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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS IN ESTEBAN-LIONS DOMAINS WITH HOLES

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Abstract. In this article, we consider the semilinear elliptic equation

$$(*)_\lambda \qquad -\Delta u + u = \lambda K(x)u^p + h(x) \text{ in } \Omega, u > 0 \text{ in } \Omega\ , u \in H_0^1(\Omega),$$

where $\lambda \geq 0$, $N \geq 2$, $1 < p < 2^* - 1$ and Ω is the upper semi-strip domain with a hole in $\mathbb{R}^N$. Under some suitable conditions on K and h, we show that there exists a positive constant λ^* such that equation $(*)_\lambda$ has at least two solutions if $\lambda \in (0, \lambda^*)$, a unique solution if $\lambda = 0$ or $\lambda = \lambda^*$ and no solution if $\lambda > \lambda^*$. We also establish the asymptotic behavior and some further properties of positive solutions of equation $(*)_\lambda$.

1. Introduction

Throughout this article, let $N \geq 2$, $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = \infty$ for $N = 2$, q_0 be a given constant such that $q_0 > N/2$ if $N \geq 4$ and $q_0 = 2$ if $N = 2, 3$, and (y, z) be the generic point of $\mathbb{R}^N$ with $y \in \mathbb{R}^{N-1}$, $z \in \mathbb{R}$. Denote by $B^N(x_0; R)$ the $N-$ball, $\mathbb{S}$ the strip domain, $\mathbb{S}^+$ the upper semi-strip domain, Ω the upper semi-strip domain with a hole as follows:

$$\begin{aligned} B^N(x_0; R) &= \{x \in \mathbb{R}^N : |x - x_0| < R\}; \\ \mathbb{S} &= \{(y, z) : |y| < r_0\}; \\ \mathbb{S}^+ &= \{(y, z) \in \mathbb{S} : z > 0\} \cup B^N(0; r_0); \\ \Omega &= \mathbb{S}^+ \setminus \overline{D}, \text{where } D \subset\subset \mathbb{S}^+ \text{ is a smooth bounded domain in } \mathbb{R}^N. \end{aligned}$$

where r_0 is a fixed positive constant and R is a positive constant.

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Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = u^p \text{ in } \Theta, \\ u > 0 \text{ in } \Theta \ , u \in H_0^1(\Theta), \end{cases} \tag{1.1}$$

where Θ is a smooth domain in $\mathbb{R}^N$ and $1 < p < 2^* - 1$.

The existence and nonexistence of solutions of equation (1.1) have been the focus of a great deal of research in recent years. By the Rellich compactness theorem, it is easy to obtain a solution of (1.1) in a bounded domain. For general unbounded domain Θ, because of the lack of compactness, the existence of solutions of (1.1) in Θ is very difficult and unclear. The breakthrough was made by Esteban-Lions [5]. They asserted that (1.1) does not admit any nontrivial solution in Esteban-Lions domain, where the definition of Esteban-Lions domain is: For a proper unbounded domain Θ in $\mathbb{R}^N$, there exists a $\chi \in \mathbb{R}^N$, $\|\chi\| = 1$ such that $n(x) \cdot \chi \geq 0$ and $n(x) \cdot \chi \not\equiv 0$ on $\partial\Theta$, where $n(x)$ is the unit outward normal vector to $\partial\Theta$ at the point x. A typical example is the upper semi-strip $\mathbb{S}^+$.

Thus, perturb (1.1) to obtain the existence of solutions in Esteban-Lions domain is of great interest to research. In this paper, we study a more general equation for the full range $\lambda \in [0, \infty)$

$$\begin{cases} -\Delta u + u = \lambda K(x) u^p + h(x) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \ u \in H_0^1(\Omega), \end{cases} \tag*{$(1.2)_\lambda$}$$

where $\lambda \geq 0$, Ω is the upper semi-strip domain with a hole, $K(x)$ is a positive, bounded and continuous function on $\overline{\mathbb{S}}$ and $h(x) \in L^2(\Omega)$. Moreover, $K(x)$ and $h(x)$ satisfy the following conditions:

$(k1)$ There exists a positive constant K_∞ such that

$$\lim_{|z| \to \infty} K(y, z) = K_\infty \text{ uniformly for } y \in B^{N-1}(0; r_0);$$

$(k2)$ There exist some constants $\gamma > \frac{p+1}{p}$ and $\vartheta > 0$ such that

$$K(y, z) \geq K_\infty - \vartheta \exp(-\gamma\sqrt{1+\mu_1}|z|) \text{ as } |z| \to \infty, \text{ uniformly for } y \in B^{N-1}(0; r_0),$$

where μ_1 is the first eigenvalue of the Dirichlet problem $-\Delta$ in $B^{N-1}(0; r_0)$;

$(h1)$ $h(x) \geq 0$, $h(x) \not\equiv 0$, $h(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$.

Our main results are as follows:

Theorem 1.1. *Assume u_0 is the unique solution of $(1.2)_0$ and conditions $(k1)$, $(k2)$ and $(h1)$ hold, then there exists a constant $\lambda^* > 0$ such that*

(i) equation $(1.2)_\lambda$ has at least two positive solutions, u_λ, U_λ and $u_\lambda < U_\lambda$ if $\lambda \in (0, \lambda^*)$;

(ii) equation $(1.2)_\lambda$ has a unique positive solution u_λ if $\lambda = 0$ or $\lambda = \lambda^*$;

(iii) equation $(1.2)_\lambda$ has no positive solutions if $\lambda > \lambda^*$,

Furthermore,

$$
\begin{aligned}
\lambda_1 &\equiv \frac{(p+1)(p-1)^{p-1} M^{\frac{p+1}{2}}}{(2p)^p \|K\|_{L^\infty(\Omega)} \|h\|^{p-1}_{L^2(\Omega)}} \\
&\le \lambda^* \le \inf_{w \in H_0^1(\Omega)\setminus\{0\}} \left(\frac{\|w\|^2}{p \displaystyle\int_\Omega K u_0^{p-1} w^2 dx} \right) \equiv \lambda_2 \qquad (1.3) \\
&\le \frac{p\|h\|^2_{L^2(\Omega)}}{(p-1)^2 \displaystyle\int_\Omega K u_0^{p+1} dx} \equiv \lambda_3
\end{aligned}
$$

where $M = \inf\{\int_{\mathbb{S}} (|\nabla u|^2 + |u|^2) dx : \int_{\mathbb{S}} |u|^{p+1} dx = 1\}$, u_λ is the unique minimal solution of equation $(1.2)_\lambda$, U_λ is the second solution of equation $(1.2)_\lambda$ constructed in Section 5.

Theorem 1.2. *Under the assumptions of Theorem 1.1. Then*

(i) u_λ is strictly increasing with respect to λ and uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in [0, \lambda^*]$ and

$$u_\lambda \to u_0 \text{ in } L^\infty(\Omega) \cap H_0^1(\Omega) \text{ as } \lambda \to 0^+.$$

(ii) U_λ is strictly decreasing with respect to λ and unbounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for $\lambda \in (0, \lambda^*)$, that is

$$\lim_{\lambda \to 0^+} \|U_\lambda\| = \lim_{\lambda \to 0^+} \|U_\lambda\|_{L^\infty(\Omega)} = \infty.$$

This paper is organized as follows. In section 2, we establish a decomposition lemma of Lions which will be used later. In section 3, we establish several lemmas for the regularity and asymptotic behavior of the solution of equation $(1.2)_\lambda$. In section 4, we apply Ekeland′s variational principle [6] to show that equation $(1.2)_\lambda$ has a solution for small $\lambda > 0$, then, by the standard barrier method to show that there is a constant $\lambda^* > 0$ such that $(1.1)_\lambda$ has a minimal solution u_λ for all $\lambda \in [0, \lambda^*]$. In section 5, we assert that there is the second solution of equation $(1.2)_\lambda$ for all $\lambda \in (0, \lambda^*)$. In section 6, we shall give some further properties of the solution of equation $(1.2)_\lambda$.

2. Preliminaries

In this paper, we denote by c and c_i ($i = 1, 2, \ldots$) the universal constants, unless otherwise specified. We set

$$\|u\| = \left(\int_\Omega (|\nabla u|^2 + |u|^2) dx\right)^{1/2},$$
$$\|u\|_{L^q(\Omega)} = \left(\int_\Omega |u|^q dx\right)^{1/q},\ 1 \le q < \infty,$$
$$\|u\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} |u(x)|,$$
$$M = \inf\{\int_{\mathbb{S}} (|\nabla u|^2 + |u|^2) dx : \int_{\mathbb{S}} |u|^{p+1} dx = 1\}.$$

Now we give some notations and some known results. In order to get the existence of positive solutions of equation $(1.2)_\lambda$, we consider the energy functional $I_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$I_\lambda(u) = \int_\Omega \Big[\frac{1}{2}(|\nabla u|^2 + |u|^2) - \frac{\lambda}{p+1} K(x)(u^+)^{p+1} - h(x)u\Big] dx,$$

where $u^\pm(x) = \max\{\pm u(x), 0\}$. It is well-known that the critical points of I_λ are the positive solutions of equation $(1.2)_\lambda$.

Now, we introduce the following elliptic equation on $\mathbb{S}$:

$$\begin{cases} -\Delta u + u = \lambda K_\infty u^p \text{ in } \mathbb{S}, \\ u \in H_0^1(\mathbb{S}),\ N \ge 2. \end{cases} \tag{2.1$_\lambda$}$$

Associated with $(2.1)_\lambda$, we consider the energy functional I_λ^∞ defined by

$$I_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{S}} \left(|\nabla u|^2 + u^2\right) dx - \frac{\lambda}{p+1}\int_{\mathbb{S}} K_\infty (u^+)^{p+1} dx,\ u \in H_0^1(\mathbb{S}).$$

By Lions [13] and Lien-Tzeng-Wang [12], we know that $(2.1)_\lambda$ has a ground state solution $\varpi_\lambda(x) > 0$ in $\mathbb{S}$ such that

$$M_\lambda^\infty = I_\lambda^\infty(\varpi_\lambda) = \sup_{t\ge 0} I^\infty(t\varpi_\lambda). \tag{2.2}$$

Now, we give the following decomposition lemma for later use.

Proposition 2.1. *Let condition $(k1)$ be satisfied and $\{u_k\}$ be a $(PS)_\beta$−sequence of I_λ in $H_0^1(\Omega)$:*

$$I_\lambda(u_k) = \beta + o(1) \text{ as } k \to \infty,$$
$$I'_\lambda(u_k) = o(1) \text{ strongly in } H^{-1}(\Omega).$$

Then there exist an integer $l \geq 0$, sequence $\{x_k^i\} \subseteq \mathbb{R}^N$ of the form $(0, z_k^i) \in \mathbb{S}$, functions $\overline{u} \in H_0^1(\Omega)$, $\overline{u}_i \in H_0^1(\mathbb{S})$, $1 \leq i \leq l$, such that for some subsequence (still denoted by) $\{u_k\}$, we have

$$\begin{cases} u_k \rightharpoonup \overline{u} \text{ weakly in } H_0^1(\Omega); \\ \beta = I_\lambda(\overline{u}) + \sum_{i=1}^{l} I_\lambda^\infty(\overline{u}_i); \\ -\Delta\overline{u} + \overline{u} = \lambda K(x)\overline{u}^p + h(x) \text{ in } H^{-1}(\Omega); \\ -\Delta\overline{u}_i + \overline{u}_i = \lambda K_\infty \overline{u}_i^p \text{ in } H^{-1}(\mathbb{S}),\ 1 \leq i \leq l; \\ \left|x_k^i\right| \to \infty,\ \ \left|x_k^i - x_k^j\right| \to \infty,\ 1 \leq i \neq j \leq l. \end{cases}$$

where we agree that in the case $l = 0$ the above holds without $\overline{u}_i$, x_k^i.

Proof. The proof can be obtained by using the arguments in Bahri-Lions [3] (also see Lien-Tzeng-Wang [12], Theorem 4.1). ∎

Now, we combine Hsu [9], Proposition 3.4 and [11], Lemma 3.6, we obtain a precise asymptotic behavior result for positive solutions of $(2.1)_\lambda$ at infinity.

Proposition 2.2 *Let ϖ_λ be a positive solution of $(2.1)_\lambda$ in an unbounded cylinder $\mathbb{S} = \omega \times \mathbb{R}^n \subseteq \mathbb{R}^{m+n}$, $m \geq 2$, $n \geq 1$ and φ be the first positive eigenfunction of the Dirichlet problem $-\Delta\varphi = \mu_1\varphi$ in ω, then for any $\varepsilon > 0$, there exist positive constants c_ε and c such that*

$$\begin{cases} \varpi_\lambda(y,z) \leq c_\varepsilon \varphi(y) \exp(-\sqrt{1+\mu_1}\,|z|)\,|z|^{-\frac{n-1}{2}+\varepsilon} \\ \varpi_\lambda(y,z) \geq c\varphi(y) \exp(-\sqrt{1+\mu_1}\,|z|)\,|z|^{-\frac{n-1}{2}} \end{cases} \quad \text{as } |z| \to \infty,\ \ y \in \varpi$$

Remark 2.3. Now, we apply Proposition 2.2 to equation $(2.1)_\lambda$ in $\mathbb{S} = B^{N-1}(0; r_0) \times \mathbb{R}$, then we can easily deduce that for any $0 < \varepsilon < 1 + \mu_1$, there exist positive constants c_ε and c such that

$$\begin{cases} \varpi_\lambda(y,z) \leq c_\varepsilon \varphi(y) \exp(-\sqrt{1+\mu_1-\varepsilon}\,|z|) \\ \varpi_\lambda(y,z) \geq c\varphi(y) \exp(-\sqrt{1+\mu_1}\,|z|) \end{cases} \quad \text{for all } x = (y,z) \in \mathbb{S} \tag{2.3}$$

where φ is the first positive eigenfunction of the Dirichlet problem $-\Delta\varphi = \mu_1\varphi$ in $B^{N-1}(0; r_0)$.

3. Asymptotic Behavior of Solutions

In this section, we present two asymptotic behavior of each solution of equation $(1.2)_\lambda$ in Ω.

Lemma 3.1. *Let conditions* $(k1)$, $(k2)$ *and* $(h1)$ *be satisfied and suppose that* $u \in H_0^1(\Omega)$ *is a weak solution* of equation $(1.2)_\lambda$ in Ω. Then we have the following results.

(i) $u \in L^q(\Omega)$ for $q \in [2, \infty)$.

(ii) There exist some positive constants c_1 and c_2, depending on q_0 and K, such that $u \in C^{0,\alpha}(\overline{\Omega}) \cap W^{2,q_0}(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \le \|u\|_{C^{0,\alpha}(\overline{\Omega})} \le c_1\|u\|_{W^{2,q_0}(\Omega)} \le c_2(\lambda\|u\|^p_{L^{pq_0}(\Omega)} + \|h\|_{L^{q_0}(\Omega)}),$$

where $q_0 > N/2$ and $\alpha = 2 - \frac{N}{q_0} - \left[2 - \frac{N}{q_0}\right]$.

(iii) $\lim_{z\to\infty} u(y, z) = 0$ uniformly in y, where $(y, z) \in \Omega$.

Proof. (i) See Hsu [10] for the proof.

(ii) Since $u \in H_0^1(\Omega)$ is a weak solution of equation $(1.2)_\lambda$ in Ω, by part (i) and $(h1)$, $\lambda K(x)u^p + h(x) \in L^{q_0}(\Omega)$ for some $q_0 > N/2$. By Gilbarg-Trudinger [8], Theorem 9.15 and Lemma 9.17, the Dirichlet problem

$$\begin{cases} -\Delta v + v = \lambda K(x)u^p + h(x) \text{ in } \Omega, \\ v \in W_0^{1,q_0}(\Omega) \cap H_0^1(\Omega), \end{cases}$$

has a unique strong solution $v \in W^{2,q_0}(\Omega) \cap W_0^{1,q_0}(\Omega) \cap H_0^1(\Omega)$ and

$$\|v\|_{W^{2,q_0}(\Omega)} \le c_1\|\lambda K(x)u^p + h(x)\|_{L^{q_0}(\Omega)}.$$

Thus, u and v satisfy weakly

$$\begin{cases} -\Delta v + v = \lambda K(x)u^p + h(x) \text{ in } \Omega, \\ -\Delta u + u = \lambda K(x)u^p + h(x) \text{ in } \Omega. \end{cases}$$

By Gilbarg-Trudinger [8], Corollary 8.2, $u = v \in W^{2,q_0}(\Omega) \cap W_0^{1,q_0}(\Omega) \cap H_0^1(\Omega)$ and

$$\|u\|_{W^{2,q_0}(\Omega)} \le c_1\|\lambda K(x)u^p + h(x)\|_{L^{q_0}(\Omega)} \le c_2(\lambda\|u\|^p_{L^{pq_0}(\Omega)} + \|h\|_{L^{q_0}(\Omega)}).$$

Let $\alpha = 2 - \frac{N}{q_0} - \left[2 - \frac{N}{q_0}\right]$. Since $q_0 > N/2$ and by Brezis [4], p.168, we have $u \in C^{0,\alpha}(\overline{\Omega})$ and

$$\|u\|_{L^\infty(\Omega)} \le \|u\|_{C^{0,\alpha}(\overline{\Omega})} \le c_1 \|u\|_{W^{2,q_0}(\Omega)} \le c_2(\lambda \|u\|^p_{L^{pq_0}(\Omega)} + \|h\|_{L^{q_0}(\Omega)}).$$

(iii) Since $u \in H_0^1(\Omega)$ is a weak solution of equation $(1.2)_\lambda$, by part (ii), $u \in C^{0,\alpha}(\overline{\Omega}) \cap W^{2,q_0}(\Omega)$. For each $R > 0$, apply Brezis [4], p.168 to obtain

$$\|u\|_{L^\infty(\Omega_R)} \le c_{q_0} \|u\|_{W^{2,q_0}(\Omega_R)},$$

where $\Omega_R = \{x = (y,z) \in \Omega | |z| > R\}$. Since $\|u\|_{W^{2,q_0}(\Omega_R)} = o(1)$ as $R \to \infty$, we have $\lim_{z\to\infty} u(y,z) = 0$ uniformly in y, where $(y,z) \in \Omega$. ∎

Lemma 3.2. *Let u be a positive solution of equation $(1.2)_\lambda$ and φ be the first positive eigenfunction of the Dirichlet problem $-\Delta\varphi = \mu_1\varphi$ in $B^{N-1}(0;r_0)$, then there exists a constant $c > 0$ such that*

$$u(y,z) \ge c\varphi(y)\exp(-\sqrt{1+\mu_1}z) \quad \text{for all } y \in B^{N-1}(0;r_0),\ z \ge \rho_0. \tag{3.1}$$

where $\rho_0 = 2\max\{\sup_{(y,z)\in D} z, 1\}$.

Proof. Let

$$\Psi(x) = \varphi(y)\exp(-\sqrt{1+\mu_1}|z|) \ \text{ for } x = (y,z) \in \Omega.$$

It is very easy to show that

$$-\Delta\Psi(x) + \Psi(x) = 0 \ \text{ for } x \in \Omega$$

From the proof of Hsu [9], Proposition 3.4, we can deduce that $u(x)\varphi^{-1}(y) > 0$ for $x \in \overline{U_{\rho_0}}$ and $u(x)\varphi^{-1}(y) \in C^1(\overline{U_{\rho_0}})$, where $U_{\rho_0} = \{x = (y,z) : y \in B^{N-1}(0;r_0),\ z \ge \rho_0\}$ If we set

$$\alpha_1 = \sup_{y\in\overline{B^{N-1}(0;r_0)}, z=\rho_0} (u(x)\Psi^{-1}(x)),$$

then $\alpha_1 > 0$ and

$$\alpha_1\Psi(x) \ge u(x) \quad \text{for } y \in \overline{B^{N-1}(0;r_0)}, z = \rho_0.$$

Let $\Phi_1(x) = \alpha_1\Psi(x)$, for $x \in \overline{\Omega}$. Then, for $z \ge \rho_0$, we have

$$\Delta(\Phi_1 - u)(x) - (\Phi_1 - u)(x) = \lambda K(x)u^p(x) + h(x) \ge 0.$$

Therefore, by means of the strong maximum principle implies that $u(x) - \Phi_1(x) \ge 0$ for $x \in \overline{U_{\rho_0}}$. This completes the proof of Lemma 3.2. ∎

4. Existence of Minimal Solution

In this section, by the barrier method, we will establish the existence of minimal positive solution u_λ for all λ in some finite interval $[0, \lambda^*]$ (i.e. for any positive solution u of equation $(1.2)_\lambda$, then $u \geq u_\lambda$).

Lemma 4.1. *Assume condition* $(k1)$ *holds. Then equation* $(1.2)_\lambda$ *has a solution* u_λ *if* $0 \leq \lambda < \lambda_1$ *where* λ_1 *is given by* (1.3).

Proof. For $\lambda = 0$, the existence question is equivalent to the existence of $u_0 \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u_0 \cdot \nabla\phi + u_0\phi = \int_\Omega h\phi \tag{4.1}$$

for all $\phi \in H_0^1(\Omega)$. Since

$$\left|\int_\Omega h\phi\right| \leq \|h\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)} \leq \|h\|_{L^2(\Omega)}\|\phi\|.$$

Hence, according to the Lax-Milgram theorem, there exists a unique $u_0 \in H_0^1(\Omega)$ satisfies (4.1). Since $0 \not\equiv h \geq 0$ in Ω, by strong maximum principle (see Gilbarg-Trudinger [8]), we conclude that $u_0 > 0$ in Ω.

We consider next the case $\lambda > 0$. We show first that for sufficiently small λ, say $\lambda = \lambda_0$, there exists $t_0 = t(\lambda_0) > 0$ such that $I_{\lambda_0}(u) > 0$ for $\|u\| = t_0$. From the definition of I_λ, we have

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{p+1}\|K\|_{L^\infty(\Omega)}M^{-\frac{p+1}{2}}\|u\|^{p+1} - \|h\|_{L^2(\Omega)}\|u\|$$

where $M = \inf\{\int_\Omega(|\nabla u|^2+|u|^2)dx : \int_\Omega |u|^{p+1}dx = 1\} = \inf\{\int_{\mathbb{S}}(|\nabla u|^2+|u|^2)dx : \int_{\mathbb{S}} |u|^{p+1}dx = 1\}$ (see Wang [14], Proposition 14).

Set

$$f(t) = \frac{1}{2}t - \lambda c_1 t^p - c_2,$$

where $c_1 = \frac{1}{p+1}\|K\|_{L^\infty(\Omega)}M^{-\frac{p+1}{2}}$ and $c_2 = \|h\|_{L^2(\Omega)}$.

It then follows that $f(t)$ achieves a maximum at $t_\lambda = (2p\lambda c_1)^{-(p-1)^{-1}}$. Set $B_\lambda = \{u \in H_0^1(\Omega) : \|u\| < t_\lambda\}$. Then for all $u \in \partial B_\lambda = \{u \in H_0^1(\Omega) : \|u\| = t_\lambda\}$,

$$I_\lambda(u) \geq t_\lambda h(t_\lambda) \geq t_\lambda[t_\lambda(p-1)/2p - c_2] > 0$$

provided that $c_2 < t_\lambda(p-1)/2p$, which is satisfied for $\lambda < \lambda_1$. Fix such a value of λ, say λ_0, and set $t_0 = t(\lambda_0)$. Let $0 \not\equiv \phi \geq 0$, $\phi \in C_0^\infty(\Omega)$ such that $\int_\Omega h\phi dx > 0$. Then

$$I_{\lambda_0}(t\phi) = \frac{t^2}{2}\|\phi\|^2 - \frac{\lambda_0}{p+1}t^{p+1}\int_\Omega K\phi^{p+1} - t\int_\Omega h\phi < 0$$

for sufficiently small $t > 0$, and it is easy to see that I_{λ_0} is bounded below on B_{t_0}. Set $\beta = \inf\{I_{\lambda_0}(u) : u \in B_{t_0}\}$. Then $\beta < 0$, and since $I_{\lambda_0}(u) > 0$ on ∂B_{t_0}, the continuity of I_{λ_0} on $H_0^1(\Omega)$ implies that there exists $0 < t_1 < t_0$ such that $I_{\lambda_0}(u) > \beta$ for all $u \in H_0^1(\Omega)$ and $t_1 \le \|u\| \le t_0$. By the Ekeland′s variational principle [6], there exists a sequence $\{u_k\}_{k=1}^{\infty} \subset B_{t_1}$ such that $I_{\lambda_0}(u_k) = \beta + o(1)$ and $I'_{\lambda_0}(u_k) = o(1)$ strongly in $H^{-1}(\Omega)$, as $k \to \infty$. By Proposition 2.1, we have that there exist a subsequence $\{u_k\}$, an integer $l \ge 0$, $\overline{u}_i > 0$, $1 \le i \le l$ (if $l \ge 1$), $u_0 > 0$ in Ω and u_0 in $\overline{B}_{t_1}$ such that

$$\begin{cases} u_k \rightharpoonup u_0 \text{ weakly in } H_0^1(\Omega), \\ -\Delta u_0 + u_0 = \lambda_0 K(x) u_0^p + h(x) \text{ in } H^{-1}(\Omega), \\ -\Delta \overline{u}_i + \overline{u}_i = \lambda_0 K_\infty \overline{u}_i^p \text{ in } H^{-1}(\mathbb{R}^N),\ 1 \le i \le l. \end{cases}$$

Moreover,

$$I_{\lambda_0}(u_k) = I_{\lambda_0}(u_0) + \sum_{i=1}^{l} I_{\lambda_0}^{\infty}(\overline{u}_i) + o(1) \text{ as } k \to \infty.$$

Note that $I_{\lambda_0}^{\infty}(\overline{u}_i) \ge M_{\lambda_0}^{\infty} > 0$ for $i = 1, 2, \cdots, l$. Since $u_0 \in B_{t_0}$, we have $I_{\lambda_0}(u_0) \ge \beta$. We conclude that $l = 0$, $I_{\lambda_0}(u_0) = \beta$ and $I'_{\lambda_0}(u_0) = 0$, i.e., u_0 is a weak positive solution of equation $(1.2)_{\lambda_0}$. ■

Now, by the standard barrier method, we get the following Lemma.

Lemma 4.2. *Assume condition* $(k1)$ *holds. Then there exists a constant* $\lambda^* > 0$ *such that for each* $\lambda \in [0, \lambda^*)$, *equation* $(1.2)_\lambda$ *has a minimal positive solution* u_λ *and* u_λ *is strictly increasing in* λ.

Proof. Denoting

$$\lambda^* = \sup\{\lambda \ge 0 : \text{ equation } (1.2)_\lambda \text{ has a positive solution }\}.$$

By Lemma 4.1, we have $\lambda^* > 0$. Now, consider $\lambda \in [0, \lambda^*)$. By the definition of λ^*, we know that there exists $\lambda' > \lambda$ such that $\lambda' < \lambda^*$ and equation $(1.1)_{\lambda'}$ has a positive solution $u_{\lambda'} > 0$, i.e.,

$$\begin{aligned} -\Delta u_{\lambda'} + u_{\lambda'} &= \lambda' K(x) u_{\lambda'}^p + h(x) \\ &> \lambda K(x) u_{\lambda'}^p + h(x). \end{aligned}$$

Then $u_{\lambda'}$ is a supersolution of equation $(1.2)_\lambda$. From $h(x) \ge 0$ and $h(x) \not\equiv 0$, it is easily verified that 0 is a subsolution of equation $(1.2)_\lambda$. By the standard barrier method, there exists a solution u_λ of equation $(1.2)_\lambda$ such that $0 \le u_\lambda \le u_{\lambda'}$. Since

0 is not a solution of equation $(1.2)_\lambda$ and $\lambda' > \lambda$, the maximum principle implies that $0 < u_\lambda < u_{\lambda'}$. Again using a result of Amann [1], we can choose a minimum positive solution u_λ of equation $(1.2)_\lambda$. This completes the proof of Lemma 4.2.

Now, we consider a solution u of equation $(1.2)_\lambda$. Let $\sigma_\lambda(u)$ be defined by

$$\sigma_\lambda(u) = \inf\{\int_\Omega (|\nabla\varphi|^2 + |\varphi|^2)dx : \varphi \in H_0^1(\Omega), \int_\Omega pKu^{p-1}\varphi^2 dx = 1\} \tag{4.2}$$

By the standard direct minimization procedure, we can show that $\sigma_\lambda(u)$ is attained by a function $\varphi_\lambda > 0$, $\varphi_\lambda \in H_0^1(\Omega)$, satisfying

$$-\Delta\varphi_\lambda + \varphi_\lambda = \sigma_\lambda(u)pKu^{p-1}\varphi_\lambda \text{ in } \Omega. \tag{4.3}$$

Lemma 4.3. *Assume condition* $(k1)$ *holds. For* $\lambda \in [0, \lambda^*)$, *let* u_λ *be the minimal solution of equation* $(1.2)_\lambda$ *and* $\sigma_\lambda(u_\lambda)$ *be the corresponding number given by* (4.2). *Then*

(i) $\sigma_\lambda(u_\lambda) > \lambda$ and is strictly decreasing in λ, $\lambda \in [0, \lambda^*)$;

(ii) $\lambda^* < \infty$, and $(1.2)_{\lambda^*}$ has a minimal solution u_{λ^*}.

Proof. Consider $u_{\lambda'}$, u_λ, where $\lambda^* > \lambda' > \lambda \geq 0$. Let φ_λ be a minimizer of $\sigma_\lambda(u_\lambda)$, then by Lemma 4.2, we obtain that

$$\int_\Omega pKu_{\lambda'}^{p-1}\varphi_\lambda^2 dx > \int_\Omega pKu_\lambda^{p-1}\varphi_\lambda^2 dx = 1,$$

and there is a constant t, $0 < t < 1$ such that

$$\int_\Omega pKu_{\lambda'}^{p-1}(t\varphi_\lambda)^2 = 1.$$

Therefore,

$$\sigma_{\lambda'}(u_{\lambda'}) \leq t^2\|\varphi_\lambda\|^2 < \|\varphi_\lambda\|^2 = \sigma_\lambda(u_\lambda), \tag{4.4}$$

showing the monotonicity of $\sigma_\lambda(u_\lambda)$, $\lambda \in [0, \lambda^*)$.

Consider now $\lambda \in (0, \lambda^*)$. Let $\lambda < \lambda' < \lambda^*$. From (4.3) and the monotonicity of u_λ, we get

$$\begin{aligned}
&\sigma_\lambda(u_\lambda)p\int_\Omega (u_{\lambda'} - u_\lambda)Ku_\lambda^{p-1}\varphi_\lambda dx \\
&= \int_\Omega \nabla(u_{\lambda'} - u_\lambda)\cdot\nabla\varphi_\lambda dx + \int_\Omega (u_{\lambda'} - u_\lambda)\varphi_\lambda dx \\
&= (\lambda' - \lambda)\int_\Omega Ku_{\lambda'}^p\varphi_\lambda dx + \lambda\int_\Omega K(u_{\lambda'}^p - u_\lambda^p)\varphi_\lambda dx \\
&> \lambda p\int_\Omega K\varphi_\lambda \int_{u_\lambda}^{u_{\lambda'}} t^{p-1}dtdx \\
&\geq \lambda p\int_\Omega Ku_\lambda^{p-1}(u_{\lambda'} - u_\lambda)\varphi_\lambda dx,
\end{aligned} \tag{4.5}$$

which implies that $\sigma_\lambda(u_\lambda) > \lambda$, $\lambda \in (0, \lambda^*)$. This completes the proof of (ii).

We show next that $\lambda^* < \infty$. Let $\lambda_0 \in (0, \lambda^*)$ be fixed. For any $\lambda \geq \lambda_0$, (4.4) and (4.5) imply

$$\sigma_{\lambda_0}(u_{\lambda_0}) \geq \sigma_\lambda(u_\lambda) > \lambda$$

for all $\lambda \in [\lambda_0, \lambda^*)$. Thus, $\lambda^* < \infty$.

By (4.2) and $\sigma_\lambda(u_\lambda) > \lambda$, we have

$$\int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2)dx - \lambda p \int_\Omega K u_\lambda^{p+1} dx > 0.$$

and also we have

$$\int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2)dx - \int_\Omega \lambda K u_\lambda^{p+1} dx - \int_\Omega h u_\lambda = 0.$$

Thus

$$\begin{aligned}\int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2)dx &= \int_\Omega \lambda K u_\lambda^{p+1} dx + \int_\Omega h u_\lambda dx \\ &< \frac{1}{p}\int_\Omega (|\nabla u_\lambda|^2 + |u_\lambda|^2)dx + \|h\|_{L^2(\Omega)}\|u_\lambda\|.\end{aligned}$$

This implies that, for all $\lambda \in (0, \lambda^*)$,

$$\|u_\lambda\| \leq \frac{p}{p-1}\|h\|_{L^2(\Omega)} \tag{4.6}$$

By (4.6) and part (i) of Lemma 4.3, the solution u_λ is strictly increasing with respect to λ; we may suppose that

$$u_\lambda \rightharpoonup u_{\lambda^*} \text{ weakly in } H_0^1(\Omega) \text{ as } \lambda \to \lambda^*.$$

This implies that

$$\left\{\begin{array}{l} \displaystyle\int_\Omega (\nabla u_\lambda \cdot \nabla\varphi + u_\lambda\varphi)dx \to \int_\Omega (\nabla u_{\lambda^*} \cdot \nabla\varphi + u_{\lambda^*}\varphi)dx, \\ \displaystyle\int_\Omega (\lambda K u_\lambda^p + h)\varphi dx \to \int_\Omega (\lambda^* K u_{\lambda^*}^p + h)\varphi dx, \end{array}\right. \quad \text{as } \lambda \to \lambda^*$$

for all $\varphi \in H_0^1(\Omega)$. Hence, u_{λ^*} is a minimal positive solution of $(1.2)_\lambda$. This completes the proof of Lemma 4.3.

Lemma 4.4. *If condition $(k1)$ holds, then $\lambda_1 \leq \lambda^* \leq \lambda_2 \leq \lambda_3$, where λ_1, λ_2 and λ_3 are given by (1.3).*

Proof. By Lemma 4.1 and the definition of λ^*, we conclude that $\lambda^* \geq \lambda_1$.

As in Lemma 4.3, we have $\sigma_\lambda(u_\lambda) > \lambda$ for all $\lambda \in (0, \lambda^*)$, so for any $w \in H_0^1(\Omega) \setminus \{0\}$, we have

$$\int_\Omega (|\nabla w + |w|^2) dx > \lambda p \int_\Omega K u_\lambda^{p-1} w^2 dx. \tag{4.7}$$

Let u_0 is the unique solution of $(1.2)_0$, then by (4.7) and $u_\lambda > u_0$ for all $\lambda \in (0, \lambda^*]$, we obtain that

$$\int_\Omega (|\nabla w + |w|^2) dx > \lambda p \int_\Omega K u_0^{p-1} w^2 dx,$$

i.e.

$$\lambda \leq \inf_{w \in H_0^1(\Omega) \setminus \{0\}} \left(\frac{\|w\|^2}{p \int_\Omega K u_0^{p-1} w^2 dx} \right) = \lambda_2. \tag{4.8}$$

This implies that $\lambda^* \leq \lambda_2$.

Take $w = u_\lambda$ in (4.7), and by (4.6) and the monotonicity of u_λ, we get that

$$\begin{aligned} \lambda_2 \leq & \quad \frac{\|u_\lambda\|^2}{p \int_\Omega K u_0^{p-1} u_\lambda^2 dx} \\ \leq & \quad \frac{p \|h\|_{L^2(\Omega)}^2}{(p-1)^2 \displaystyle\int_\Omega K u_0^{p+1} dx} = \lambda_3. \end{aligned}$$

5. Existence of Second Solution

The existence of a second solution of equation $(1.2)_\lambda$, $\lambda \in (0, \lambda^*)$, will be established via the mountain pass theorem. When $0 < \lambda < \lambda^*$, we have known that equation $(1.2)_\lambda$ has a minimal positive solution u_λ by Lemma 4.2, then we need only to prove that equation $(1.2)_\lambda$ has another positive solution in the form of $U_\lambda = u_\lambda + v_\lambda$, where v_λ is a solution of the following equation:

$$\begin{cases} -\Delta v + v = \lambda K[(v + u_\lambda)^p - u_\lambda^p] \text{ in } \Omega, \\ v \in H_0^1(\Omega), v > 0 \text{ in } \Omega. \end{cases} \tag{5.1$_\lambda$}$$

The corresponding variational functional of $(5.1)_\lambda$ is

$$J_\lambda(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2 + v^2) - \lambda \int_\Omega \int_0^{v^+} K[(s + u_\lambda)^p - u_\lambda^p] ds dx, \ v \in H_0^1(\Omega).$$

To verify the conditions of the mountain pass theorem, we need the following lemmas.

Lemma 5.1. *For any $\epsilon > 0$, there is a positive constant c_ϵ such that*

$$(u_\lambda + s)^p - u_\lambda^p - p u_\lambda^{p-1} s \le \epsilon u_\lambda^{p-1} s + c_\epsilon s^p \text{ for all } s \ge 0.$$

Proof. From the fact

$$\lim_{s\to 0} \frac{(u_\lambda + s)^p - u_\lambda^p - p u_\lambda^{p-1} s}{s} = 0 \text{ and } \lim_{s\to\infty} \frac{(u_\lambda + s)^p - u_\lambda^p - p u_\lambda^{p-1} s}{s^p} = 1,$$

it is easy to see that the assertion is correct.

Lemma 5.2. *If condition $(k1)$ holds, then there exist positive constants ρ and β, such that*

$$J_\lambda(v) \ge \beta > 0,\ v \in H_0^1(\Omega),\ \|v\| = \rho. \tag{5.2}$$

Proof. By Lemma 5.1, we have

$$\begin{aligned} J_\lambda(v) &= \frac{1}{2}\int_\Omega (|\nabla v|^2 + v^2)dx - \frac{1}{2}\lambda p \int_\Omega K u_\lambda^{p-1} (v^+)^2 dx \\ &-\lambda \int_\Omega \int_0^{v^+} K[(u_\lambda + s)^p - u_\lambda^p - p u_\lambda^{p-1} s] ds dx \\ &\ge \frac{1}{2}\Big[\int_\Omega (|\nabla v|^2 + v^2)dx - \lambda p \int_\Omega K u_\lambda^{p-1} (v^+)^2 dx\Big] \\ &-\lambda \int_\Omega K\Big[\frac{\epsilon}{2} u_\lambda^{p-1} (v^+)^2 + c_\epsilon \frac{(v^+)^{p+1}}{p+1}\Big] dx. \end{aligned} \tag{5.3}$$

Furthermore, from the definition $\sigma_\lambda(u_\lambda)$ in (4.3), we have

$$\int_\Omega (|\nabla v|^2 + v^2)dx \ge \sigma_\lambda(u_\lambda) p \int_\Omega K u_\lambda^{p-1} (v^+)^2 dx,$$

and, therefore, by (5.3) we obtain

$$J_\lambda(v) \ge \frac{1}{2}\sigma_\lambda(u_\lambda)^{-1}(\sigma_\lambda(u_\lambda) - \lambda - \frac{\epsilon}{2}\lambda)\|v\|^2 - \lambda c_\epsilon (p+1)^{-1} \int_\Omega K (v^+)^{p+1} dx. \tag{5.4}$$

Since $\sigma_\lambda(u_\lambda) > \lambda$, by part (i) of Lemma 4.3, the boundedness of K, and the Sobolev inequality imply that for small $\epsilon > 0$,

$$J_\lambda(v) \ge \frac{1}{4}\sigma_\lambda(u_\lambda)^{-1}(\sigma_\lambda(u_\lambda) - \lambda)\|v\|^2 - \lambda c \|v\|^{p+1},$$

and the conclusion in Lemma 5.2 follows.

Similar to Proposition 2.1, for the energy functional J_λ, we also have the following result:

Lemma 5.3. *Assume condition* $(k1)$ *holds. Let* $\{v_k\}$ *be a* $(PS)_\beta$ *sequence of* J_λ *in* $H_0^1(\Omega)$*:*

$$\begin{aligned} J_\lambda(v_k) &= \beta + o(1) \text{ as } k \to \infty, \\ J'_\lambda(v_k) &= o(1) \text{ strong in } H^{-1}(\Omega). \end{aligned} \tag{5.5}$$

Then there exist an integer $l \geq 0$, *sequence* $\left\{x_k^i\right\} \subseteq \mathbb{R}^N$ *of the form* $(0, z_k^i) \in \mathbb{S}$, *functions* $v_\lambda \in H_0^1(\Omega)$, $\overline{v}_i \in H_0^1(\mathbb{S})$, $1 \leq i \leq l$, *such that for some subsequence (still denoted by)* $\{v_k\}$, *we have*

$$\begin{cases} v_k \rightharpoonup v_\lambda \text{ weakly in } H_0^1(\Omega); \\ \beta = J_\lambda(v_\lambda) + \displaystyle\sum_{i=1}^{l} I_\lambda^\infty(\overline{v}_i^p); \\ -\Delta v_\lambda + v_\lambda = \lambda K[(v_\lambda + u_\lambda)^p - u_\lambda^p] \text{ in } H^{-1}(\Omega); \\ -\Delta \overline{v}_i^p + \overline{v}_i^p = \lambda K_\infty \overline{v}_i^p \text{ in } H^{-1}(\mathbb{S}),\ 1 \leq i \leq l; \\ \left|x_k^i\right| \to \infty,\ \left|x_k^i - x_k^j\right| \to \infty,\ 1 \leq i \neq j \leq l. \end{cases}$$

where we agree that in the case $l = 0$ *the above holds without* $\overline{v}_i$, x_k^i.

Now, let δ be small enough, D^δ a δ-tubular neighborhood of D such that $D^\delta \subset\subset \Omega$. Let $\eta : \mathbb{S} \to [0,1]$ be a C^∞ cut-off function such that $0 \leq \eta \leq 1$ and

$$\eta(x) = \begin{cases} 0, & \text{if } x \in D \cup (\mathbb{S} \setminus \mathbb{S}^+); \\ 1, & \text{if } x \in (\Omega \setminus \overline{D}^\delta) \cap \{x = (y,z) \in \mathbb{S} | z \geq r_0\}. \end{cases}$$

Let $\tau \geq 0$, $e_N = (0, 0, \cdots, 0, 1) \in \mathbb{R}^N$ and ϖ_λ be a ground state solution of $(2.1)_\lambda$, denote

$$\begin{aligned} &\tau_0 = 2 \sup_{x \in D^\delta} |x| + r_0 + 1, \\ &z_0 = \max\{\tau_0, r_0\}, \\ &U_0 = \{(y,z) \in \mathbb{S} : 0 \leq z \leq z_0\}, \\ &\varpi_{\lambda,\tau(x)} = \varpi_\lambda(x - \tau e_N), \\ &\eta_\tau(x) = \eta(x + \tau e_N). \end{aligned}$$

Clearly, $\eta \varpi_{\lambda,\tau} \in H_0^1(\Omega)$.

Lemma 5.4. *Assume conditions* $(k1)$*,* $(k2)$ *and* $(h1)$ *hold, then there exist* $t_0 > 0$*,* $\tau_* \geq \tau_0$ *such that* $J_\lambda(t\eta\varpi_{\lambda,\tau}) < 0$ *for all* $\tau \geq \tau_*$*,* $t \geq t_0$*.*

Proof. By ϖ_λ is a ground state solution of $(2.1)_\lambda$, then we have

$$
\begin{aligned}
&J_\lambda(t\eta\varpi_{\lambda,\tau})\\
&= \frac{1}{2}t^2\int_\Omega(|\nabla(\eta\varpi_{\lambda,\tau})|^2+|\eta\varpi_{\lambda,\tau}|^2)dx-\frac{1}{p+1}t^{p+1}\int_\Omega\lambda K(x)(\eta\varpi_{\lambda,\tau})^{p+1}dx\\
&\quad -\int_\Omega\int_0^{t\eta\varpi_{\lambda,\tau}}\lambda K(x)[(s+u_\lambda)^p-u_\lambda^p-s^p]dsdx\\
&\leq \frac{1}{2}t^2\int_{\mathbb{S}}(-\Delta\varpi_\lambda+\varpi_\lambda)(\eta_\tau^2\varpi_\lambda)dx+\frac{1}{2}t^2\int_{\mathbb{S}}|\nabla\eta_\tau|^2|\varpi_\lambda|^2dx\\
&\quad -\frac{1}{p+1}t^{p+1}\int_{\mathbb{S}}\lambda K(x)\eta(x)\varpi_\lambda^{p+1}(x-\tau e_N)dx\\
&\leq \frac{1}{2}t^2\int_{\mathbb{S}}\lambda K_\infty\varpi_\lambda^{p+1}dx+\frac{1}{2}t^2(\max_{x\in\mathbb{S}}|\nabla\eta|^2)\int_{\mathbb{S}}|\varpi_\lambda|^2dx\\
&\quad -\frac{t^{p+1}}{p+1}\int_{\mathbb{S}}\lambda K(x)\eta(x)\varpi_\lambda^{p+1}(x-\tau e_N)dx,
\end{aligned}\tag{5.6}
$$

Set $B_1(\tau e_N) = \{x = (y, z) \in \mathbb{S} : y \in B^{N-1}(0; r_0/2), |z - \tau| < 1\}$. By condition $(k1)$, there exists $\tau_* \geq \tau_0$ such that $K(x) \geq \frac{K_\infty}{2}$ for $x \in B_1(\tau e_N)$ for all $\tau \geq \tau_*$ and note that $\eta(x) \equiv 1$ on $B_1(\tau e_N)$ for $\tau \geq \tau_*$, then we obtain that

$$
\begin{aligned}
&\int_{\mathbb{S}}\lambda K(x)\eta(x)\varpi_\lambda^{p+1}(x-\tau e_N)dx\\
&\geq \int_{B_1(\tau e_N)}\frac{\lambda}{2}K_\infty\varpi_\lambda^{p+1}(x-\tau e_N)dx\\
&= \int_{\{x=(y,z)\in\mathbb{S}:y\in B^{N-1}(0;r_0),|z|\leq 1\}}\frac{\lambda}{2}K_\infty\varpi_\lambda^{p+1}(x)dx\\
&= c > 0
\end{aligned}\tag{5.7}
$$

where c independent of τ. Combining (5.6) and (5.7), there exist c_1, c_2, independent of τ, such that

$$
J_\lambda(t\eta\varpi_{\lambda,\tau}) \leq c_1t^2 - c_2t^{p+1} \text{ for all } \tau \geq \tau_*. \tag{5.8}
$$

From (5.8), we conclude the result.

Lemma 5.5. *Assume condition* $(k1)$, $(k2)$ *and* $(h1)$ *hold, then there exists* $\tau^* > 0$, *such that the following inequality holds for* $\tau \geq \tau^*$,

$$0 < \sup_{t \geq 0} J_\lambda(t\eta\varpi_{\lambda,\tau}) < I_\lambda^\infty(\varpi_\lambda) = M_\lambda^\infty. \tag{5.9}$$

Proof. From (5.2), we easily see that the left hand of (5.9) holds and we need only to show that the right hand of (5.9) holds. By Lemma 5.4, we have that there exists a constant $t_2 > 0$ such that

$$\sup_{t \geq 0} J_\lambda(t\eta\varpi_{\lambda,\tau}) = \sup_{0 \leq t \leq t_2} J_\lambda(t\eta\varpi_{\lambda,\tau}), \text{ for any } \tau \geq \tau_*.$$

Since J_λ is continuous in $H_0^1(\Omega)$ and $J_\lambda(0) = 0$, there exists a constant $t_1 > 0$ such that

$$J_\lambda(t\eta\varpi_{\lambda,\tau}) < M_\lambda^\infty, \text{ for any } \tau \in (0, \infty) \text{ and } 0 \leq t < t_1.$$

Then, to prove (5.9) we now only to prove the following inequality

$$\sup_{t_1 \leq t \leq t_2} J_\lambda(t\eta\varpi_{\lambda,\tau}) < M_\lambda^\infty, \text{ for } \tau \text{ large enough .}$$

By the definition of J_λ, we get

$$\begin{aligned} J_\lambda(t\eta\varpi_{\lambda,\tau}) = {} & \frac{t^2}{2}\int_\Omega (|\nabla(\eta\varpi_{\lambda,\tau})|^2 + |\eta\varpi_{\lambda,\tau}|^2)dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{S}} \lambda K_\infty(\eta\varpi_{\lambda,\tau})^{p+1}dx \\ & + \frac{t^{p+1}}{p+1}\int_\Omega \lambda(K_\infty - K(x))(\eta\varpi_{\lambda,\tau})^{p+1}dx \\ & - \int_\Omega \int_0^{t\eta\varpi_{\lambda,\tau}} \lambda K(x)[(s+u_\lambda)^p - u_\lambda^p - s^p]dsdx. \end{aligned}$$

Since ϖ_λ is a ground state solution of $(2.1)_\lambda$, then we have

$$\begin{aligned} J_\lambda(t\eta\varpi_{\lambda,\tau}) \leq {} & \frac{t^2}{2}\int_{\mathbb{S}} (-\Delta\varpi_\lambda + \varpi_\lambda)(\eta_\tau^2\varpi_\lambda)dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{S}} \lambda K_\infty \varpi_\lambda^{p+1}dx \\ & + \frac{t_2^2}{2}\int_{\mathbb{S}} |\nabla\eta|^2|\varpi_{\lambda,\tau}|^2dx \\ & + \frac{t_2^{p+1}}{p+1}\int_{\mathbb{S}} \lambda K_\infty(1-\eta^{p+1})\varpi_{\lambda,\tau}^{p+1}dx \\ & + \frac{t_2^{p+1}}{p+1}\int_{\mathbb{S}} \lambda(K_\infty - K(x))^+(\eta\varpi_{\lambda,\tau})^{p+1}dx \\ & - \int_\Omega \int_0^{t\eta\varpi_{\lambda,\tau}} \lambda K(x)[(s+u_\lambda)^p - u_\lambda^p - s^p]dsdx. \end{aligned} \tag{5.10}$$

It follows from (2.3) that for any $0<\epsilon<1+\mu_1$, there exist some constants $\tau_1\geq\tau_*$ and $c_\epsilon>0$, independent of τ, such that, for all $\tau\geq\tau_1$,

$$\frac{t_2^2}{2}\int_{\mathbb{S}}|\nabla\eta|^2|\varpi_{\lambda,\tau}|^2dx\leq\frac{t_2^2}{2}\int_{U_0}|\nabla\eta|^2|\varpi_{\lambda,\tau}|^2dx\leq c_\epsilon\exp(-2\sqrt{1+\mu_1-\epsilon}\tau), \tag{5.11}$$

$$\begin{aligned}
&\frac{t_2^{p+1}}{p+1}\int_{\mathbb{S}}\lambda K_\infty(1-\eta^{p+1})\varpi_{\lambda,\tau}^{p+1}dx\\
\leq\ &\frac{t_2^{p+1}}{p+1}\int_{D^\delta\cup\{(y,z)\in\mathbb{S}:z\leq r_0\}}\lambda K_\infty\varpi_{\lambda,\tau}^{p+1}dx\\
\leq\ &c\int_{\{(y,z)\in\mathbb{S}:z\leq z_0\}}\varphi(y)\exp(-(p+1)\sqrt{1+\mu_1-\epsilon}|z|)dx\\
\leq\ &c_\epsilon\exp(-2\sqrt{1+\mu_1-\epsilon}\tau).
\end{aligned}\tag{5.12}$$

Now, we fix a constant α with $\frac{1}{\gamma}<\alpha<\frac{p}{p+1}$. By condition $(k2)$ and (2.3), there exists $\tau_2>0$ such that, for all $\tau\geq\tau_2$,

$$\begin{aligned}
&\frac{t_2^{p+1}}{p+1}\int_{\mathbb{S}}\lambda(K_\infty-K(x))^+(\eta\varpi_{\lambda,\tau})^{p+1}dx\\
\leq\ &c\Big(\int_{\mathbb{S}\cap\{|z|\geq\alpha\tau\}}+\int_{\mathbb{S}\cap\{|z|\leq\alpha\tau\}}\Big)(K_\infty-K(x))^+\varpi_\lambda^{p+1}(x+\tau e_N)dx\\
\leq\ &c_1\exp(-\alpha\gamma\sqrt{1+\mu_1}\tau)+2\tilde{c}_\epsilon\alpha\tau\exp(-(p+1)(1-\alpha)\sqrt{1+\mu_1-\epsilon}\tau)
\end{aligned}\tag{5.13}$$

where c_1 and $\tilde{c}_\epsilon$ are some positive constants independent of τ.

Set $B_1(\tau e_N)=\{x=(y,z)\in\mathbb{S}:y\in B^{N-1}(0;r_0/2),|z-\tau|<1\}$. By the definition of z_0, we have that $B_1(\tau e_N)\subset\subset\Omega$ for all $\tau\geq z_0$. Noting that $(a+b)^p\geq a^p+b^p$, for all $a\geq0$, $b\geq0$, $p>1$. Then for $\tau\geq z_0$, we have $\eta(x)=1$ on $B_1(\tau e_N)$ and

$$\begin{aligned}
&\int_\Omega\int_0^{t\eta\varpi_{\lambda,\tau}}\lambda K(x)[(s+u_\lambda)^p-u_\lambda^p-s^p]dsdx\\
\geq&\int_{B_1(\tau e_N)}\int_0^{t\varpi_{\lambda,\tau}}\lambda K(x)[(s+u_\lambda)^p-u_\lambda^p-s^p]dsdx\\
=&\int_{B_1(\tau e_N)}\int_0^{t\varpi_{\lambda,\tau}}\lambda K(x)\Big([(s+u_\lambda)^{p-1}-s^{p-1}]s+[(s+u_\lambda)^{p-1}-u_\lambda^{p-1}]u_\lambda\Big)dsdx\\
\geq&\int_{B_1(\tau e_N)}\int_0^{t\varpi_{\lambda,\tau}}\lambda K(x)[(s+u_\lambda)^{p-1}-u_\lambda^{p-1}]u_\lambda dsdx\\
=&\int_{B_1(\tau e_N)}\lambda K(x)\Big[\frac{(t\varpi_{\lambda,\tau}+u_\lambda)^p-u_\lambda^p}{p\varpi_{\lambda,\tau}}-tu_\lambda^{p-1}\Big]\varpi_{\lambda,\tau}u_\lambda dx.
\end{aligned}\tag{5.14}$$

By part (iii) of Lemma 3.1, there exist $\tau_3 \geq z_0 + \tau_2$ and $\beta > 0$, such that

$$\frac{(t\varpi_{\lambda,\tau} + u_\lambda)^p - u_\lambda^p}{p\varpi_{\lambda,\tau}} - tu_\lambda^{p-1} \geq \beta, \text{ for } \tau \geq \tau_3, \ x \in B_1(\tau e_N), \ t \in [t_1, t_2],$$

then by $(k1)$, (3.1) and (5.14), there exist $\tau_4 \geq \tau_3$ such that $K(x) \geq \frac{K_\infty}{2}$ for $x \in B_1(\tau e_N)$ and

$$\begin{aligned} &\int_\Omega \int_0^{t\eta\varpi_{\lambda,\tau}} \lambda K(x)[(s+u_\lambda)^p - u_\lambda^p - s^p] ds dx \\ &\geq \frac{1}{2}\lambda\beta K_\infty \int_{B_1(\tau e_N)} \varpi_\lambda(x - \tau e_N) u_\lambda(x) dx \\ &\geq c \int_{B_1(\tau e_N)} \varpi_\lambda(x - \tau e_N) \exp(-(\tau+1)\sqrt{1+\mu_1}) dx \\ &\geq c_2 \exp(-\sqrt{1+\mu_1}\tau), \end{aligned} \tag{5.15}$$

where $c_2 > 0$ is independent of τ for all $\tau \geq \tau_4$ and $t \in [t_1, t_2]$.

By (5.10) and use (5.11)-(5.15), we get, for $\tau \geq \tau_4 + \tau_*$ and $t \in [t_1, t_2]$,

$$\begin{aligned} J_\lambda(t\eta\varpi_{\lambda,\tau}) \leq M_\lambda^\infty &+ 2c_\epsilon \exp(-2\sqrt{1+\mu_1-\epsilon}\tau) + c_1 \exp(-\alpha\gamma\sqrt{1+\mu_1}\tau) \\ &+ 2\tilde{c}_\epsilon \alpha\tau \exp(-(p+1)(1-\alpha)\sqrt{1+\mu_1-\epsilon}\tau) - c_2 \exp(-\sqrt{1+\mu_1}\tau) \end{aligned}$$

where c_ϵ, $\tilde{c}_\epsilon$, c_i, $1 \leq i \leq 2$, are independent of τ.

Since $0 < \frac{1}{\gamma} < \alpha < \frac{p}{p+1}$, we have that there exists a constant $\epsilon_0 > 0$ such that

$$\nu_\epsilon = \min\{2\sqrt{1+\mu_1-\epsilon}, \alpha\gamma\sqrt{1+\mu_1}, (p+1)(1-\alpha)\sqrt{1+\mu_1-\epsilon}\} > \sqrt{1+\mu_1}$$

for all positive constant $\epsilon \leq \epsilon_0$.

Therefore, we can find some $\tau^* > \tau_4 + \tau_*$ large enough such that for all $\tau \geq \tau^*$

$$\begin{aligned} &2c_{\epsilon_0} \exp(-2\sqrt{1+\mu_1-\epsilon_0}\tau) + c_1 \exp(-\alpha\gamma\sqrt{1+\mu_1}\tau) \\ &+ 2\tilde{c}_{\epsilon_0}\alpha\tau \exp(-(p+1)(1-\alpha)\sqrt{1+\mu_1-\epsilon_0}\tau) - c_2 \exp(-\sqrt{1+\mu_1}\tau) < 0 \end{aligned}$$

and (5.9) is proved.

Proposition 5.6. *Assume conditions* $(k1)$ *and* $(k2)$ *hold, then equation* $(5.1)_\lambda$ *has at least one solution for* $\lambda \in (0, \lambda^*)$.

Proof. For the constant τ^* in Lemma 5.5, and by Lemma 5.4, we know that there is $t_0 > 0$ such that $J_\lambda(t_0\eta u_{\tau^*}) < 0$. We set

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \ \gamma(1) = t_0\eta u_{\tau^*}\},$$

then, by (5.2) and (5.9) we get

$$0 < \beta \le c = \inf_{\gamma \in \Gamma} \max_{0 \le s \le 1} J_\lambda(\gamma(s)) < M_\lambda^\infty. \tag{5.16}$$

Applying the mountain pass lemma of Ambrosetti-Rabinowitz [2], there exists a $(PS)_c-$sequence $\{v_k\}$ such that

$$J_\lambda(v_k) \to c \text{ and } J'_\lambda(v_k) \to 0 \text{ in } H^{-1}(\Omega).$$

By Lemma 5.3, there exist a subsequence, still denoted by $\{v_k\}$, an integer $l \ge 0$, a solution v_λ of $(5.1)_\lambda$ and solutions $\overline{v}_\lambda^i$ of $(2.1)_\lambda$, for $1 \le i \le l$, such that

$$c = J_\lambda(v_\lambda) + \sum_{i=1}^{l} I_\lambda^\infty(\overline{v}_\lambda^i). \tag{5.17}$$

By the strong maximum principle, to complete the proof, we only need to prove $v_\lambda \not\equiv 0$ in Ω. In fact, by (5.16) and (5.17), we have

$$c = J_\lambda(v_\lambda) \ge \beta > 0 \text{ if } l = 0, M_\lambda^\infty > c \ge J_\lambda(v_\lambda) + M_\lambda^\infty \text{ if } l \ge 1.$$

This implies $v_\lambda \not\equiv 0$ in Ω.

6. Properties of Solutions

In this section, we always assume that conditions $(k1)$, $(k2)$ and $(h1)$ hold. Denote by $Q = \{(\lambda, u) : u \text{ solves equation } (1.2)_\lambda, \lambda \in [0, \lambda^*]\}$. By Lemma 3.1, we have $Q \subset L^\infty(\Omega) \cap H_0^1(\Omega)$.

For each $(\lambda, u) \in Q$, let $\sigma_\lambda(u)$ denote the number defined by (4.2), which is the first eigenvalue of the problem (4.3).

Lemma 6.1. *Let (λ, u) and $(\lambda, u_\lambda) \in Q$, where u_λ is the minimal solution of equation $(1.2)_\lambda$ for $\lambda \in (0, \lambda^*)$. Then*

(i) $\sigma_\lambda(u) > \lambda$ if and only if $u = u_\lambda$;

(ii) $\sigma_\lambda(U_\lambda) < \lambda$, where U_λ is the second solution of equation $(1.2)_\lambda$ constructed in Section 5.

Proof. (i) Now, let $\psi \ge 0$ and $\psi \in H_0^1(\Omega)$. Since u and u_λ are the solution of equation $(1.2)_\lambda$, then

$$\begin{aligned}
\int_\Omega \nabla\psi \cdot \nabla(u_\lambda - u)dx + \int_\Omega \psi(u_\lambda - u)dx &= \lambda \int_\Omega K(u_\lambda^p - u^p)\psi dx \\
&= \lambda \int_\Omega \Big(\int_u^{u_\lambda} t^{p-1}dt\Big) pK\psi dx \\
&\ge \lambda \int_\Omega pKu^{p-1}(u_\lambda - u)\psi dx.
\end{aligned} \tag{6.1}$$

Let $\psi = (u - u_\lambda)^+ \geq 0$ and $\psi \in H_0^1(\Omega)$. If $\psi \not\equiv 0$, then (6.1) implies

$$-\int_\Omega (|\nabla\psi|^2 + \psi^2)dx \geq -\lambda \int_\Omega pKu^{p-1}\psi^2 dx$$

and, therefore, the definition of $\sigma_\lambda(u)$ implies

$$\begin{aligned}\int_\Omega (|\nabla\psi|^2 + \psi^2)dx &\leq \lambda \int_\Omega pKu^{p-1}\psi^2 dx \\ &< \sigma_\lambda(u) \int_\Omega pKu^{p-1}\psi^2 dx \\ &\leq \int_\Omega (|\nabla\psi|^2 + \psi^2)dx,\end{aligned}$$

which is impossible. Hence $\psi \equiv 0$, and $u = u_\lambda$ in Ω. On the other hand, by Lemma 4.3, we also have that $\sigma_\lambda(u_\lambda) > \lambda$. This completes the proof of (i).

(ii) By part (i), we get that $\sigma_\lambda(U_\lambda) \leq \lambda$ for $\lambda \in (0, \lambda^*)$. We claim that $\sigma_\lambda(U_\lambda) = \lambda$ can not occur. We proceed by contradiction. Set $w = U_\lambda - u_\lambda$; we have

$$(6.2) \qquad -\Delta w + w = \lambda K[U_\lambda^p - (U_\lambda - w)^p],\ w > 0 \text{ in } \Omega.$$

By $\sigma_\lambda(U_\lambda) = \lambda$, we have that the problem

$$(6.3) \qquad -\Delta\phi + \phi = \lambda pKU_\lambda^{p-1}\phi, \qquad \phi \in H_0^1(\Omega)$$

possesses a positive solution ϕ_1.

Multiplying (6.2) by ϕ_1 and (6.3) by w, integrating and subtracting we deduce that

$$\begin{aligned}0 &= \int_\Omega \lambda K[U_\lambda^p - (U_\lambda - w)^p - pU_\lambda^{p-1}w]\phi_1 dx \\ &= -\frac{1}{2}p(p-1)\int_\Omega \lambda K\xi_\lambda^{p-2}w^2\phi_1 dx,\end{aligned}$$

where $\xi_\lambda \in (u_\lambda, U_\lambda)$. Thus $w \equiv 0$, that is $U_\lambda = u_\lambda$ for $\lambda \in (0, \lambda^*)$. This is a contradiction. Hence, we have that $\sigma_\lambda(U_\lambda) < \lambda$ for $\lambda \in (0, \lambda^*)$.

Remark 6.2. Since $\sigma_\lambda(U_\lambda) < \lambda$, one may employ a similar argument to the used for u_λ to show that U_λ is strictly decreasing in λ, $\lambda \in (0, \lambda^*)$.

Lemma 6.3. *Let u_λ be the minimal solution of equation $(1.2)_\lambda$ for $\lambda \in [0, \lambda^*]$ and $\sigma_\lambda(u_\lambda) > \lambda$. Then for any $g(x) \in H^{-1}(\Omega)$, problem*

$$(6.4)_\lambda \qquad -\Delta w + w = \lambda pKu_\lambda^{p-1}w + g(x), w \in H_0^1(\Omega)$$

has a solution.

Proof. Consider the functional

$$\Phi(w) = \frac{1}{2}\int_\Omega (|\nabla w|^2 + w^2)dx - \frac{1}{2}\lambda p \int_\Omega K u_\lambda^{p-1} w^2 dx - \int_\Omega g(x) w dx,$$

where $w \in H_0^1(\Omega)$. From Hölder inequality and Young's inequality, we have, for any $\epsilon > 0$, that

$$\begin{aligned}\Phi(w) &\geq \frac{1}{2}(1 - \lambda\sigma_\lambda(u_\lambda)^{-1})\|w\|^2 - \frac{1}{2}\epsilon\|w\|^2 - \frac{C_\epsilon}{2}\|g\|^2_{H^{-1}(\Omega)} \\ &\geq -C\|g\|^2_{H^{-1}(\Omega)}\end{aligned} \tag{6.5}$$

if we choose ϵ small.

Now, let $\{w_n\} \subset H_0^1(\Omega)$ be the minimizing sequence of variational problem

$$d = \inf\{\Phi(w) | w \in H_0^1(\Omega)\}.$$

From (6.5) and $\sigma_\lambda(u_\lambda) > \lambda$, we can also deduce that $\{w_n\}$ is bounded in $H_0^1(\Omega)$, if we choose ϵ small. So we may suppose that

$$\begin{aligned}&w_n \rightharpoonup w \text{ weakly in } H_0^1(\Omega) \text{ as } n \to \infty, \\ &w_n \to w \text{ a.e. in } \Omega \text{ as } n \to \infty.\end{aligned}$$

By Fatou's Lemma,

$$\|w\|^2 \leq \liminf \|w_n\|^2.$$

The weak convergence and the fact that $u_\lambda(x) \to 0$ as $|x| \to \infty$ imply

$$\int_\Omega g w_n dx \to \int_\Omega g w dx, \int_\Omega K u_\lambda^{p-1} w_n^2 dx \to \int_\Omega K u_\lambda^{p-1} w^2 dx \text{ as } n \to \infty.$$

Therefore

$$\Phi(w) \leq \lim_{n\to\infty} \Phi(w_n) = d,$$

and hence $\Phi(w) = d$ which gives that w is a solution of $(6.4)_\lambda$.

Lemma 6.4. *Suppose u_{λ^*} is a solution of $(1.1)_{\lambda^*}$, then $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$ and the solution u_{λ^*} is unique.*

Proof. Define $F : \mathbb{R} \times H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ by

$$F(\lambda, u) = \Delta u - u + \lambda K(u^+)^p + h(x).$$

Since $\sigma_\lambda(u_\lambda) \geq \lambda$ for $\lambda \in (0, \lambda^*)$, so $\sigma_{\lambda^*}(u_{\lambda^*}) \geq \lambda^*$. If $\sigma_{\lambda^*}(u_{\lambda^*}) > \lambda^*$, the equation $F_u(\lambda^*, u_{\lambda^*})\phi = 0$ has no nontrivial solution. From Lemma 6.3, F_u maps $\mathbb{R} \times H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Applying the implicit function theorem to F, we can find a neighborhood $(\lambda^* - \delta, \lambda^* + \delta)$ of λ^* such that equation $(1.2)_\lambda$ possesses a solution u_λ if $\lambda \in (\lambda^* - \delta, \lambda^* + \delta)$. This is contradictory to the definition of λ^*. Hence, we obtain that $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$.

Next, we are going to prove that u_{λ^*} is unique. In fact, suppose $(1.1)_{\lambda^*}$ has another solution $U_{\lambda^*} \geq u_{\lambda^*}$. Set $w = U_{\lambda^*} - u_{\lambda^*}$; we have

$$-\Delta w + w = \lambda^* K[(w + u_{\lambda^*})^p - u_{\lambda^*}^p],\ w > 0 \text{ in } \Omega. \tag{6.6}$$

By $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$, we have that the problem

$$-\Delta\phi + \phi = \lambda^* pKu_{\lambda^*}^{p-1}\phi, \qquad \phi \in H_0^1(\Omega) \tag{6.7}$$

possesses a positive solution ϕ_1.

Multiplying (6.6) by ϕ_1 and (6.7) by w, integrating and subtracting we deduce that

$$\begin{aligned} 0 &= \int_\Omega \lambda^* K[(w + u_{\lambda^*})^p - u_{\lambda^*}^p - pu_{\lambda^*}^{p-1}w]\phi_1 dx \\ &= \frac{1}{2}p(p-1)\int_\Omega \lambda^* K\xi_{\lambda^*}^{p-2}w^2\phi_1 dx, \end{aligned}$$

where $\xi_{\lambda^*} \in (u_{\lambda^*}, u_{\lambda^*} + w)$. Thus $w \equiv 0$.

Proposition 6.5. *Let u_λ be the minimal solution of equation $(1.2)_\lambda$. Then u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in [0, \lambda^*]$ and*

$$u_\lambda \to u_0 \textit{ in } L^\infty(\Omega) \cap H_0^1(\Omega) \textit{ as } \lambda \to 0,$$

where u_0 is the unique positive solution of $(1.1)_0$.

Proof. By Lemma 3.1, 4.2, and 6.4, we can deduce $\|u_\lambda\|_{L^\infty(\Omega)} \leq \|u_{\lambda^*}\|_{L^\infty(\Omega)} \leq c$ for $\lambda \in [0, \lambda^*]$. By (4.6), we have that $\|u_\lambda\| \leq \dfrac{p}{p+1}\|h\|_{L^2(\Omega)}$. Hence, u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for $\lambda \in [0, \lambda^*]$.

Now, let $w_\lambda = u_\lambda - u_0$, then w_λ satisfies the following equation

$$-\Delta w_\lambda + w_\lambda = \lambda K u_\lambda^p \text{ in } \Omega, \tag*{$(6.8)_\lambda$}$$

and by u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, we have that

$$\begin{aligned} \|w_\lambda\|^2 &= \int_\Omega \lambda K u_\lambda^p w_\lambda dx \\ &\leq \lambda\|K\|_{L^\infty(\Omega)}\|u_\lambda\|_{L^\infty(\Omega)}^{p-1}\|u_\lambda\|_{L^2(\Omega)}\|w_\lambda\|_{L^2(\Omega)} \\ &\leq c\lambda, \end{aligned} \tag{6.9}$$

where c is independent of λ. Hence, we obtain that $u_\lambda \to u_0$ in $H_0^1(\Omega)$ as $\lambda \to 0$.

By Lemma 3.1, $u_\lambda \in L^q(\Omega)$ for all $q \in [2, \infty)$ and u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, then we have that for any $q \in [2, \infty)$, there exists a positive constant c_q, independent of u_λ, $\lambda \in [0, \lambda^*]$, such that

$$\|Ku_\lambda^p\|_{L^q(\Omega)} \le c_q. \tag{6.10}$$

Now, let $q' = N/2 + 1 > N/2$. Apply the proof of part (i) and (ii) in Lemma 3.1, to equation $(6.8)_\lambda$ and by (6.9) and (6.10), we obtain that

$$\begin{aligned}\|w_\lambda\|_{L^\infty(\Omega)} &\le c_1 \|w_\lambda\|_{W^{2,q'}(\Omega)} \\ &\le c_2 \|\lambda K u_\lambda^p\|_{L^{q'}(\Omega)} \\ &\le c\lambda,\end{aligned}$$

where c is independent of λ. Hence, we obtain that $u_\lambda \to u_0$ in $L^\infty(\Omega)$ as $\lambda \to 0$.

Proposition 6.6. *For $\lambda \in (0, \lambda^*)$, let U_λ be the positive solution of equation $(1.2)_\lambda$ with $U_\lambda > u_\lambda$, then U_λ is unbounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, that is*

$$\lim_{\lambda \to 0} \|U_\lambda\| = \lim_{\lambda \to 0} \|U_\lambda\|_{L^\infty(\Omega)} = \infty.$$

Proof. First, we show that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $H_0^1(\Omega)$. Since $U_\lambda = u_\lambda + v_\lambda$, we only need to show that $\{v_\lambda : \lambda > 0\}$ is unbounded in $H_0^1(\Omega)$. If not, then

$$\|v_\lambda\| \le M \tag{6.11}$$

for all $\lambda \in (0, \lambda^*)$. Since for any $\delta > 0$, $\{U_\lambda\}_{\lambda \ge \delta}$ is bounded in $H_0^1(\Omega)$, we may assume $\lambda \in (0, \delta]$.

Choose $\lambda_n \downarrow 0$ and let v_{λ_n} be the corresponding solutions constructed by Proposition 5.6. By the Hölder inequality and the Sobolev embedding theorem, we obtain that

$$\begin{aligned}\int_\Omega (|\nabla v_{\lambda_n}|^2 + |v_{\lambda_n}|^2) dx &= \int_\Omega \lambda_n K [U_{\lambda_n}^p - u_{\lambda_n}^p] v_{\lambda_n} dx \\ &\le c\lambda_n \|U_{\lambda_n}\|_{L^{p+1}(\Omega)}^p \|v_{\lambda_n}\|_{L^{p+1}(\Omega)} \\ &\le c\lambda_n \|U_{\lambda_n}\|^p \|v_{\lambda_n}\| \\ &\le c_1 \lambda_n\end{aligned}$$

for some constant c_1, independent of v_{λ_n}, where we have used (6.11) and the boundedness of $\{u_{\lambda_n}\}$ in $H_0^1(\Omega)$. Hence, we have $\lim\limits_{n \to \infty} \|v_{\lambda_n}\|^2 = 0$. It implies that

$$\lim_{n \to \infty} \|v_{\lambda_n}\|_{L^2(\Omega)} = 0. \tag{6.12}$$

On the other hand, we notice that $U_\lambda = u_\lambda + v_\lambda$ is decreasing and u_λ is increasing in λ. Therefore, v_λ is decreasing in λ, which implies

$$v_{\lambda_n} \geq v_\delta \text{ for all } n.$$

Then we obtain that

$$\|v_{\lambda_n}\|_{L^2(\Omega)} \geq \|v_\delta\|_{L^2(\Omega)} > 0 \text{ for all } n.$$

which contradicts (6.12). This implies that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $H_0^1(\Omega)$.

Now, we show that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $L^\infty(\Omega)$. We proceed by contradiction. Assume to the contrary that there exists $c_0 > 0$ such that

$$\|U_\lambda\|_{L^\infty(\Omega)} \leq c_0 < \infty \text{ for all } \lambda \in (0, \lambda^*).$$

Since U_λ is a solution of equation $(1.2)_\lambda$, we have that

$$\begin{aligned}\|U_\lambda\|^2 &= \int_\Omega \lambda K U_\lambda^{p+1} dx + \int_\Omega h U_\lambda dx \\ &\leq \lambda c_0^{p-1} \|K\|_{L^\infty(\Omega)} \|U_\lambda\|^2_{L^2(\Omega)} + \|h\|_{L^2(\Omega)} \|U_\lambda\|_{L^2(\Omega)} \\ &\leq c_1 \lambda \|U_\lambda\|^2 + c_2 \|U_\lambda\|,\end{aligned}$$

where c_1 and c_2 are independent of λ. If we choose $\lambda_0 = \min\{\lambda^*, \frac{1}{2c_1}\}$, then there exists $c > 0$, independent of λ, such that $\|U_\lambda\| \leq c$ for all $\lambda \leq \lambda_0$. This is a contradiction to that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $H_0^1(\Omega)$. This completes the proof of Proposition 6.6.

Proof of Theorem 1.1 and Theorem 1.2. Theorem 1.1 now follows from Lemma 4.2, 4.3, 4,4, 6.1, 6.4, and Proposition 5.6. The conclusion of Theorem 1.2 follows immediately from Lemma 4.2, Remark 6.2 and Proposition 6.5, 6.6.

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