|v_0\|$ and $l(c_2) = \|v_1\|$, by (4.17) and (4.19), Lemma 4.1 is applicable to concluding that

$$\begin{aligned} & DX(p_0)^{-1}P_{p_0,p} \circ P_{p,q}D^3X(q) \\ &= DX(p_0)^{-1}P_{p_0,p} \sum_{l=0}^{\infty} \frac{1}{l!} D^{l+3}X(p)v_1^l P_{p,q}^3 \\ &= DX(p_0)^{-1} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{j=0}^{\infty} \frac{1}{j!} D^{l+j+3}X(p_0)v_0^j P_{p_0,p}^{l+3}v_1^l P_{p,q}^3. \end{aligned} \quad (4.20)$$

Noting that

$$\frac{\|DX(p_0)^{-1}D^{l+j+3}X(p)\|}{(l+j+3)!} \leq \gamma(X, p_0)^{l+j+2},$$

one has from (4.20) that

$$\begin{aligned} & \|DX(p_0)^{-1}P_{p_0,p} \circ P_{p,q}D^3X(q)\| \\ & \leq \sum_{l=0}^{\infty} \frac{(l+3)!}{l!} \sum_{j=0}^{\infty} \frac{(l+j+3)!}{j!(l+3)!} \gamma(X, p_0)^{l+j+2} \|v_0\|^j \|v_1\|^l. \end{aligned} \quad (4.21)$$

Using Lemma 4.2 to calculate the quantity on the right-hand side of inequality (4.21), we get that

$$\|DX(p_0)^{-1}P_{p_0,p} \circ P_{p,q}D^3X(q)\| \leq \frac{6\gamma(X, p_0)^2}{(1 - \gamma(X, p_0)(\|v_0\| + \|v_1\|))^4}. \quad (4.22)$$

Since $\|v_0\| = l(c_1)$, $\|v_1\| = l(c_2)$ and $\gamma = \gamma(X, p_0)$, (4.16) follows from (4.22). The proof is complete. ■

Recall that $p_0 \in M$ is such that $DX(p_0)^{-1}$ exists and that

$$\beta = \|DX(p_0)^{-1}X(p_0)\|, \quad \alpha = \gamma\beta,$$

where $\gamma = \gamma(X, p_0)$. Then, by Theorem 3.1, Lemma 4.4 and (2.22), we have the following corollary.

Corollary 4.1. *Suppose that*

$$\alpha = \beta\gamma \leq 3 - 2\sqrt{2}.$$

Then the sequence $\{p_n\}$ generated by (2.3) with initial point p_0 is well-defined for all $\lambda \in [0, 2]$ and converges to a singular point p^ of X in $\overline{\mathbf{B}}(p_0, r_1)$, where r_1 is given by (2.21).*

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